A C T A U N I V E R S I T A T I S L O D Z I E N S I S
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SOME PROBLEMS IN RESIDUAL ANALYSIS

1. INTRODUCTION

For a long time, regression analysis has proved to be a very fruitful statistical instrument for analysing the relationships between economic and social phenomena. The classical simple linear regression equation relating an endogenous variable y_i to exogenous variable x_i may look like

$$y_i = \beta_o + \beta_1 x_i + u_i, \tag{1.1}$$

where β_o, β_1 are parameters and u_i is the disturbance. Usually the disturbance term is assumed to have certain properties in order to carry out the statistical inference. A departure from these assumptions gives rise to some problems.

The main objectives of this paper are to probe into the adequacy of these assumptions when the model is applied to the analysis of economic behavioral relations. Some new results are obtained.

2. AUTO-CORRELATION IN THE DISTURBANCES

Case (a). The errors in (1.1) are assumed to follow the first-order Markov scheme, i.e.

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(2.1) $$u_i = \rho u_{i-1} + \varepsilon_i,$$

where $E(\varepsilon_i) = 0$, $E(\varepsilon_i^2) = \sigma_\varepsilon^2$ for all i, $E(\varepsilon_i \varepsilon_j) = 0$ for all $i \neq j$ and $0 \leqslant \rho < 1$.

Now to estimate the parameters in (1.1) we may write (2.1) as

$$\varphi(B) u_i = \varepsilon_i,$$

where

$$\varphi(B) = 1 - \rho B,$$

where B is the back shift operator such that $Bu_i = u_{i-1}$, so (1.1) may be written as

(2.2) $$y_i = \beta_0 + \beta_1 x_i + \frac{1}{\varphi(B)} \varepsilon_i.$$

Then, the least squares solution of the problem is carried out by minimizing

$$v = \sum_{i=1}^{n} \varepsilon_i^2 = \sum_{i=1}^{n} \left[\varphi(B)(y_i - \beta_0 - \beta_1 x_i)\right]^2,$$

we find that

$$\frac{\partial v}{\partial \beta_0} = -2 \sum_{i=1}^{n} [\varphi(B)]^2 (y_i - \beta_0 - \beta_1 x_i),$$

$$\frac{\partial v}{\partial \beta_1} = -2 \sum_{i=1}^{n} [\varphi(B)]^2 (y_i - \beta_0 - \beta_1 x_i) x_i,$$

$$\frac{\partial v}{\partial \rho} = -2 \sum_{i=1}^{n} \varphi(B) B (y_i - \beta_0 - \beta_1 x_i)^2.$$

Using the approximation proposed by N u r i [4]:

$$[\varphi(B)]^2 \simeq 1 - 2\varrho B \quad \text{and} \quad \varphi(B)B = B - \varrho B^2,$$

the least squares equation takes the form

$$\Sigma y_i - 2\hat{\varrho}\Sigma y_{i-1} = n\hat{\beta}_o(1-2\hat{\varrho}) + \hat{\beta}_1(\Sigma x_i - 2\hat{\varrho}\Sigma x_{i-1}),$$

$$(2.3) \quad \Sigma y_i x_i - 2\hat{\varrho}y_{i-1}x_i = \hat{\beta}_o(\Sigma x_i - 2\hat{\varrho}\Sigma x_{i-1}) + \hat{\beta}_1(\Sigma x_i^2 - 2\hat{\varrho}\Sigma x_{i-1}^2),$$

$$\Sigma z_{i-1} - \hat{\varrho}\Sigma z_{i-2} = 0, \quad \text{where} \quad z_i = (y_i - \hat{\beta}_o - \hat{\beta}_1 x_i)^2.$$

An initial value $\hat{\beta}^{(o)}$ of $\beta = \begin{pmatrix}\beta_o\\ \beta_1\end{pmatrix}$ is obtained by the ordinary least squares method for the model (1.1). Having computed $\hat{\beta}^{(o)}$, an initial value $\hat{\varrho}^{(o)}$ of ϱ is obtained. After obtaining the initial value of (2.3) is employed to obtain the first approximate estimate $\hat{\beta}^{(1)}$ of β.

These steps are continued to conclude that $\hat{\beta}^{(m)}$, and $\hat{\varrho}^{(m)}$ are the solutions of β and ϱ at the m-th stage whenever we have

$$|\hat{\beta}^{(m)} - \hat{\beta}^{(m-1)}| < \delta_1, \quad \text{and} \quad |\hat{\varrho}^{(m)} - \hat{\varrho}^{(m-1)}| < \delta_2,$$

for some specified numbers δ_1, δ_2 (see [1]).

Following P i e r c e [5], we can show that

$$\hat{\alpha}' = (\hat{\beta}_o \;\; \hat{\beta}_1 \;\; \hat{\varrho})$$

is distributed asymptotically normally with mean α and covariance matrix Σ, where

$$\Sigma \simeq \frac{\sigma_\varepsilon^2}{n}\begin{bmatrix} D & 0 \\ 0 & \Gamma_o \end{bmatrix}^{-1}, \quad D = \begin{bmatrix} d_{00} & d_{01} \\ d_{10} & d_{11} \end{bmatrix}, \quad \Gamma_o = E(u_i)^2 = \frac{\sigma_\varepsilon^2}{1-\varrho^2},$$

$$d_{hj} = \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n} w_{hi}w_{ji}, \; h,j = 0,1 \quad \text{and} \quad w_{hi} = \varphi(B)x_{hi}, x_{oi} = 1,$$

$x_{1i} = x_i$, so

$$\Sigma = \frac{\sigma_\varepsilon^2}{n}\begin{bmatrix} D^{-1} & 0 \\ 0 & \frac{1-\rho^2}{\sigma_\varepsilon^2} \end{bmatrix}.$$

Case (b). The disturbances in (1.1) are auto-correlated through a moving average process of order one, i.e.

$$u_i = \varepsilon_i - \theta_1 \varepsilon_{i-1}, \quad |\theta_1| < 1 \tag{2.4}$$

or

$$u_i = \theta(B)\,\varepsilon_i,$$

where

$$\theta(B) = 1 - \theta_1 B \quad \text{and} \quad \varepsilon_i \sim IN\left(0,\ \sigma_\varepsilon^2\right).$$

Therefore, the model (1.1) can be written as

$$y_i = \beta_0 + \beta_1 x_i + \theta(B)\varepsilon_i, \quad i = 1, 2, \ldots, n$$

and so, the least squares solution of the problem is carried out by minimizing

$$v = \sum_{i=1}^{n} \varepsilon_i^2 = \sum_{i=1}^{n} \frac{1}{[\theta(B)]^2}\,(y_i - \beta_0 - \beta_1 x_i)^2. \tag{2.5}$$

Similarly, to case (a), we can find

$$\frac{\partial v}{\partial \beta_0} = -2 \sum_{i=1}^{n} \frac{1}{[\theta(B)]^2}\,(y_i - \beta_0 - \beta_1 x_i),$$

$$\frac{\partial v}{\partial \beta_1} = -2 \sum_{i=1}^{n} \frac{1}{[\theta(B)]^2}\,(y_i - \beta_0 - \beta_1 x_i)x_i,$$

and

$$\frac{\partial v}{\partial \theta_1} = 2 \sum_{i=1}^{n} \frac{1}{[\theta(B)]^3} B (y_i - \beta_o - \beta_1 x_i)^2.$$

Using the approximations proposed by N u r i [4]:

$$\frac{1}{[\theta(B)]^2} = 1 + 2\theta_1 B \quad \text{and} \quad \frac{B}{[\theta(B)]^3} = B + 3\theta_1 B^2$$

the least squares equation takes the form

$$\Sigma y_i + 2\hat{\theta}_1 \Sigma_{i-1} = n\, \hat{\beta}_o (1 + 2\theta_1) + \hat{\beta}_1 (\Sigma x_i + 2\theta_1 \Sigma x_{i-1}),$$

$$\Sigma y_i x_i + 2\theta_1 \Sigma y_{i-1} x_i = \hat{\beta}_o (\Sigma x_i + 2\theta_1 \Sigma x_{i-1}) +$$

$$+ \hat{\beta}_1 (\Sigma x_i^2 + 2\theta_1 \Sigma x_{i-1}^2).$$

and

(2.6) $$\Sigma z_{i-1} + 3\theta_1 \Sigma z_{i-2} = 0.$$

Using the same argument as in case (a) one can obtain approximate solutions for the estimate and that

$$\hat{\alpha}^{*\prime} = (\hat{\beta}_o \quad \hat{\beta}_1 \quad \hat{\theta}_1)$$

is asymptotically normally distributed with mean α^* and covariance matrix Σ^*, where

$$\Sigma^* \approx \frac{\sigma_\varepsilon^2}{n} \begin{bmatrix} D^{*-1} & 0 \\ 0 & \dfrac{1-\theta_1^2}{\sigma_\varepsilon^2} \end{bmatrix}, \quad D^* = \begin{bmatrix} d_{oo}^* & d_{01}^* \\ d_{10}^* & d_{11}^* \end{bmatrix},$$

where

$$d_{hj}^* = \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} w_{hi}^* w_{ji}^*, \quad h,j = 0,1$$

and

$$w^{*}_{hi} = \Theta(B)x_{hi}, x_{oi} = 1, \quad x_{1i} = x_i.$$

3. HETEROSCEDASTICITY

This usually happens when the variance of the disturbances is not an identical constant. This arises frequently in the analysis of crosssection data, as in the case of sampling lineary across high-income and low-income sections of a city (see [2]).

Case (a). In the model (1.1) let us assume that the variance of u_i is directly proportional to some power of x_i, that is

$$\text{var}\,(u_i) = \sigma_u^2\, x_i^{\alpha}, \tag{3.1}$$

where α denotes the heteroscedasticity parameter.

Using the generalized least squares (GLS) we can estimate the coefficient of regression

$$\hat{\beta}' = (\hat{\beta}_o \;\; \hat{\beta}_1) = (x'\Omega^{-1}x)^{-1}x'\Omega^{-1}y,$$

where

$$X = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ 1 & x_n \end{bmatrix} \text{ and } \Omega = \begin{bmatrix} x_1^{\alpha} & & & & & 0 \\ & x_2^{\alpha} & & & & \\ & & \cdot & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ 0 & & & & & x_n^{\alpha} \end{bmatrix}.$$

So, after a few steps, we obtain

$$\hat{\beta}_o = \frac{\left(\sum \frac{y_i}{x_i^{\alpha}}\right)\left(\sum \frac{1}{x_i^{\alpha-2}}\right) - \left(\sum \frac{y_i}{x_i^{\alpha-1}}\right)\left(\sum \frac{1}{x_i^{\alpha-1}}\right)}{\left(\sum \frac{1}{x_i^{\alpha-1}}\right)^2 - \left(\sum \frac{1}{x_i^{\alpha}}\right)\left(\sum \frac{1}{x_i^{\alpha-2}}\right)} \tag{3.2}$$

and

$$(3.3)\qquad \hat{\beta}_1 = \frac{\left(\sum \frac{y_i}{x_i^{\alpha-1}}\right)\left(\sum \frac{1}{x_i^{\alpha}}\right) - \left(\sum \frac{y_i}{x_i^{\alpha}}\right)\left(\sum \frac{1}{x_i^{\alpha-1}}\right)}{\left(\sum \frac{1}{x_i^{\alpha-1}}\right)^2 - \left(\sum \frac{1}{x_i^{\alpha}}\right)\left(\sum \frac{1}{x_i^{\alpha-2}}\right)}.$$

Case (b). In cross-section regression analysis, the problems of autocorrelation and heteroscedasticity may often stem from a common cause. They may, therefore, be reasonably expected to occur simultaneously. So in equation (2.1) with ρ_1 instead of ρ let us assume that var $(\varepsilon_i) = \sigma_i^2$ for all values of i and

$$\frac{1}{n}\sum_{i=1}^{n} \sigma_i^2 \to s_o$$

(a constant) when $n \to \infty$.

Now to obtain the least squares estimate of ρ_1, we minimize v, where

$$(3.4)\qquad v = \sum_{i=2}^{n} \varepsilon^2{}_i = \sum_{i=2}^{n} (u_i - \rho_1 u_{i-1})^2.$$

Differentiating with respect to ρ_1 to obtain

$$(3.5)\qquad \sum_{j=0}^{1} \hat{\rho}_j \sum_{i=2}^{n} u_{i-j}u_{i-1} = 0, \quad \hat{\rho}_o = -1.$$

Letting $c_{j1} = \frac{1}{n-1}\sum u_{i-j}u_{i-1}$, equation (3.5) becomes

$$(3.6)\qquad \sum_{j=0}^{1} \hat{\rho}_j (n-1) c_{j1} = 0.$$

Similarly,

$$c_s = \frac{1}{n-s} \sum_{i=1}^{n-s} u_i u_{i+s}, \quad s = 0, 1,$$

so

$$E(c_s) = \frac{1}{n-s} \sum_{j=0}^{\infty} \psi_j \psi_{j+s} \sum_{i=1}^{n-s} \sigma_{i-j}^2,$$

where ψ_j is the correlation coefficient between ε_i and ε_{i-j}. Thus

$$E(c_s) = \gamma_s s_o \quad \text{as} \quad n \to \infty,$$

where

$$\gamma_s = \sum_{j=0}^{\infty} \psi_j \psi_{j+s},$$

also, it is not difficult to see that

$$\operatorname{var}(c_s) \leq \frac{5}{n-s} \left(\sum_{j=0}^{\infty} \psi_j \right)^4 \sigma_T^4,$$

where

$$\sigma_T^2 = \max_i \sigma_i^2$$

(see [3]).

Because for our case $\psi_j = \rho^j$, thus

$$\operatorname{var}(c_s) \to 0 \quad \text{as} \quad n \to \infty,$$

since $\sum_{j=0}^{\infty} |\psi_j|$ is bounded. So, c_s is a consistent estimator of $s_o \gamma s$.

On the other hand, to find the distribution of $\hat{\rho}_1$, we first write the model (2.1) as

$$(3.7) \qquad \sum_{j=0}^{1} \rho_j u_{i-j} + \varepsilon_i = 0, \quad \rho_0 = -1.$$

Let

$$(3.8) \qquad z = -(n-1)^{-\frac{1}{2}} \sum_{i=1}^{n} \left(\sum_{j=0}^{1} \rho_j u_{i-j} \right) u_{i-1} = (n-1)^{-\frac{1}{2}} \sum_{i=1}^{n} \varepsilon_i u_{i-1}.$$

So, z is a sum of independent random variables ε_i each of which is independent of u_{i-1}, hence

(i) $$E(z) = 0$$

and

(ii) $$E(z^2) = (n-1)^{-1} \sum E(\varepsilon_i^2) E(u_{i-1}^2),$$

where

$$E(\varepsilon_i^2) = \sigma_i^2 \quad \text{and} \quad E(u_{i-1}^2) = \sum_{j=0}^{\infty} \psi_j \sigma_{i-j-1}^2.$$

Therefore it can be said that z is asymptotically normally distributed with zero mean and σ_z^2 variance, where

$$\sigma_z^2 = \frac{1}{n-1} \sum_{j=0}^{\infty} \psi_j^2 \sum_{i=1}^{n} \sigma_i^2 \, \sigma_{i-j-1}^2.$$

From equation (3.8) z can be written as

$$(3.9) \qquad z = (n-1)^{\frac{1}{2}} \sum_{j=0}^{1} \rho_j c_{j1},$$

by adding equation (3.6) to equation (3.9) we get

$$(3.10) \qquad z = (n-1)^{\frac{1}{2}} \sum_{j=0}^{1} c_{j1} (\hat{\rho}_j - \rho_j).$$

So, we can conclude that

$$(n-1)^{\frac{1}{2}} \quad (\hat{\rho}_1 - \rho_1)$$

is asymptotically normally distributed with zero mean and variance v^*, where

$$v^* = \frac{1}{s_0^2 \gamma_0^2 (n-1)} \sum_{j=0}^{\infty} \psi_j^2 \sum_{i=1}^{n} \sigma_i^2 \sigma_{i-j}^2 .$$

4. CONCLUSIONS AND ILLUSTRATION

In cross-section regression analysis the problems of auto-correlation and heteroscedasticity may often stem from a common cause. In this paper a theoretical study was done, because we believe that the nature of the model can be better understood by examining its mathematical form. Also it is of interest to illustrate with a small sample these problems. An effort was made to construct some suitable numerical examples.

Illustration:

(i) Consider the following simple regression

$$y_i = \beta x_i + u_i, \quad i = 1, 2$$

with auto-correlated disturbances. It is not difficult to show that

$$(4.1) \qquad \text{var}(\hat{\beta}) = \sigma_u^2 \frac{\sum_{i=1}^{2} x_i^2 + \rho x_1 x_2}{\left(\sum_{i=1}^{2} x_i^2\right)^2} = \frac{\sigma_u^2}{\sum x_i^2} + \frac{2\rho x_1 x_2 \sigma_u^2}{\left(\sum x_i^2\right)^2}$$

and so, we can notice that, the first factor in equation (4.1) is the variance of the ordinary least squares estimator of β in the absence of first order auto-correlation in u_i's.

(ii) Where $Eu_i u_j = 0$ and $\text{var}(u_i) = \sigma_u^2 x_i^{\alpha}$, it is clear that

(4.2) $$\operatorname{var}(\hat{\beta}) = \frac{\sigma_u^2}{\left(\sum x_i^2\right)^2} \sum x_i^{\alpha+2} = \frac{\sigma_u^2}{\sum x_i^2} \cdot \frac{\sum x_i^{\alpha+2}}{\sum x_i^2}.$$

So, when the heteroscedasticity parameter $\alpha = 0$, we obtain the variance of the ordinary least squares estimator of β.

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WYBRANE PROBLEMY ANALIZY RESZT

Celem artykułu jest analiza konsekwencji przyjęcia założenia, że proces $\{u_i\}$ jest procesem AR (1) autoregresyjnym rzędu pierwszego oraz, że proces $\{u_i\}$ jest procesem MA(1) ze średnią ruchomą rzędu pierwszego, gdzie u_i są addytywnymi składnikami losowymi równania regresji postaci $y_i = \beta_o + \beta_i x_i + u_i$, $i = 1, \ldots, n$.

Zainteresowanie autorów skupiło się na ustaleniu konsekwencji przyjęcia tych założeń dla postaci asymptotycznego rozkładu iteracyjnych estymatorów metody najmniejszych kwadratów parametrów β_o, β_1 oraz parametrów przyjętych postaci procesów AR (1) i MA (1). Udowodniono asymptotyczną normalność tych estymatorów.