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A NOTE ON THE SYSTEM OF COMPLEMENTS OF SETS SPARSE AT A POINT

In this note we prove that the system of complements of sets sparse at a point on the right satisfies a strong intersection condition of the form "$S_x \cap S_y \cap (y, y + (y - x)^{\alpha} \neq \emptyset$" for each $\alpha \in (0, 1)$.

In [1] the definition of a strong intersection condition was introduced. This condition is related to T h o m s o n s intersection condition (see [3]). Let $\mathbb{S} = \{\mathbb{S}(x);\ x \in R\}$ be a family of collections of subsets of the real line. We say that $\mathbb{S}$ satisfies a strong intersection condition of the form "$S_x \cap S_y \neq \emptyset$" ("$S_x \cap S_y \cap (x, y) \neq \emptyset$", etc.) if, for any $x \in R$ and $S \in \mathbb{S}(x)$, there is a positive number $\delta(x, S)$ such that $S_x \cap S_y \neq \emptyset$ ($S_x \cap S_y \cap (x, y) \neq \emptyset$, etc.) whenever $S_x \in \mathbb{S}(x)$, $S_y \in \mathbb{S}(y)$ and $0 < y - x < \min\{\delta(x, S_x), \delta(y, S_y)\}$.

If E is a measurable subset of the real line, then $|E|$ denotes the Lebesgue measure of E. By a right upper (lower) density of E at a point x we mean

$$d^+(E, x) = \limsup_{h \to 0^+} \frac{|E \cap (x, x + h)|}{h}$$

$$(d_+(E, x) = \liminf_{h \to 0^+} \frac{|E \cap (x, x + h)|}{h}$$

We define the family $\mathbb{S}^+ = \{\mathbb{S}^+(x);\ x \in R\}$ in the following way

$$A \in \mathbb{S}^+(x) \iff x \in A$$

and there is a measurable set $E \subset A$ such that, for each measurable F with $d^+(F, x) = 1$ and $d_+(F, x) > 0$, we have $d^+(E \cap F, x) = 1$ and $d_+(E \cap F, x) > 0$.

From ([2], Theorem 3.1) it follows that a measurable set E belongs to $\mathcal{S}^+(x)$ if and only if

(*) for each $\varepsilon > 0$, there is $k = k(x, \varepsilon) > 0$ such that each interval $(a, b) \subset (x, x + k)$ with $a - x \leq k(b - x)$ contains at least one point y such that $|E \cap (x, y)| > (1 - \varepsilon)(y - x)$.

If $R \setminus E \in \mathcal{S}^+(x)$, then E is called sparse at x on the right (see [2]).

In [1] it was proved that $\mathcal{S}^+$ satisfies a strong intersection condition of the form "$S_x \cap S_y \neq \emptyset$", and that there is no $\lambda \geq 0$ such that $\mathcal{S}^+$ satisfies a strong intersection condition of the form "$S_x \cap S_y \cap [y, y + \lambda(y - x)] \neq \emptyset$" (and even an intersection condition of this form).

THEOREM. For each $\alpha \in (0, 1)$, the family $\mathcal{S}^+$ satisfies a strong intersection condition of the form

$$\text{"}S_x \cap S_y \cap (y, y + (y - x)^\alpha) \neq \emptyset\text{"}.$$

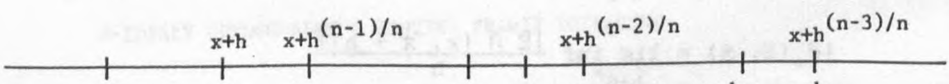
P r o o f. Let n be a natural number such that $\frac{n - 3}{n} > \alpha$. Evidently, $n \geq 4$. Let $x \in R$ and $S \in \mathcal{S}^+(x)$. We can assume that S is measurable. Thus S fulfils condition (*). We can additionally assume that $k(x, \varepsilon) \leq \varepsilon$ for each $\varepsilon > 0$.

Put

$$\delta(x, S) = [k(x, \tfrac{1}{4}k(x, \tfrac{1}{4}))]^n.$$

Let $S_x \in \mathcal{S}^+(x)$, $S_y \in \mathcal{S}^+(y)$ and $0 < y - x < \min\{\delta(x, S_x), \delta(y, S_y)\}$. We must show that $S_x \cap S_y \cap (y, y + (y - x)^\alpha) \neq \emptyset$.

Put $h = y - x$, $a = x + h^{(n-2)/n}$ and $b = x + h^{(n-3)/n}$.



We shall consider two cases:

(i) $k(x, \frac{1}{4}) \leq k(y, \frac{1}{4})$.

Since

$$b = x + h^{(n-3)/n} < x + [\delta(x, S_x)]^{(n-3)/n}$$

$$= x + [k(x, \tfrac{1}{4}k(x, \tfrac{1}{4}))]^{n-3}$$

$$\le x + k(x, \tfrac{1}{4}k(x, \tfrac{1}{4})),$$

therefore $(a, b) \subset (x, x + k(x, \frac{1}{4}k(x, \frac{1}{4})))$. Moreover,

$$a - x = h^{(n-2)/n} = (b - x)h^{1/n}$$

$$< (b - x)k(x, \tfrac{1}{4}k(x, \tfrac{1}{4}))$$

and, by (*), there exists $b_1 \in (a, b)$ such that

$$|S_x \cap (x, b_1)| > (1 - \tfrac{1}{4}k(x, \tfrac{1}{4}))(b_1 - x).$$

Hence

(1) $$|(x, b_1) \setminus S_x| < \tfrac{1}{4}k(x, \tfrac{1}{4})(b_1 - x) \le \tfrac{1}{4}k(y, \tfrac{1}{4})(b_1 - x).$$

Put $a_1 = x + (b_1 - x)k(y, \frac{1}{4})$. Then

$$\frac{a_1 - y}{b_1 - y} < \frac{a_1 - x}{b_1 - x} = k(y, \tfrac{1}{4}).$$

On the other hand,

$$b_1 < b = x + h^{(n-3)/n} < x + [k(y, \tfrac{1}{4}k(y, \tfrac{1}{4}))]^{n-3}$$

$$< x + k(y, \tfrac{1}{4}) < y + k(y, \tfrac{1}{4})$$

and

(2) $$a_1 - x = (b_1 - x)k(y, \tfrac{1}{4}) > (a - x)k(y, \tfrac{1}{4}k(y, \tfrac{1}{4}))$$

$$= h^{(n-2)/n}[\delta(y, S_y)]^{1/n} > h^{(n-2)/n}h^{1/n} = h^{(n-1)/n}.$$

This means that $(a_1, b_1) \subset (y, y + k(y, \frac{1}{4}))$ and, by (*), there is $z \in (a_1, b_1)$ such that

(3) $$|S_y \cap (y, z)| > \tfrac{3}{4}(z - y).$$

From (1) it follows that

$$(4)\qquad |S_x \cap (x, z)| = (z - x) - |(x, z) \setminus S_x|$$

$$\geq z - x - |(x, b_1) \setminus S_x| > z - x - \tfrac{1}{4}k(y, \tfrac{1}{4})(b_1 - x)$$

$$= z - x - \tfrac{1}{4}(a_1 - x) > \tfrac{3}{4}(z - x).$$

Since $h^{1/n} < k(x, \frac{1}{4}k(x, \frac{1}{4})) < \frac{1}{4}$, condition (2) implies

$$z - y > a_1 - y > h^{(n-1)/n} - h > 3h.$$

Thus, from (4) it follows that

$$(5)\qquad |S_x \cap (y, z)| \geq |S_x \cap (x, z)| - (y - x) > \tfrac{3}{4}(z - x) - h$$

$$> \tfrac{3}{4}(z - y) - \tfrac{1}{3}(z - y) = \tfrac{5}{12}(z - y).$$

Inequalities (3) and (5) imply

$$|S_x \cap S_y \cap (y, z)| > (\tfrac{3}{4} + \tfrac{5}{12} - 1)(z - y) > 0.$$

Hence

$$S_x \cap S_y \cap (y, y + (y - x)^{\alpha})$$

$$\supset S_x \cap S_y \cap (y, y + (y - x)^{(n-3)/n})$$

$$\supset S_x \cap S_y \cap (y, z) \neq \emptyset.$$

(ii) $k(x, \frac{1}{4}) \geq k(y, \frac{1}{4})$.

Since

$$(a, b) \subset (y, y + k(y, \tfrac{1}{4}k(y, \tfrac{1}{4})))$$

and

$$a - y = h^{(n-2)/n} - h < (h^{(n-3)/n} - h)h^{1/n}$$

$$< (b - y)k(y, \tfrac{1}{4}k(y, \tfrac{1}{4})),$$

there is $b_1 \in (a, b)$ such that

$$|S_y \cap (y, b_1)| > (1 - \tfrac{1}{4}k(y, \tfrac{1}{4}))(b_1 - y).$$

Hence

$$(6)\qquad |(y, b_1)\setminus S_y| < \tfrac{1}{4}k(y, \tfrac{1}{4})(b_1 - y) \le \tfrac{1}{4}k(x, \tfrac{1}{4})(b_1 - y).$$

Put $a_1 = x + (b_1 - x)k(x, \frac{1}{4})$. Similarly to case (i) we get

$b_1 < x + k(x, \frac{1}{4})$ and

$$(7)\qquad a_1 - x > h^{(n-1)/n}.$$

This means that $(a_1, b_1) \subset (x, x + k(x, \frac{1}{4}))$ and, by (*), there is $z \in (a_1, b_1)$ such that

$$(8)\qquad |S_x \cap (x, z)| > \tfrac{3}{4}(z - x).$$

In the same way as in case (i), from (8) it follows that

$$(9)\qquad |S_x \cap (y, z)| > \tfrac{5}{12}(z - y).$$

On the other hand, (6) and (7) imply

$$(10)\qquad |S_y \cap (y, z)| = (z - y) - |(y, z) \setminus S_y|$$

$$\ge z - y - |(y, b_1)\setminus S_y| > z - y - \tfrac{1}{4}k(x, \tfrac{1}{4})(b_1 - y)$$

$$> z - y - \tfrac{1}{4}k(x, \tfrac{1}{4})(b_1 - x) = z - y - \tfrac{1}{4}(a_1 - x)$$

$$= z - y - \tfrac{1}{4}(a_1 - y)\left(1 + \frac{h}{a_1 - y}\right)$$

$$> (z - y)\left(1 - \tfrac{1}{4}\left(1 + \frac{h}{a_1 - y}\right)\right)$$

$$> (z - y)\left(1 - \tfrac{1}{4}\left(1 + \frac{h}{h^{(n-1)/n} - h}\right)\right)$$

$$= (z - y)\left(1 - \tfrac{1}{4}\left(\frac{1}{1 - h^{1/n}}\right)\right) > (z - y)\left(1 - \frac{1}{4}\,\frac{1}{1 - \frac{1}{4}}\right)$$

$$= \tfrac{2}{3}(z - y).$$

From (9) and (10) we conclude that

$$|S_x \cap S_y \cap (y, z)| > \left(\tfrac{5}{12} + \tfrac{2}{3} - 1\right)(z - y) > 0$$

and hence

$$S_x \cap S_y \cap (y, y + (y - x)^{\alpha}) \supset S_x \cap S_y \cap (y, z) \neq \emptyset.$$

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UWAGA O RODZINIE DOPEŁNIEŃ ZBIORÓW RZADKICH W PUNKCIE

W pracy dowodzimy, że dla dowolnej liczby $\alpha \in (0, 1)$, rodzina dopełnień zbiorów prawostronnie rzadkich w punkcie spełnia silny warunek przekroju typu "$S_x \cap S_y \cap (y, y + (y - x)^{\alpha}) \neq \emptyset$".