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ON THE SETS A + A AND A - A

In the paper, an example of a closed set F of real numbers satisfying the conditions F + F = [0, 2] and F - F ≠ [-1, 1] is presented. It is a negative answer to the problem posed by M. Laczkovich. Also, some necessary condition for sets of this type is formulated.

For the arbitrary set $A \subset R$ we will use the following notation:

$$A + A = \{x + y,\ x \in A,\ y \in A\}$$

and

$$A - A = \{x - y,\ x \in A,\ y \in A\}.$$

It is well-known that for Cantor set C, C + C = [0, 2] and C - C = = [-1; 1]. For the set F = {1} ∪ [0; 1/2], F + F ≠ [0, 2] and F - F = [-1; 1]. S. Picard [1] showed in 1942 that there is a set X such that X + X = R and X - X is of measure zero. M. Laczkovich asked if it is true for closed sets that condition F + F = [0; 2] implies F - F = [-1; 1]. The answer is negative.

Example. Let

$$F = [0; 2/20] \cup [3/20; 4/20] \cup [15/40; 25/40]$$
$$\cup \{15/20\} \cup [17/20; 1].$$

F is obviously closed. Moreover,

$$[0; 2/20] + [0; 2/20] = [0; 4/20],$$
$$[0; 2/20] + [3/20; 4/20] = [3/20; 6/20],$$
$$[3/20; 4/20] + [3/20; 4/20] = [6/20; 8/20],$$
$$[15/40; 25/40] + [0; 2/20] = [15/40; 29/40],$$
$$[15/40; 25/40] + [3/20; 4/20] = [21,40; 33/40],$$
$$[15/40; 25/40] + [15/40; 25/40] = [30/40; 50/40],$$
$$[17/20; 1] + [15/40; 25/40] = [49/40; 65/40],$$
$$\{15/20\} + [17/20; 1] = [32/20; 35/20],$$
$$[17/20; 1] + [17/20; 1] = [34/20; 2],$$

so $F + F = [0; 2]$. It is easy to check that $51/80 \notin F - F$, so $F - F \neq [-1; 1]$.

Now, we will prove that if F is closed and has at most four components, then the implication $F + F = [0; 2] \rightarrow F - F = [-1; 1]$ holds. The theorem will be preceded by the following

LEMMA. If a closed set $F \subset [0; 1]$ satisfies condition: There exists a component $(\alpha; \beta)$ of $[0; 1]\setminus F$ such that

(1) $\beta - \alpha > \min(\alpha; 1 - \beta)$,

then $F + F \neq [0; 2]$.

Proof.

$$F + F \subset [0; 2\alpha] \cup [\beta; 1 + \alpha] \cup [2\beta; 2].$$

By (1),

$$\beta - \alpha > \alpha \quad \text{or} \quad \beta - \alpha > 1 - \beta.$$

Thus

$$F + F \neq [0; 2].$$

THEOREM 1. Let $F \subset [0; 1]$ be a closed set such that:

(a) $[0; 1]\setminus F$ has at most three components;

(b) $F - F \neq [-1; 1]$,

then $F + F \neq [0; 2]$.

Proof. We started with a case when the set $[0; 1]\setminus F$ has exactly three components. Suppose that there exists a closed set F such that $[0; 1]\setminus F$ has three components, $F - F \neq [-1; 1]$ and

(2) $F + F = [0; 2]$

$$F = [x_o; y_o] \cup [x_1; x_1] \cup [x_2; y_2] \cup [x_3, y_3]$$

where

$$0 = x_o \leq y_o < x_1 \leq y_1 < x_2 \leq y_2 \leq y_3 = 1.$$

Let us introduce the following notations:

$$I_1 = [x_o; y_o], \quad I_2 = [x_1; y_1], \quad I_3 = [x_2; y_2],$$

$$I_4 = [x_3; y_3], \quad U_1 = (y_o; x_o), \quad U_2 = (y_1; x_2),$$

$$U_3 = (y_2; x_3).$$

We can assume that

(3) $|I_1| \geq |I_4|$.

If it were not the truth, then one should consider the set $1 - F$.

Since $F - F \neq [-1; 1]$, therefore there exists a number $d \in (0; 1)$ such that $(d + F) \cap F = \emptyset$ and $d \in U_1 \cup U_2 \cup U_3$.

First, let us consider the case

(*) $d \in U_1$.

Since

(4) $d + I_1 = [d; d + y_o] \subset U_1$,

we have

(5) $|I_1| < |U_1|$.

By Lemma, $F + F \neq [0; 2]$ which contradicts (2).

Now, consider the case:

(**) $d \in U_3$.

Using the same argumentation as in (*), we obtain

(6) $|I_1| < |U_3|$

By (6) and (3),

(7) $|I_4| < |U_3|$.

Component U_3 satisfies the assumptions of Lemma so $F + F \neq [0; 2]$ which contradicts (2).

It remains the case:

(***) $d \in U_2$,

$d + I_1 = [d + x_o; d + y_o] \subset U_2$.

Thus

(8) $|I_1| < |U_2|$.

Let us consider three following subcases:

$d + x_1 \in U_2$,

$d + x_1 \in (1; \infty)$,

$d + x_1 \in U_3$.

If $d + x_1 \in U_2$, then $d + y_1 \in U_2$ and $|U_2| > d - y_1 - d = y_1$, so by Lemma, $F + F \neq [0; 2]$. If $d + x_1 \in (1; \infty)$ and $d + y_o \in U_o$, then

(9) $|U_1| > |I_3| + |U_3| + |I_4|$.

Since $|U_1| \le |I_1| < |U_2|$ (which is a consequence of (8) and Lemma), we obtain

(10) $|U_2| > |U_1| > |I_3| + |U_3| + |I_4|$

and U_2 satisfies the assumption of Lemma.

If $d + x_1 \in U_3$, then

(11) $|I_2| < |U_3|$

and

(12) $|I_3| < |U_1|$.

Let $\rho(A; B) \overset{def}{=} \inf\{|x - y|;\ x \in A,\ y \in B\}$. By (8) and (3)

(13) $\rho(I_2 + I_4;\ I_3 + I_4) = |U_2| - |I_4| > 0.$

It is obvious that only the set $I_3 + I_3$ can cover the gap between sets $I_2 + I_4$ and $I_3 + I_4$. But then $2y_2 \geq x_2 + x_3$, so

(14) $|I_3| \geq |U_3|$.

Similarly, $\rho(I_1 + I_2;\ I_1 + I_3) = |U_2| - |I_1| > 0$ and for the set $I_2 + I_2$, we obtain the inequality $2x_1 \leq y_o + y_1$ which imply

(15) $|U_1| \leq |I_2|$.

By (11), (12), (14), (15),

$$|U_3| \leq |I_3| < |U_1| \leq |I_2| < |U_3|.$$

This contradiction establishes the theorem in case when the set $[0, 1]\setminus F$ has exactly three components. If $[0, 1]\setminus F$ has one or two components the proof is analogous to the proofs of cases (*) and (**).

REMARK. If the closed sed $F \subset [0, 1]$ satisfies the condition $F = 1 - F$, then

$$F + F = [0; 2] \iff F - F = [-1; 1].$$

REFERENCE

[1] P i c a r d S., *Sur des ensembles parfaits,* Mem. de Univ. de Neuchatel, 16 (1942).

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O ZBIORACH A + A i A - A

W pracy przedstawiony został przykład zbioru domkniętego F liczb rzeczywistych spełniający warunki $F + F = [0, 2]$ i $F - F \neq [-1, 1]$. Jest to negatywna odpowiedź na problem postawiony przez M. Laczkovicha. Sformułowano także pewien warunek konieczny dla zbiorów tego typu.