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THE CONVERGENCE OF MULTI-INDEX SEQUENCES OF BAIRE FUNCTIONS

This paper deals with the convergence of multi-index sequences of Baire functions with respect to category bases. It is a generalization of some theorems concerning the convergence of double measurable sequences of real functions, presented in [2]. The abstract generalizations in category bases possess the most natural application in the case of measure and category.

W. Sierpiński, while solving Sikorski problem (see [3]), proved the theorem on the convergence almost everywhere of double sequences of measurable functions in the Lebesgue sense. E. Wagner (see [2]) generalized the result of Sierpiński by considering the convergence of double sequences of measurable functions with respect to a σ-field with that the convergence takes place, everywhere except a set belonging to a σ-ideal. She defined the condition (F) which turned out to be necessary and sufficient for the convergence of a double sequence of measurable functions.

J. C. Morgan II (see [1]) carried over completely the results of Wagner to quite arbitrary category bases in such a way that measurable functions become abstract Baire functions.

Let us begin with some definitions.

DEFINITION 1 (cf. [1]). A pair $(X, \mathcal{S})$, where X is a set and $\mathcal{S}$ is a family of subsets of X, is called a category base if the nonempty sets $\mathcal{S}$, called regions, satisfy the following axioms:

1. Each point of X belongs to some region.

2. Let A be a region and let $\mathcal{D}$ be any nonempty family of disjoint regions.

(a) If $A \cap (\bigcup\mathcal{D})$ contains a region, then there is a region $D \in \mathcal{D}$ such that $A \cap D$ contains a region;

(b) If $A \cap (\bigcup\mathcal{D})$ contains no region, then there is a region $B \subset A$ which is disjoint from each region in $\mathcal{D}$.

DEFINITION 2 (cf. [1]). We shall say that a set A is nowhere dense in a base $(X, \mathcal{S})$ if a quite arbitrary region contains a subregion disjoint from A.

DEFINITION 3 (cf. [1]). We shall say that a set A is of the first category in a base $(X, \mathcal{S})$ if $A = \bigcup_{n=1}^{\infty} A_n$ where A_n is nowhere dense for each $n \in N$. Otherwise the set A is of the second category.

DEFINITION 4 (cf. [1]). We shall say that a set $A \subset X$ is a Baire set in a base $(X, \mathcal{S})$ if an arbitrary region P contains a subregion P' such that the set $A \cap P'$ or $(X - A) \cap P'$ is a set of the first category.

The family of Baire sets in a category base $(X, \mathcal{S})$, denoted by $\mathcal{B}(\mathcal{S})$ forms the σ-field containing all regions and the family of all sets of the first category. If, moreover, the family of pairwise disjoint regions is at most countable, then the family $B(\mathcal{S})$ is the smallest σ-field possessing the above property. In this case, the Baire sets have a special representation (see [1]).

DEFINITION 5 (cf. [1]). We shall say that a function $f: X \to R$ is a Baire function in a base $(X, \mathcal{S})$ if, for an arbitrary real number a, $\{x: f(x) < a\} \in \mathcal{B}(\mathcal{S})$.

Morgan pointed out that, with every measurable space $(X, \mathcal{A})$ with a σ-ideal $\mathcal{I} \subset \mathcal{A}$ such that each subfamily of the pairwise disjoint sets of the family $\mathcal{S} = \mathcal{A} \setminus \mathcal{I}$ is at most countable, we can associate a category base $(X, \mathcal{S})$ such that the family of the first category sets is identical with the σ-ideal $\mathcal{I}$.

This theorem is the natural reason to create some notations and definitions not only in measurable spaces, but also in spaces with category bases.

DEFINITION 6 (cf. [1], [3]). We shall say that a category base $(X, \mathcal{S})$ satisfies the condition (F) if, for every second category Baire set D and every double sequence $\{B_{j,n}\}_{j,n\in N}$ of Baire sets satisfying

(a) $B_{j,n} \subset B_{j,n+1}$ for all $j, n \in N$;

(b) $D = \bigcup_{n=1}^{\infty} B_{j,n}$ for all $j \in N$;

(c) $B_{j_1,n_1} \supset B_{j_2,n_2}$ if $j_1 < j_2$ and $j_1 + r_1 = j_2 + r_2$,

there exists a sequence $\{j_p\}_{p\in N}$ of positive integers for which $\limsup_{p\in N} B_{j,n_p}$ is a second category set.

Further the convergence essentially everywhere of a sequence of real-valued functions in a category base (X, S) means the convergence everywhere except a set of the first category. The connection between the convergence of double sequences of Baire functions and the fulfilmnet of the condition (F) by a category base (X, S) is given by the following theorem (see [1], [3]).

THEOREM A. Let $\{f_{m,n}\}_{m,n\in N}$ be a double sequence of real-valued Baire functions defined on X. The convergence essentially everywhere to a function f of all sequences $\{f_{m_k,n_k}\}_{k\in N}$, where $m_k \xrightarrow[k\to\infty]{} \infty$, $n_k \xrightarrow[k\to\infty]{} \infty$, implies the convergence essentially everywhere to f of a double sequence $\{f_{m,n}\}_{m,n\in N}$ if and only if (X,S) satisfies the condition (F).

Now, we are going to consider n-index sequences $\{f_{k_1, \ldots, k_n}\}_{k_1, \ldots, k_n\in N}$ of real-valued Baire functions. We shall prove an analogue of the previous theorem.

LEMMA. For every positive integer n, there exists a one-to--one function φ of a set $A = \{(k_1, \ldots, k_n): k_1, \ldots, k_n \in N\}$ onto a set $B = \{(l_1, l_2): l_1, l_2 \in N\}$ satisfying the following conditions:

(a) $\forall_{M\in R} [(k_1 \geq M \wedge k_2 \geq M \wedge \ldots \wedge k_n \geq M) \Leftrightarrow (l_1(k_1, \ldots, k_n) \geq M \wedge l_2(k_1, \ldots, k_n) \geq M)]$,

(b) $(k_1(s) \xrightarrow[s\to\infty]{} \infty \wedge \ldots \wedge k_n(s) \xrightarrow[s\to\infty]{} \infty) \Leftrightarrow (l_1(k_1(s), \ldots, k_n(s)) \xrightarrow[s\to\infty]{} \infty \wedge l_2(k_1(s), \ldots, k_1(s)) \xrightarrow[s\to\infty]{} \infty)$.

P r o o f. For every positive integer m, put

$$A_m = \{(k_1, \ldots, k_n): k_1, \ldots, k_n \in N;\ \min(k_1, \ldots, k_n) = m\},$$

$$B_m = \{(l_1, l_2): l_1, l_2 \in N;\ \min(l_1, l_2) = m\}.$$

It is easy to see that $A_{m_1} \cap A_{m_2} = \emptyset$ for any disjoint positive integers m_1, m_2, and that $\bigcup_{m=1}^{\infty} A_m = A$. Similarly, $B_{m_1} \cap B_{m_2} = \emptyset$ for any distinct positive integers m_1, m_2, and $\bigcup_{m=1}^{\infty} B_m = B$. The sets A_m and B_m have the same cardinality for every integer m, thus there exists a one-to-one mapping φ_m of A_m onto B_m. If we define a function $\varphi: A \to B$ such that $\varphi(x) = \varphi_m(x)$ for each $x \in A_m$ we immediately observe that φ is defined in a correct way. Moreover, we see that $\varphi(A_m) = B_m$ for every integer m.

We shall prove conditions (a) and (b).

Let $M \in R$ and let $k_1, \ldots, k_n$ be positive integers such that $k_1 \geq M, \ldots, k_n \geq M$. Then $(k_1, \ldots, k_n) \in A_s$ where $s \in M$. Since $\varphi(k_1, \ldots, k_n) = (l_1(k_1, \ldots, k_n), l_2(k_1, \ldots, k_n)) \in B_s$, therefore $\min(l_1, l_2) = s \geq M$. Hence $l_1 \geq M$ and $l_2 \geq M$.

Suppose now that $l_1 \geq M$ and $l_2 \geq M$. Then $(l_1, l_2) \in B_s$, $s \geq M$. Since $\varphi^{-1}(l_1, l_2) = (k_1, \ldots, k_n) \in A_s$, therefore $\min(k_1, \ldots, k_n) = s \geq M$. Hence $k_1 \geq M, \ldots, k_n \geq M$ and condition (a) is satisfied.

Condition (b) is a simple consequence of (a).

Let $(X, \mathcal{S})$ be an arbitrary category base.

THEOREM 1. Let $\{f_{k_1, \ldots, k_n}\}_{k_1, \ldots, k_n \in N}$ be an n-index sequence of real Baire functions defined on X for any positive integer $n \geq 2$. The convergence essentially everywhere to a function f of all sequences $\{f_{k_1(s), \ldots, k_n(s)}\}_{s \in N}$, where $k_1(s) \xrightarrow[s\to\infty]{} \infty, \ldots, k_n(s) \xrightarrow[s\to\infty]{} \infty$, implies the convergence essentially everywhere to f of an n-idex sequence

$\{f_{k_1, \ldots, k_n}\}_{k_1, \ldots, k_n \in N}$ if and only if the category base (X, S) satisfies the condition (F).

P r o o f. Sufficiency. Let us suppose that (X, S) satisfies the condition (F). Let φ be the function defined in our Lemma. Let us define a double sequence $\{F_{l_1,l_2}\}_{l_1,l_2 \in N}$ of Baire functions in the following way:

$$F_{l_1,l_2}(x) = f_{k_1, \ldots, k_n}(x) \quad \text{for } x \in X, \text{ where}$$

$$(k_1, \ldots, k_n) = \varphi^{-1}(l_1, l_2).$$

The sequence $\{F_{l_1,l_2}\}_{l_1,l_2 \in N}$ has the property that, for arbitrary sequences $\{l_1(s)\}_{s \in N}$, $\{l_2(s)\}_{s \in N}$ tending to infinity, the sequence $\{F_{l_1(s),l_2(s)}\}_{s \in N}$ converges essentially everywhere to f. Indeed, putting $(k_1(s), \ldots, k_n(s)) = \varphi^{-1}(l_1(s), \; l_2(s))$ and using condition (b) of the Lemma, we have the sequence $\{k_1, (s)\}_{s \in N}, \ldots, \{k_n(s)\}_{n \in N}$ tending to infinity. Hence, by the assumption of the theorem and the definition of the sequence $\{F_{l_1,l_2}\}_{l_1,l_2 \in N}$, we have just obtained that the sequence $\{F_{l_1(s),l_2(s)}\}_{s \in N}$ is convergent essentially everywhere to f. Applying Theorem A, we conclude that the sequence $\{F_{l_1,l_2}\}_{l_1,l_2 \in N}$ is convergent essentially everywhere to f. According to condition (a) of the Lemma, we have

$$\lim_{k_1, \ldots, k_n \to \infty} f_{k_1, \ldots, k_n}(x) = \lim_{l_1, l_1 \to \infty} F_{l_1,l_2}(x) \quad \text{for } x \in X,$$

which means the convergence of the sequence $\{f_{k_1, \ldots, k_n}\}_{k_1, \ldots, k_n \in N}$ essentially everywhere.

Necessity. Let us suppose that the category base (X, S) does not satisfy the condition (F). Then, analogously as in the proof of Theorem A, there exists a double sequence $\{F_{l_1,l_2}\}_{l_1,l_2 \in N}$ of Baire functions such that, for all sequences of positive integers $\{l_1(s)\}_{s \in N}$, $\{l_2(s)\}_{s \in N}$ tending to infinity, the sequence

$\{F_{l_1(s),l_2(s)}\}_{s\in N}$ is convergent essentially everywhere to 0, whereas the double sequence $\{F_{l_1,l_2}\}_{l_1,l_2\in N}$ is not. Putting $f_{k_1,\dots,k_n}(x) = F_{l_1,l_2}(x)$, where $(k_1, \dots, k_n) = \varphi^{-1}(l_1,l_2)$ and φ is the function defined in the Lemma, we have, by condition (2) of the Lemma, that $\lim\limits_{s\to\infty} f_{k_1(s),\dots,k_n(s)}(x) = 0$ essentially everywhere. Simultaneously, by condition (a) of the Lemma, we have

$$\lim_{k_1,\dots,k_n\to\infty} f_{k_1,\dots,k_n}(x) = \lim_{l_1,l_2\to\infty} F_{l_1,l_2}(x),$$

which implies that the n-index sequence $\{f_{k_1,\dots,k_n}\}_{k_1,\dots,k_n\in N}$ is not convergent essentially everywhere to 0.

E x a m p l e 1. Let $(X,\mathcal{A})$ be a measurable space such that $\mathcal{A}$ is the σ-field of the subsets of X of σ-finite measure μ and $\mathcal{J}\subset\mathcal{A}$ is the σ-ideal of subsets of μ-measure zero. It is well known that each subfamily of the family $\mathcal{S} = \mathcal{A}\setminus\mathcal{J}$ of pairwise disjoint sets is at most countable thus $(X, \mathcal{S})$ is a category base such that the family of $\mathcal{A}$-measurable real-valued functions is the family of Baire functions in the category base $(X, \mathcal{S})$. Similarly as in [2], we conclude that the category base $(X, \mathcal{S})$ satisfies the condition (F) so that Theorem 2 can be applied to the sequence of measurable functions in the common sense.

E x a m p l e 2. If X is an arbitrary complete and separable space and $\mathcal{S}$ is the family of open balls, then $(X, \mathcal{S})$ is a category base (cf. [1]) such that the family of Baire functions is the family of functions having the Baire property. The category base $(X, \mathcal{S})$ satisfies the condition (F) (cf. [3]), so we have that Theorem 2 can be applied in this case.

The example below shows that the sequences $\{k_1(s),\dots,k_n(s)\}_{s\in N}$ cannot be replaced by sequences monotonously tending to infinity.

Let $(\mathcal{R}, \mathcal{S})$ be the category base in the real line, described in Example 1, with respect to the Lebesgue measure μ.

E x a m p l e 3. For an arbitrary positive integer $n \geq 2$, there exists an n-idex sequence $\{f_{k_1,\dots,k_n}\}_{k_1,\dots,k_n\in N}$ of continuous functions defined on the real line, satisfying the following conditions:

1° for any sequences $\{k_1(s)\}_{s\in N}, \ldots, \{k_n(s)\}_{n\in N}$ of positive integers tending to infinity, the sequence $\{f_{k_1(s),\ldots,k_n(s)}\}_{s\in N}$ is convergent essentially everywhere to 0.

2° $\mu\{x: \sim(\lim_{k_1,\ldots,k_n\to\infty} f_{k_1,\ldots,k_n}(x) = 0)\} > 0.$

P r o o f. For n = 2, Sierpiński pointed out a double sequence $\{g_{k_1,k_2}\}_{k_1,k_2\in N}$ fulfilling conditions (1°) and (2°). Putting

$$f_{k_1,\ldots,k_n}(x) = g_{k_1,k_2}(x)$$

for any positive integer $n > 2$, we conclude the desired properties.

This remark is essential because, in some category bases, the requirement of the convergence with respect to subsequences monotonously tending to infinity is sufficient.

Let $(X, \mathcal{S})$ be the category base described in Example 2. W a g n e r proved the following theorem (see [3]).

THEOREM B. If $\{f_{m,n}\}_{m,n\in N}$ is a double sequence of Baire functions fulfilling the following condition: for all increasing sequences $\{m_k\}_{k\in N}$ and $\{n_k\}_{k\in N}$ of positive integers, $\lim_{k\to\infty} f_{m_k,n_k}(x) = 0$ essentially everywhere on X; then $\lim_{m,n\to\infty} f_{m,n}(x) = 0$ essentially everywhere on X.

Quite analogously as Theorem 1, by using the Lemma we can generalize Theorem B to multi-index sequences of functions.

THEOREM 2. If, for a positive integer $n \geq 2$, $\{f_{k_1,\ldots,k_n}\}$ is an n-index sequence of Baire functions fulfilling the following condition: for all increasing sequences $\{k_1(s)\}, \ldots, \{k_n(s)\}$ of positive integers, $\lim_{s\to\infty} f_{k_1(s),\ldots,k_n(s)}(x) = 0$ essentially everywhere on X; then $\lim_{k_1,\ldots,k_n\to\infty} f_{k_1,\ldots,k_n}(x) = 0$ essentially everywhere on X.

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ZBIEŻNOŚĆ WIELOWSKAŹNIKOWYCH CIĄGÓW FUNKCJI BAIRE´A

W pracy rozważa się zbieżność wielowskaźnikowych ciągów funkcji Baire´a według kategorii względem baz kategorialnych w sensie Morgana. Uogólnia się twierdzenie o zbieżności z ciągów dwuwskaźnikowych na ciągi wielowskaźnikowe, zakładając zbieżność odpowiednich podciągów.