A C T A U N I V E R S I T A T I S L O D Z I E N S I S
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EXTREMAL PROPERTIES OF STARLIKE FUNCTIONS IN THE RING $0<|z|<1$

Let $\wp(A, B)$, $-1 \leq B < A \leq 1$, denote the family of functions P, $P(0) = 1$, holomorphic in the disc $K = \{z : |z| < 1\}$ and such that $P(z) = [1 + A\omega(z)]/[1 + B\omega(z)]$ for some function ω, $\omega(0) = 0$, $|\omega(z)| < 1$, holomorphic in K. Next, let $\Sigma^*(A, B)$ be the family of functions $F(z) = \frac{1}{z} + a_o + a_1z + a_2z^2 + \dots$ holomorphic in the ring $Q = \{z : 0 < |z| < 1\}$ and such that $-zF'(z)/F(z) \in \wp(A, B)$ for $z \in Q$. In the paper the functionals re $\{P(z) - zP'(z)/P(z)\}$ for $P \in \wp(A, B)$ and $z \in K$, $|F(z)|$, $|F'(z)|$ for $F \in \Sigma^*(A, B)$ and $z \in Q$ are estimated. Moreover, the radius of convexity of the family $\Sigma^*(A, B)$ is determined. Finally, two properties of functions of some class of meromorphic close-to-convex functions are proved.

I. INTRODUCTION

For fixed A, B, $-1 < A \leq 1$, $-1 \leq B < A$, let us denote by $\wp(A, B)$ the family of functions

$$P(z) = 1 + p_1z + \dots \tag{1.1}$$

holomorphic in the disc $K = \{z : |z| < 1\}$ and such that

$$P(z) = \frac{1 + A\omega(z)}{1 + B\omega(z)}$$

for some function ω holomorphic in K and satisfying the conditions $\omega(0) = 0$, $|\omega(z)| < 1$, for $z \in K$. The class $\wp(A, B)$ was introduced and examined by W. J a n o w s k i in paper [5]. In a little different way, this class, with $B \neq -1$, was introduced by J a k u b o w s k i and investigated in papers [2],

[3] and [4]. Of course, $\wp(1, -1) \equiv \wp$ where $\wp$ is the family of functions of form (1.1) and such that $\operatorname{re} P(z) > 0$ for $z \in K$. Moreover, from the inequalities adopted for A and B it follows that $\wp(A, B) \subset \wp$.

Next, let $\Sigma^*(A, B)$ stand for the family of functions

$$F(z) = \frac{1}{z} + a_o + a_1 z + a_2 z^2 + \ldots \tag{1.2}$$

holomorphic in the ring $Q = \{z : 0 < |z| < 1\}$ and such that

$$-\frac{zF'(z)}{F(z)} = P(z) \tag{1.3}$$

for some function $P \in \wp(A, B)$ and each $z \in Q$.

By assuming certain special values of the parameters A and B, we shall obtain the classes considered by other authors, for instance, in papers [4], [6], [10], [12], [14], [15], [16]. When $A = 1$, $B = -1$, we obtain the well-known class Σ^* studied, among others, by C l u n i e [1] and R o b e r t s o n [13].

In the present paper there are an estimation of the functional

$$H(P) = \operatorname{re}\{P(z) - \frac{zP'(z)}{P(z)}\}, \quad z \in K, \tag{1.4}$$

where $P \in \wp(A, B)$, as well as estimations of the modulus of the function and the modulus of its derivative in the family $\Sigma^*(A, B)$. Besides, the radius of convexity of the family $\Sigma^*(A, B)$ is determined. The method of investigations we take up here is an adaptation of that used in paper [5], with modifications resulting from the form of functional (1.4) and from the properties of the class $\Sigma^*(A, B)$. Some reasoning in proofs, being a repetition of that included in [5], will be omitted.

To analogous questions paper [10] is devoted. It concerns, however, the properties of k-symmetric functions of the class $\wp_k(M)$ (i.e. functions P holomorphic in the disc K, satisfying the condition $|P(z) - M| < M$ for $z \in K$, where $M > \frac{1}{2}$ is a fixed number) and the properties of k-symmetric functions of the class $\Sigma_k^*(M)$ (i.e. functions F holomorphic in Q and satisfying the condition $|-zF'(z)/F(z) - M| < M$ for $z \in Q$). The results included there are identical, for $k = 1$, with the corresponding results obtained in the present paper when $A = 1$ and $B = \frac{1}{M} - 1$.

Finally, we shall define the class $J(A, B; M, N)$ of meromorphic close-to-convex functions f, generated by functions of the classes $\wp(A, B)$ and $\Sigma^*(M, N)$. For functions of the class $J(A, B; M, N)$, we shall prove some property of coefficients and some geometrical property connected with the angle of inclination of the tangent to the image of the circle $|z| = r$, $0 < r < 1$, under the mapping by means of a function f.

II. ESTIMATION OF THE FUNCTIONAL H(P)

Let $p \in \wp$. Then the function

$$P(z) = \frac{(1 + A)p(z) + 1 - A}{(1 + B)p(z) + 1 - B} \tag{2.1}$$

belongs to the class $\wp(A, B)$ and vice versa [5]. Besides, in paper [5] *(lemma 2)* it was proved that if z is any fixed point of the disc K, then, for any function $P \in \wp(A, B)$,

$$|P(z) - c| \le \varrho \tag{2.2}$$

where

$$c = c(r) = \frac{1 - ABr^2}{1 - B^2r^2}, \quad \varrho = \varrho(r) = \frac{(A - B)r}{1 - B^2r^2}, \quad r = |z|. \tag{2.3}$$

Denote by $\wp_\lambda(A, B)$ a subclass of the family $\wp(A, B)$ containing all functions of form (2.1), where

$$p(z) = \frac{1}{2}(1 + \lambda)\frac{1 + \varepsilon_1 z}{1 - \varepsilon_1 z} + \frac{1}{2}(1 - \lambda)\frac{1 + \varepsilon_2 z}{1 - \varepsilon_2 z} \tag{2.4}$$

$-1 \le \lambda \le 1$, $|\varepsilon_j| = 1$, $j = 1, 2$.

Let $G(u, v)$ be an analytic function in the half-plane re $u > 0$ and in the plane (v), such that $|G'_u|^2 + |G'_v|^2 > 0$ at each point (u, v). Since each boundary function with respect to the functional $G(p(z), zp'(z))$, $|z| = r$, in the family is of form (2.4) (cf. [13]), therefore the boundary function of the family $\wp(A, B)$ with respect to the functional $G(P(z), zP'(z))$, $|z| = r$, belongs to the family $\wp_\lambda(A, B)$. Consequently, the determination of the extremum of the functional re $G(P(z), zP'(z))$ in the family $\wp(A, B)$ is reduced to the determination of the extremum of this functional in the subclass $\wp_\lambda(A, B)$.

The functions $P \in \wp_\lambda(A, B)$ have the property that

$$zP'(z) = \frac{1}{A - B}[-BP^2(z) + (A + B)P(z) - A] - \frac{\varrho^*}{2\varrho}[\varrho^2 - |P(z) - c|^2]\eta^*,$$

where c, ρ are given by (2.3), $\rho^* = \frac{2r}{1 - r^2}$, $r = |z|$, $|\eta^*| = 1$ (cf. [5], *lemma 4*).

So,

$$L(P(z)) = P(z) - \frac{zP'(z)}{P(z)} = \frac{AP^2(z) - (A + B)P(z) + A}{(A - B)P(z)} +$$

$$+ \frac{\rho^*}{2\rho} \frac{\rho^2 - |P(z) - c|^2}{P(z)} \eta^*.$$

Denoting $P(re^{i\varphi}) = se^{it}$, $s > 0$, $\operatorname{im} t = 0$, we shall get

$$\operatorname{re} L(se^{it}) = c_1 s \cos t - c_2 + c_1 s^{-1} \cos t +$$

$$+ (-c_3 s + c_4 \cos t - c_5 s^{-1})\varepsilon$$

where

$$c_1 = \frac{A}{A - B}, \quad c_2 = \frac{A + B}{A - B},$$

$$c_3 = \frac{\rho^*}{2\rho} = \frac{1 - B^2r^2}{(A - B)(1 - r^2)}, \quad c_4 = \frac{\rho^* c}{\rho} = 2\frac{1 - ABr^2}{(A - B)(1 - r^2)}$$

$$c_5 = \frac{\rho^*}{2\rho}(c^2 - \rho^2) = \frac{1 - A^2r^2}{(A - B)(1 - r^2)} \tag{2.5}$$

and $\varepsilon = \operatorname{re}(e^{-it}\eta^*)$.

The determination of the extremum of the functional $H(P) = \operatorname{re} L(P(z))$ is thus reduced to finding the extremum of a function of two variables s, t of the form

$$\Phi(s, t; \varepsilon) = (c_1 s + \varepsilon c_4 + c_1 s^{-1})\cos t - (c_2 + \varepsilon c_3 s + \varepsilon c_5 s^{-1}), \tag{2.6}$$

since $-1 \le \varepsilon \le 1$ and $\rho^2 - |P(z) - c|^2 \ge 0$, the minimum of the function Φ can be attained with $\varepsilon = -1$, its maximum - with $\varepsilon = 1$.

The function Φ is defined in the set

$$D = \{(s, t) : c - \rho < s < c + \rho, \ -\gamma(s) < t < \gamma(s)\} \tag{2.7}$$

and on its boundary ∂D, with that

$$\gamma(s) = \operatorname{arc\,cos} \frac{s^2 + c^2 - \rho^2}{2cs}, \quad 0 \le \gamma(s) \le \gamma(s^*) \tag{2.8}$$

where $s^{*2} = c^2 - \rho^2$.

If Φ attains its extremum at some point $(s', t') \in D$, then s' and t' are solutions of the system of equations $\frac{\partial\Phi}{\partial s} = 0$, $\frac{\partial\Phi}{\partial t} = 0$ with the unknown quantities s and t, that is, of the system

$$c_1(1 - s^{-2}) \cos t - \varepsilon(c_3 - c_5 s^{-2}) = 0, \quad \sin t = 0,$$

or of the system

$$c_1(1 - s^{-2}) \cos t - \varepsilon(c_3 - c_5 s^{-2}) = 0, \quad c_1 s + \varepsilon c_4 + c_1 s^{-1} = 0.$$

It is easy to check that the numbers s', t' do not satisfy the second system. Consequently, the problem of determining the extremum of the function Φ in the set D is equivalent to the analogous problem for a function

$$\Phi_o(s; \varepsilon) = \Phi(s, 0; \varepsilon), \quad s \in I, \quad \varepsilon = \pm 1, \tag{2.9}$$

where $I = \{s : c - \rho < s < c + \rho\}$. Besides, we have

$$\inf H(P) \geq \min_s \Phi_o(s; -1),$$

$$\sup H(P) \leq \max_s \Phi_o(s; 1).$$

From (2.9) and (2.6) we have

$$\Phi_o(s; \varepsilon) = (c_1 - \varepsilon c_3)s + (c_1 - \varepsilon c_5)s^{-1} + \varepsilon c_4 - c_2 \tag{2.10}$$

Let $A \neq 1$. Then, in view of (2.10) and (2.5),

$$(A - B)(1 - r^2)\Phi_o'(s; \varepsilon) = u(\varepsilon) - v_1(\varepsilon)r^2 - - [u(\varepsilon) - v_2(\varepsilon)r^2]s^{-2}$$

where

$$u(\varepsilon) = A - \varepsilon, \quad v_1(\varepsilon) = A - \varepsilon B^2, \quad v_2(\varepsilon) = A - \varepsilon A^2. \tag{2.11}$$

In can easily be shown that, for $r \in (0, 1)$,

$$\varepsilon[u(\varepsilon) - v_k(\varepsilon)r^2] < 0, \quad k = 1,2 \tag{2.12}$$

and that the function $\Phi_o(s; -1)$ decreases when $s < s_{-1}$, and increases when $s > s_{-1}$; whereas the function $\Phi_o(s; 1)$ increases when $s < s_1$, and decreases when $s > s_1$, where

$$s_\varepsilon = s_\varepsilon(r; A, B) = \sqrt{\frac{u(\varepsilon) - v_2(\varepsilon)r^2}{u(\varepsilon) - v_1(\varepsilon)r^2}}, \quad \varepsilon = \pm 1.$$

Hence, the function $\Phi_o(s; -1)$ attains its minimum in s_{-1} if $s_{-1} \in I$, and the function $\Phi_o(s; 1)$ attains its maximum in s_1 if $s_1 \in I$. It $s_\varepsilon \notin I$, then $\Phi_o(s; \varepsilon)$ attains its extremum in $c - \rho$ or $c + \rho$. One must then determine the parameters r, A, B for which $s_\varepsilon \in I$. For the purpose, let $\sigma_\varkappa(r) = [c(r) + \varkappa\rho(r)]^2$, $\varkappa = \pm 1$, $\mu_\varepsilon(r) = s_\varepsilon^2(r; A, B)$, where $c(r)$ and $\rho(r)$ are defined by

(2.3). We have $\sigma_{-1}(0) = \mu_\varepsilon(0) = \sigma_1(0) = 1$, $\sigma_{-1}(1) < \mu_\varepsilon(1) < \sigma_1(1)$, $\varepsilon = \pm 1$ and $\sigma'_{-1}(r) < 0$, $\sigma'_1(r) > 0$; $\mu'_\varepsilon(r) \geq 0$ if $A + B \leq 0$; $\mu'_\varepsilon(r) > 0$ if $A + B > 0$. So, for each $r \in (0, 1)$, $\sigma_{-1}(r) < \mu_\varepsilon(r)$ if $A + B \leq 0$, $\mu_\varepsilon(r) < \sigma_1(r)$ if $A + B > 0$, that is, $c - \varrho < s_\varepsilon$ when $A + B \leq 0$, $s_\varepsilon < c + \varrho$ when $A + B > 0$.

Let

$$h(A, B; \varepsilon) = 2(A - 2\varepsilon)^2 B^4 + 2A(3A^5 + 5\varepsilon A - 6)B^3 + A(7A^3 + 7\varepsilon A^2 - A - 13\varepsilon)B^2 + 2A^2(2A^3 - 15A + 11\varepsilon)B + A^2(A^4 - 5\varepsilon A^3 - A^2 - 13\varepsilon A + 16), \quad \varepsilon = \pm 1. \quad (2.13)$$

It can be proved that the equation $h(A, B; \varepsilon) = 0$ with the unknown B has exactly one solution $B = B_o(A; \varepsilon)$, $-1 \leq B_o(A; -1) < A$, $-A < B_o(A; 1) < A$.

Denote

$$D_1 = \{(A, B) : (-1 < A < -0{,}8, \ -1 \leq B < A) \cup (-0{,}8 \leq A < A_o, \ -1 \leq B < B_o(A; -1))\}, \quad (2.14)$$

$$D_2 = \{(A, B) : (-0{,}8 < A < A_o, \ B_o(A; -1) \leq B < A) \cup (A_o \leq A < 0{,}8, \ -1 \leq B < A) \cup (0{,}8 < A < 1, \ -1 \leq B < B_o(A; 1))\}.$$

$$D_3 = \{(A, B) : 0{,}8 < A < 1, \ B_o(A; 1) < B < A\},$$

$$A_o = \tfrac{1}{3}\left[\sqrt[3]{-181 + 8\sqrt{702}} + \sqrt[3]{-181 - 8\sqrt{702}} + 2\right] \approx -0{,}7281 \quad (2.15)$$

and

$$g(r; \varepsilon, \varkappa) = A(A + B)r^3 + 2\varkappa A(1 + \varepsilon B)r^2 - (A + B)(A - 2\varepsilon)r + - 2\varkappa(A - \varepsilon). \quad (2.16)$$

Let $A + B > 0$. The inequality $\sigma_{-1}(r) < \mu_1(r)$ is equivalent to $g(r; 1, -1) < 0$. Having done arduous but elementary calculations, we have that if $(A, B) \in D_2$ and $A + B > 0$,

$\sigma_{-1}(r) < \mu_1(r)$ for $r \in (0, 1)$;

if $(A, B) \in D_3$,

$\sigma_{-1}(r) < \mu_1(r)$ for $r \in (0, r_1^*) \cup (r_1^{**}, 1)$,

$\sigma_{-1}(r) > \mu_1(r)$ for $r \in \langle r_1^*, r_1^{**} \rangle$,

where r_1^* and r_1^{**} are roots of the equation $g(r; 1, -1) = 0$. In the same way we obtain the remaining conditions determining the position of s_ε with respect to the interval I.

If $(A, B) \in D_1$,

$$c - \varrho < s_1 < c + \varrho \quad \text{for} \quad r \in (0, 1),$$
$$c - \varrho < s_{-1} < c + \varrho \quad \text{for} \quad r \in (0, r^*_{-1}) \cup (r^{**}_{-1}, 1),$$
$$c - \varrho < c + \varrho \le s_{-1} \quad \text{for} \quad r \in \langle r^*_{-1}, r^{**}_{-1} \rangle; \tag{2.17}$$

if $(A, B) \in D_2$,

$$c - \varrho < s_\varepsilon < c + \varrho, \quad \varepsilon = \pm 1 \quad \text{for} \quad r \in (0, 1);$$

if $(A, B) \in D_3$,

$$c - \varrho < s_{-1} < c + \varrho \quad \text{for} \quad r \in (0, 1),$$
$$c - \varrho < s_1 < c + \varrho \quad \text{for} \quad r \in (0, r^*_1) \cup (r^{**}_1, 1),$$
$$s_1 \le c - \varrho < c + \varrho \quad \text{for} \quad r \in \langle r^*_1, r^{**}_1 \rangle, \tag{2.18}$$

where r^*_ε, r^{**}_ε, $\varepsilon = \pm 1$, are solutions of the equation $g(r; \varepsilon, -\varepsilon) = 0$ in the interval $(0, 1)$ (cf. (2.16)).

If $A = 1$, then, in virtue of (2.10) and (2.5),

$$\Phi_o(s; \varepsilon) = (c_1 - \varepsilon c_3)s + c_1(1 - \varepsilon)s^{-1} + \varepsilon c_4 - c_2, \tag{2.19}$$

so, $s_{-1}(r; 1, B) \in I$. Since $c_1 - c_3 \le 0$, the function $\Phi_o(s; 1)$ is decreasing or constant in the interval I.

The above reasoning proves the verity of the following lemma.

Lemma 1. If $(A, B) \in D_1$, then

$$\min_{s \in I} \Phi_o(s; \varepsilon) = \Phi_o(s_{-1}; 1) \quad \text{for} \quad r \in (0, r^*_{-1}) \cup (r^{**}_{-1}, 1),$$

$$\max_{s \in I} \Phi_o(s; \varepsilon) = \Phi_o(s_1; 1) \quad \text{for} \quad r \in (0, 1),$$

if $(A, B) \in D_2$, then, for each $r \in (0, 1)$,

$$\min_{s \in I} \Phi_o(s; \varepsilon) = \Phi_o(s_{-1}; -1),$$

$$\max_{s \in I} \Phi_o(s; \varepsilon) = \Phi_o(s_1; 1),$$

if $(A, B) \in D_3$, then

$$\min_{s \in I} \Phi_o(s; \varepsilon) = \Phi_o(s_{-1}; -1) \quad \text{for} \quad r \in (0, 1),$$

$$\max_{s \in I} \Phi_o(s; \varepsilon) = \Phi_o(s_1; 1) \quad \text{for} \quad r \in (0, r^*_1) \cup (r^{**}_1, 1)$$

where r^*_ε, r^{**}_ε, $\varepsilon = \pm 1$, are roots of the equation $g(r; \varepsilon, -\varepsilon) = 0$ (cf. (2.16)).

Let now $(s, t) \in \partial D$. Since $\Phi(s, t; \varepsilon) = \Phi(s, -t; \varepsilon)$, therefore, in view of (2.8), $\Phi(s, t; \varepsilon) = \Phi(s, \gamma(s); \varepsilon) = \Phi_1(s)$, $s \in J$, where

$$\begin{aligned} \Phi_1(s) &= c_1(s + s^{-1}) \cos \gamma(s) - c_2, \\ \cos \gamma(s) &= \frac{1}{c_4}(c_3 s + c_5 s^{-1}) \end{aligned} \tag{2.20}$$

and $J = \{s : c - \varrho \le s \le c + \varrho\}$ (cf. (2.5)-(2.8)).

Lemma 2. Let

$$s_o = s_o(r) = \sqrt[4]{\frac{c_5}{c_3}} = \sqrt[4]{\frac{1 - A^2r^2}{1 - B^2r^2}}$$

and

$$\begin{aligned} Z_1 &= \{(A, B) : 0 < A \le 1, \ -A \le B < A\}, \\ Z_2 &= \{(A, B) : 0 < A < 1, \ -1 \le B < -A\}, \\ Z_3 &= \{(A, B) : -1 < A \le 0, \ -1 \le B < A\}. \end{aligned}$$

Then

$$\min_s \Phi_1(s) = \begin{cases} \Phi_1(s_o) & \text{when } (A, B) \in Z_1 \cup Z_2, \\ \Phi_1(c + \varrho) & \text{when } (A, B) \in Z_3, \end{cases}$$

$$\max_s \Phi_1(s) = \begin{cases} \Phi_1(c - \varrho) & \text{when } (A, B) \in Z_1, \\ \Phi_1(c + \varrho) & \text{when } (A, B) \in Z_2, \\ \Phi_1(s_o) & \text{when } (A, B) \in Z_3, \end{cases} \tag{2.21}$$

P r o o f. Differentiating the function Φ_1, we shall get, by (2.20),

$$\Phi_1'(s) = \frac{1}{c_4}\left[c_1(1 - s^{-2})(c_3 s + c_5 s^{-1}) + c_1(s + s^{-1})(c_3 - c_5 s^{-2})\right] \tag{2.22}$$

that is,

$$\Phi_1'(s) = \frac{2c_1}{c_4 s^3}(c_3 s^4 - c_5). \tag{2.23}$$

It follows from (2.5) that, for any admissible values of A, B, r, we have $c_3 > 0$, $c_4 > 0$, $c_5 > 0$, and $c_1 \ge 0$ when $A \ge 0$, whereas $c_1 < 0$ when $A < 0$. Consequently, if $A > 0$, then, in view of (2.23), Φ_1 decreases for $s < s_o(r)$, increases - for $s > s_o(r)$; with $A < 0$, Φ_1 increases for $s < s_o(r)$, and decreases for $s > s_o(r)$.

Moreover, $\Phi_1(c+\rho) > \Phi_1(c-\rho)$ when $(A, B) \in Z_2$, $\Phi_1(c+\rho) \le \Phi_1(c-\rho)$ when $(A, B) \in Z_1 \cup Z_3$ and $c-\rho < s_o(r) < c+\rho$ when $(A, B) \in Z_1 \cup Z_2 \cup Z_3$.

So, formulae (2.21) take place.

In virtue of *lemmas 1* and *2* we shall determine the extremal values of the function Φ in the set $D \cup \partial D$ (cf. (2.6)-(2.8)). We shall prove

Lemma 3. If $(A, B) \in D_1$, then

$$\min_{s,t} \Phi(s, t; \varepsilon) = \begin{cases} \Phi(s_{-1}, 0; -1) & \text{for } r \in (0, r^*_{-1}) \cup (r^{**}_{-1}, 1), \\ \Phi(c+\rho, 0; -1) & \text{for } r \in \langle r^*_{-1}, r^{**}_{-1} \rangle, \end{cases}$$

$$\max_{s,t} \Phi(s, t; \varepsilon) = \Phi(s_1, 0; 1) \quad \text{for } r \in (0, 1);$$

if $(A, B) \in D_2$, then, for each $r \in (0, 1)$,

$$\min_{s,t} \Phi(s, t; \varepsilon) = \Phi(s_{-1}, 0; -1),$$

$$\max_{s,t} \Phi(s, t; \varepsilon) = \Phi(s_1, 0; 1);$$

if $(A, B) \in \overline{D_3} = D_3 \cup \{(A, B) : A = 1, -1 \le B < 1\}$, then

$$\min_{s,t} \Phi(s, t; \varepsilon) = \Phi(s_{-1}, 0; -1) \quad \text{for } r \in (0, 1),$$

$$\max_{s,t} \Phi(s, t; \varepsilon) = \begin{cases} \Phi(s_1, 0; 1) & \text{for } r \in (0, r^*_1) \cup (r^{**}_1, 1), \\ \Phi(c-\rho, 0; 1) & \text{for } r \in \langle r^*_1, r^{**}_1 \rangle, \end{cases}$$

where r^*_ε, r^{**}_ε, $\varepsilon = \pm 1$, are roots of the equation $g(r, \varepsilon, -\varepsilon) = 0$ (cf. (2.16)).

P r o o f. We shall first demonstrate that the number $\Phi_1(s_o)$, occurring in formulae (2.21), is not the extremal value of the function Φ_o where $\Phi_o(s; \varepsilon) = \Phi(s, 0; \varepsilon)$, $\varepsilon = \pm 1$. For the purpose we shall prove that

$$\varepsilon[\Phi_o(s_o; \varepsilon) - \Phi_1(s_o)] > 0, \qquad \varepsilon = \pm 1. \tag{2.24}$$

On account of (2.10) and (2.20),

$$\Phi_o(s; \varepsilon) - \Phi_1(s) = (c_1 s + \varepsilon c_4 + c_1 s^{-1})(1 - \cos\gamma(s)).$$

In virtue of (2.22), (2.20) and the equality $\Phi_1'(s_o) = 0$,

$$c_1(s_o + s_o^{-1}) = -c_4 \frac{c_1(s_o^2 - 1)}{c_3 s_o^2 - c_5} \cos\gamma(s_o);$$

consequently,

$$\varepsilon[\Phi_o(s_o; \varepsilon) - \Phi_1(s_o)] = c_4(1 - \cos\gamma(s_o))C(s_o),$$

where

$$C(s) = 1 - \varepsilon\frac{c_1(s^2 - 1)}{c_3 s^2 - c_5}\cos\gamma(s). \tag{2.25}$$

Since $c_4 > 0$, $0 < \cos\gamma(s) < 1$, it suffices that

$$\varepsilon\frac{c_1(s_o^2 - 1)}{c_3 s_o^2 - c_5} < 1.$$

We have

$$c_1(s_o^2 - 1) = c_1\frac{s_o^4 - 1}{s_o^2 + 1} = \frac{c_1}{c_3}\,\frac{c_5 - c_3}{s_o^2 + 1},$$

$$c_3 s_o^2 - c_5 = \frac{c_3^2 s_o^4 - c_5^2}{c_3 s_o^2 + c_5} = \frac{c_5(c_3 - c_5)}{c_3 s_o^2 + c_5};$$

thus,

$$\varepsilon\frac{c_1(s_o^2 - 1)}{c_3 s_o^2 - c_5} - 1 = -\frac{E}{c_3 c_5(s_o^2 + 1)} \tag{2.26}$$

where $E = c_3(\varepsilon c_1 + c_5)s_o^2 + c_5(\varepsilon c_1 + c_3)$. Since $c_3 c_5(s_o^2 + 1) > 0$, it is sufficient to show that $E > 0$. To this end, note that $\varepsilon c_1 + c_5 > 0$ and, by (2.12),

$$\varepsilon c_1 + c_3 = \frac{-\varepsilon[u(\varepsilon) - v_1(\varepsilon)r^2]}{(A - B)(1 - r^2)} > 0.$$

Thus, $E > 0$. In view of (2.26) and (2.25), we obtain inequality (2.24).

Let $A \neq 1$. It is easy to verify that, for each $r \in (0, 1)$ and $\varepsilon, \varkappa = \pm 1$,

$$\varepsilon[\Phi(s_\varepsilon, 0; \varepsilon) - \Phi(c + \varkappa\varrho, 0; \varepsilon)] > 0$$

and

$$\Phi(c + \varkappa\varrho, 0; \varepsilon) = \Phi_1(c + \varkappa\varrho).$$

Consequently,

$$\varepsilon[\Phi(s_\varepsilon, 0; \varepsilon) - \Phi_1(c + \varkappa\varrho)] > 0. \tag{2.27}$$

Inequality (2.27) proves that the numbers $\Phi_1(c + \varkappa\varrho)$, $\varkappa = \pm 1$, (cf. (2.21)) cannot be the extremal values of the function Φ in the set $D \cup \partial D$.

Moreover, by (2.17) and in view of the monotonicity of the function $\Phi_o(s; -1)$, it results that the minimal value of the function Φ in the set $D \cup \partial D$ is $\Phi(c + \varrho, 0; -1)$ for $r \in \langle r^*_{-1}, r^{**}_{-1}\rangle$. Analogously, by (2.18), we get that $\max\limits_{s,t} \Phi(s, t; \varepsilon) = \Phi(c - \varrho, 0; 1)$ for $r \in \langle r^*_1, r^{**}_1\rangle$.

When $A = 1$ and $\varepsilon = 1$, the number $\Phi_1(c - \varrho) = \Phi(c - \varrho, 0; 1)$ is the maximal value of the function Φ in the set $D \cup \partial D$, for it follows from (2.19) that the function $\Phi_o(s; 1) = \Phi(s; 0, 1)$ is decreasing $(B \neq -1)$ or constant $(B = -1)$ in the interval $[s - \varrho, s + \varrho]$.

This proves the verity of *lemma 3*.

The results obtained in *lemma 3* allow us to formulate the following

Theorem 1. Let $H(P)$ be the functional defined by formula (1.4). Then, for any function $P \in \mathcal{P}(A, B)$ and $|z| = r$, $0 < r < 1$, we have:

1^o if $(A, B) \in D_1$, then

$$H(P) \geq \begin{cases} X(r; A, B, -1) & \text{for } r \in (0, r^*_{-1}) \cup (r^{**}_{-1}, 1), \\ Y(r; A, B, 1) & \text{for } r \in \langle r^*_{-1}, r^{**}_{-1}\rangle, \end{cases}$$

$$H(P) \leq X(r, A, B, 1) \quad \text{for } r \in (0, 1), \tag{2.28}$$

2^o if $(A, B) \in D_2$, then, for each $r \in (0, 1)$,

$$X(r; A, B, -1) \leq H(P) \leq X(r; A, B, 1), \tag{2.29}$$

3^o if $(A, B) \in \overline{D_3}$, then

$$H(P) \geq X(r; A, B, -1) \quad \text{for } r \in (0, 1),$$

$$H(P) \leq \begin{cases} X(r; A, B, 1) & \text{for } r \in (0, r^*_1) \cup (r^{**}_1, 1), \\ Y(r; A, B, -1) & \text{for } r \in \langle r^*_1, r^{**}_1\rangle, \end{cases} \tag{2.30}$$

where

$$H(r; A, B, \varepsilon) = \frac{2\varepsilon[1 - ABr^2 - \sqrt{\mathcal{A}\cdot\mathcal{L}}]}{(A - B)(1 - r^2)} - \frac{A + B}{A - B}, \tag{2.31}$$

$$Y(r;\ A,\ B,\ \varkappa) = \frac{1 + \varkappa(A + B)\,r + A^2r^2}{(1 + \varkappa Ar)(1 + \varkappa Br)} \tag{2.32}$$

$$\mathfrak{A} = \mathfrak{A}(r;\ A,\ B,\ \varepsilon) = A - \varepsilon - (A - \varepsilon B^2)r^2,$$
$$\mathcal{L} = \mathcal{L}(r;\ A,\ B,\ \varepsilon) = A - \varepsilon - (A - \varepsilon A^2)r^2, \tag{2.33}$$

r_ε^*, r_ε^{**}, $\varepsilon = \pm 1$, are roots of the equation $g(r;\ \varepsilon, -\varepsilon) = 0$ (cf. (2.16)) in the interval $\langle 0, 1\rangle$, D_i, $i = 1, 2, 3$, are given by formulae (2.14).

Estimates (2.28)-(2.30) are sharp, the equalities $H(P) = Y(r;\ A,\ B,\ \varkappa)$ i $H(P) = X(r;\ A,\ B,\ \varepsilon)$ are attained at the point $z = re^{i\varphi}$, $0 < r < 1$, $0 \le \varphi < 2\pi$, by the functions,

$$P_\varkappa^*(z) = \frac{1 + \varkappa Ae^{-i\varphi}z}{1 + \varkappa Be^{-i\varphi}z}, \qquad \varkappa = \pm 1, \tag{2.34}$$

$$P_\varepsilon^{**}(z) = \frac{1 - (1 - \varepsilon A)\delta_\varepsilon e^{-i\varphi}z - \varepsilon Ae^{-2i\varphi}z^2}{1 - (1 - \varepsilon B)\delta_\varepsilon e^{-i\varphi}z - \varepsilon Be^{-2i\varphi}z^2}, \tag{2.35}$$

respectively, where

$$\delta_\varepsilon = \frac{1}{r}\,\frac{(1 - \varepsilon Br)^2 s_\varepsilon - (1 - \varepsilon Ar^2)}{(1 - \varepsilon B)s_\varepsilon - (1 - \varepsilon A)}, \qquad \varepsilon = \pm 1,$$
$$s_\varepsilon = \sqrt{\mathcal{L} \cdot \mathfrak{A}^{-1}},$$
$$\mathfrak{A} \ne 0.$$

P r o o f. For $s = s_\varepsilon$, $\varepsilon = \pm 1$, or for $s = c + \varkappa\varrho$, $\varkappa = \pm 1$, we get, respectively, $\Phi(s_\varepsilon,\ 0;\ \varepsilon) = X(r;\ A,\ B,\ \varepsilon)$ and $\Phi(c + \varkappa\varrho,\ 0;\ \varepsilon) = Y(r;\ A,\ B,\ \varkappa)$; so, on account of lemma 3, estimates (2.28)-(2.30) are true.

The proof of the sharpness of these estimates is identical with that in paper [5].

III. ESTIMATIONS OF THE MODULUS OF A FUNCTION AND THE MODULUS OF ITS DERIVATIVE IN THE CLASS $\Sigma^*(A, B)$

We shall prove

Theorem 2. If $F \in \Sigma^*(A, B)$, then, for $|z| = r$, $0 < r < 1$,

$$S(r;\ -1) \le |F(z)| \le S(r;\ 1) \tag{3.1}$$

where

$$S(r;\ \sigma) = \begin{cases} \frac{1}{r}(1 - \sigma Br)^{-(A - B)/B} & \text{when } B \neq 0, \\ \frac{1}{r}\, e^{\sigma Ar} & \text{when } B = 0,\ |\sigma| = 1. \end{cases}$$

Estimates (3.1) are sharp, equalities in (3.1) are attained at the point $z = re^{i\varphi}$, $0 < r < 1$, $0 \le \varphi < 2\pi$, by the functions $F^*(z;\ -1)$, $F^*(z;\ 1)$, respectively, where

$$F^*(z;\ \sigma) = \begin{cases} \frac{1}{z}(1 - \sigma Be^{-i\varphi}z)^{-(A - B)/B} & \text{when } B \neq 0, \\ \frac{1}{z}\exp\,(\sigma Ae^{-i\varphi}z) & \text{when. } B = 0. \end{cases} \tag{3.2}$$

P r o o f. If $F \in \Sigma^*(A,\ B)$, then from (1.3) we have

$$zF(z) = \exp\,\Big(\int_0^z \frac{1 - P(\zeta)}{\zeta}\, d\zeta\Big), \quad P \in \wp\,(A,\ B).$$

Hence

$$|F(z)| = \frac{1}{|z|}\exp\,\Big(\mathrm{re}\int_0^1 \frac{1 - P(zt)}{t}\, dt\Big).$$

Consequently,

$$\begin{aligned} |F(z)| &\le \frac{1}{|z|}\exp\,\Big(\int_0^1 \max_{|zt| = rt} \mathrm{re}\, \frac{1 - P(zt)}{t}\, dt\Big), \\ |F(z)| &\ge \frac{1}{|z|}\exp\,\Big(\int_0^1 \min_{|zt| = rt} \mathrm{re}\, \frac{1 - P(zt)}{t}\, dt\Big). \end{aligned} \tag{3.3}$$

From (2.2) and (2.3) it follows that

$$\max_{|zt| = rt} \mathrm{re}\, \frac{1 - P(zt)}{t} = \frac{(A - B)r}{1 - Brt}, \tag{3.4}$$

$$\min_{|zt| = rt} \mathrm{re}\, \frac{1 - P(zt)}{t} = -\frac{(A - B)r}{1 + Brt}. \tag{3.5}$$

So, integrating (3.4) and (3.5) with respect to t in the interval (0, 1), we shall obtain estimate (3. 1) in view of (3.3).

Adopting, in particular, $A = 1 - 2\alpha$, $0 \le \alpha < 1$, $B = -1$ in (3.1), we shall get the result of P o m m e r e n k e [12], whereas substituting $A = 1$, $B = -1$, we obtain the estimate in the class Σ^*

$$\frac{(1 - r)^2}{r} \le |F(z)| \le \frac{(1 + r)^2}{r}.$$

Let, for $\varepsilon = \pm 1$,

$$E_{1,\,\varepsilon} = \{(A, B) : 0 < A < 1,\ A - \varepsilon B^2 > 0\}$$

$$E_{2,\,\varepsilon} = \{(A, B) : A(A - \varepsilon B^2) < 0\}$$

$$E_{3,\,\varepsilon} = \{(A, B) : -1 < A < 0,\ A - \varepsilon B^2 < 0\}$$

and

$$K_1(x, y) = \frac{1}{2}\sqrt{xy}\,\ln\left|\frac{t - \sqrt{\frac{x}{y}}}{t + \sqrt{\frac{x}{y}}}\right| \quad \text{for } xy > 0,$$

$$K_2(x, y) = \sqrt{-xy}\ \mathrm{arc\,tg}\sqrt{-\frac{x}{y}}\,t \quad \text{for } xy < 0,$$

where $t = \sqrt{\dfrac{u - v_1 r^2}{u - v_2 r^2}}$, $u = u(\varepsilon)$, $v_k = v_k(\varepsilon)$, $k = 1, 2$, (cf. (2.11)).

Let

$$U(r;\ A, B, \varepsilon) = \int\frac{1}{r}\,[1 - X(r;\ A, B, \varepsilon)]dr, \tag{3.6}$$

$$V(r;\ A, B, \varkappa) = \int\frac{1}{r}\,[1 - Y(r;\ A, B, \varkappa)]dr, \tag{3.7}$$

where $X(r;\ A, B, \varepsilon)$ and $Y(r;\ A, B, \varkappa)$ are defined by formulae (2.31) and (2.32).

Performing the integration in (3.6), we shall get

$$U(r;\ A, B, \varepsilon) = \frac{2}{A - B}\Big[(A - \varepsilon)\ln r + \varepsilon\,\frac{1 - AB}{2}\ln(1 - r^2) + {}$$
$$+ \varepsilon J_1 + a_1\Big], \qquad \varepsilon = \pm 1, \tag{3.8}$$

where

$$J_1 = -\varepsilon\left[uK_1(1, 1) + \varepsilon K_1(1 - B^2, 1 - A^2) - J_2\right] + a_2 \tag{3.9}$$

$$J_2 = \begin{cases} K_1(v_1, v_2) + a_3 & \text{when } (A, B) \in E_{1,\,\varepsilon} \\ \varepsilon K_2(v_2, v_1) + a_4 & \text{when } (A, B) \in E_{2,\,\varepsilon} \\ -K_1(v_1, v_2) + a_5 & \text{when } (A, B) \in E_{3,\,\varepsilon} \end{cases} \tag{3.10}$$

Moreover,

$$J_1 = \begin{cases} (1 - B^2)\,[K_1(1, 1) - K_1(1 - B^2, 1)] + a_5 & \text{when } A - B^2 = 0 \\ K_1(1, 1) - K_1(1 - B^2, 1) + a_6 & \text{when } A(1 - \varepsilon A) = 0 \end{cases} \tag{3.11}$$

Performing the integration in (3.7), we shall obtain

$$V(r;\ A, B, \varkappa) = \begin{cases} \ln(1 + \varkappa Ar)(1 + \varkappa Br)^{-A/B} + a_7 & \text{when } B \neq 0 \\ \ln(1 + \varkappa Ar) - \varkappa Ar + a_8 & \text{when } B = 0 \end{cases} \tag{3.12}$$

In formulae (3.8)-(3.12) a_i, $i = 1, \ldots, 8$, stand for arbitrary constants.

Theorem 3. If $F \in \Sigma^*(A, B)$ and $r = |z|$, $0 < r < 1$, then

$$\exp \mathfrak{M}(r; A, B) \le r^2|F'(z)| \le \exp \mathfrak{N}(r; A, B), \tag{3.13}$$

where if $(A, B) \in D_1$,

$$\mathfrak{M}(r; A, B) = T(r, 0; 1) \quad \text{for} \quad r \in (0, 1), \tag{3.14}$$

$$\mathfrak{M}(r; A, B) = \begin{cases} T(r, 0; -1) & \text{for } r \in (0, r^*_{-1}), \\ T(r^*_{-1}, 0; -1) + W(r, r^*_{-1}; 1) & \text{for } r \in \langle r^*_{-1}, r^{**}_{-1}\rangle, \\ T(r^*_{-1}, 0; -1) + W(r^{**}_{-1}, r^*_{-1}; 1) + T(r, r^{**}_{-1}; -1) & \\ \qquad \text{for } r \in (r^{**}_{-1}, 1); \end{cases} \tag{3.15}$$

if $(A, B) \in D_2$ and $r \in (0, 1)$,

$$\mathfrak{M}(r; A, B) = T(r, 0; 1) \qquad \mathfrak{N}(r; A, B) = T(r, 0; -1); \tag{3.16}$$

if $(A, B) \in \overline{D_3}$,

$$\mathfrak{M}(r; A, B) = \begin{cases} T(r, 0; 1) & \text{for } r \in (0, r^*_1), \\ T(r^*_1, 0; 1) + W(r, r^*_1; -1) & \text{for } r \in \langle r^*_1, r^{**}_1\rangle, \\ T(r^*_1, 0; 1) + W(r^{**}_1, r^*_1; -1) + T(r, r^{**}_1; 1) & \\ \qquad \text{for } r \in (r^{**}_1, 1), \end{cases} \tag{3.17}$$

$$\mathfrak{N}(r; A, B) = T(r, 0; -1) \quad \text{for} \quad r \in (0, 1), \tag{3.18}$$

$T(x, y; \varepsilon) = U(x; A, B, \varepsilon) - U(y; A, B, \varepsilon)$, $W(x, y; \varkappa) = V(x; A, B, \varkappa) - V(y; A, B, \varkappa)$ (cf. (3.6) and (3.12)), r^*_ε, r^{**}_ε, $\varepsilon = \pm 1$ are roots of the equations $g(r; \varepsilon, -\varepsilon) = 0$ (cf. (2.13)) in the interval $\langle 0, 1\rangle$, D_i, $i = 1, 2, 3$, are given by formulae (2.14).

Estimates (3.13) are sharp when $\mathfrak{M}(r; A, B) = T(r, 0; 1)$ or $\mathfrak{N}(r; A, B) = T(r, 0; -1)$. The extremal function is of the form

$$F^*(z) = \frac{1}{z} \exp \int_0^z \frac{1 - P^{**}_\varepsilon(\zeta)}{\zeta}\, d\zeta, \qquad \varepsilon = \pm 1,$$

where P^{**}_ε is defined by (2.35).

P r o o f. If $F \in \Sigma^*(A, B)$, then, in virtue of (1.3), we have

$$-\left(1 + \frac{zF''(z)}{F'(z)}\right) = P(z) - \frac{zP'(z)}{P(z)}, \quad P \in \wp(A, B). \tag{3.19}$$

Since

$$\operatorname{re} \frac{zF''(z)}{F'(z)} = r \frac{\partial}{\partial r} \ln |z^2F'(z)| - 2, \qquad |z| = r < 1,$$

therefore, on account of (3.19),

$$rR(r) \geq 1 - \max \operatorname{re} \left[P(z) - \frac{zP'(z)}{P(z)}\right]$$

and

$$rR(r) \leq 1 - \min \operatorname{re} \left[P(z) - \frac{zP'(z)}{P(z)}\right]$$

where

$$R(r) = \frac{\partial}{\partial r} \ln (r^2|F'(z)|).$$

Applying *theorem 1,* we shall thus obtain:

1° if $(A, B) \in D_1$,

$$R(r) \geq \frac{1}{r} [1 - X(r; A, B, 1)] \quad \text{for} \quad r \in (0, 1), \tag{3.20}$$

$$R(r) \leq \begin{cases} \frac{1}{r} [1 - X(r; A, B, -1)] & \text{for} \quad r \in (0, r^*_{-1}) \cup (r^{**}_{-1}, 1), \\ \frac{1}{r} [1 - Y(r; A, B, 1)] & \text{for} \quad r \in \langle r^*_{-1}, r^{**}_{-1}\rangle, \end{cases} \tag{3.21}$$

2° if $(A, B) \in D_2$ and $r \in (0, 1)$,

$$R(r) \geq \frac{1}{r} [1 - X(r; A, B, 1)], \tag{3.22}$$

$$R(r) \leq \frac{1}{r} [1 - X(r; A, B, -1)], \tag{3.23}$$

3° if $(A, B) \in \overline{D_3}$

$$R(r) \geq \begin{cases} \frac{1}{r} [1 - X(r; A, B, 1)] & \text{for} \quad r \in (0, r^*_1) \cup (r^{**}_1, 1), \\ \frac{1}{r} [1 - Y(r; A, B, -1)] & \text{for} \quad r \in \langle r^*_1, r^{**}_1\rangle, \end{cases} \tag{3.24}$$

$$R(r) \leq \frac{1}{r} [1 - X(r; A, B, -1)] \quad \text{for} \quad r \in (0, 1). \tag{3.25}$$

Integrating inequality (3.20) with respect to r from 0 to r, we shall get, in view of (3.6), (3.8)-(3.11) formula (3.4). Analogously, we get formulae (3.16), (3.18) and the first formulae in (3.15) and (3.17).

If $r \in \langle r^*_{-1}, r^{**}_{-1}\rangle$ then, by (3.21),

$$\ln (r^2|F'(z)|) \leq \int_0^{r^*_{-1}} \frac{1}{r} [1 - X(r; A, B, -1)]\, dr +$$

$$+ \int_{r^*_{-1}}^{r} \frac{1}{r} [1 - Y(r; A, B, 1)]\, dr = U(r^*_{-1}; A, B, -1) +$$

$- U(0; A, B, -1) + V(r; A, B, 1) - V(r^*_{-1}; A, B, 1)$.

This implies the second of formulae (3.15). The remaining estimations of the proposition are obtained similarly.

In special cases of values of the parameters A and B we obtain: when $A = 1$, $B = \frac{1}{M} - 1$, $M \geq 1$, the result of W i a t r o w s k i [15], when $A = 1$, $B = -1$, L ö w n e r's estimate [9]

$$\frac{1 - r^2}{r^2} \leq |F'(z)| \leq \frac{1}{r^2(1 - r^2)}.$$

IV. THE RADIUS OF CONVEXITY OF THE FAMILY $\Sigma^*(A, B)$

As we know, the radius of convexity of the family $\Sigma^*(A, B)$ is defined by the formula

$$\text{r. c. } \Sigma^*(A, B) = \inf_{F \in \Sigma^*(A,B)} \{\sup [r : \text{re } (-(1 + \frac{zF''(z)}{F'(z)})) > 0, \; |z| < r]\}.$$

Since $\Sigma^*(A, B)$ is a compact family, therefore r. c. $\Sigma^*(A, B)$ is equal to the greatest value of r, $0 < r \leq 1$, such that

$$\text{re } [-(1 + \frac{zF''(z)}{F'(z)})] \geq 0 \tag{4.1}$$

for each $0 < |z| \leq r$ and each function $F \in \Sigma^*(A, B)$. In other words, r. c. $\Sigma^*(A, B)$ is equal to the smallest root r_o, $0 < r_o \leq 1$, of the equation $\omega(r) = 0$ where

$$\omega(r) = \min \{\text{re}[-(1 + \frac{zF''(z)}{F'(z)})], \; |z| = r < 1, \; F \in \Sigma^*(A, B)\}. \tag{4.2}$$

In virtue of (4.1), (3.19) and *theorem 1*,

$$\omega(r) = \begin{cases} \frac{u(r)}{u_1(r)} \text{ when } (A, B) \in D_1 \text{ and } r \in (0, r^*_{-1}) \cup (r^{**}_{-1}, 1), \\ \qquad \text{or } (A, B) \in D_2 \cup \overline{D_3} \text{ and } r \in (0, 1), \\ \frac{v(r)}{v_1(r)} \text{ when } (A, B) \in D_1 \text{ and } r \in \langle r^*_{-1}, r^{**}_{-1} \rangle, \end{cases} \tag{4.3}$$

where

$$u(r) = d_1 r^4 - 2d_2 r^2 + d_3,$$
$$u_1(r) = (1 - r^2)[2\sqrt{\alpha\varkappa} + 2(1 - ABr^2) + (A + B)(1 - r^2)] > 0,$$
$$v(r) = A^2 r^2 + (A + B)r + 1, \tag{4.4}$$
$$v_1(r) = (1 + Ar)(1 + Br) > 0,$$

$d_1 = 4A^2 + 3A + B, \quad d_2 = 2A^2 + 5A + 2 - B, \quad d_3 = 3A + B + 4.$

Let $B = B_o(A; -1)$ be the solution of the equation $h(A, B; -1) = 0$ (cf. (2.13)) in the interval $\langle -1, A)$, and $B = B_1(A) = -(A^2 + A + 1)$. Denote by A_1 the only root of the equation $B_o(A; -1) = B_1(A)$ in the interval $(-0,8; A_o)$ where A_o is given by (2.15), and let

$$E_1 = \{(A, B) : (-1 < A \le -0,8, \quad B_1(A) \le B < A) \cup \cup (-0,8 < A < A_1, \quad B_1(A) \le B < B_o(A, -1))\},$$

$$E_2 = \{(A, B) : (-1 < A \le A_1, \quad -1 \le B < B_1(A)) \cup \cup (A_1 < A < A_o, \quad -1 \le B < B_o(A, -1))\}.$$

Obviously $E_1 \cup E_2 = D_1$.

Since, for all admissible values of A and B, $d_2^2 - d_1 d_3 > 0$, $d_2 > 0$, $d_3 > 0$, $u(1) < 0$, so, the polynomial $u(r)$ has in the interval $(0, 1)$ exactly one real zero r_1, with that $u(r) > 0$ for $0 \le r < r_1$, and $u(r) < 0$ for $r_1 < r < 1$, where

$$r_1 = \sqrt{\frac{3A+B+4}{2A^2+5A+2-B+2(1+A)\sqrt{A^2+1-2B}}}. \tag{4.5}$$

Analogously, it is easy to verify that if $(A, B) \in E_1$, then $v(r) > 0$ for $r \in (0, 1)$ and if $(A, B) \in E_2$, then the polynomial $v(r)$ has in the interval $(0, 1)$ exactly one real zero and $v(r) > 0$ for $0 \le r < r_2$, $v(r) < 0$ for $r_2 < r < 1$ where

$$r_2 = 2\left[\sqrt{(B-A)(3A+B)} - (A+B)\right]^{-1}. \tag{4.6}$$

From the above considerations and from the fact that $\operatorname{sgn} u(r^*_{-1}) = \operatorname{sgn} v(r^{**}_{-1})$ and $\operatorname{sgn} u(r^{**}_{-1}) = \operatorname{sgn} v(r^{**}_{-1})$ we get

Lemma 4. If $(A, B) \in E_1 \cup D_2 \cup \overline{D_3}$, or $(A, B) \in E_2$ and $r_2 < r^*_{-1}$ or $r_2 > r^{**}_{-1}$, then $\omega(r) > 0$ for $0 \le r < r_1$, $\omega(r_1) = 0$, and $\omega(r) < 0$ for $r_1 < r < 1$. If $(A, B) \in E_2$ and $r^*_{-1} < r < r^{**}_{-1}$, then $\omega(r) > 0$ for $0 \le r < r_2$, $\omega(r_2) = 0$, and $\omega(r) < 0$ for $r_2 < r < 1$.

P r o o f. If $(A, B) \in D_2 \cup \overline{D_3}$, then, in view of (4.3), the proposition is obvious. If $(A, B) \in E_1$, then, in view of $v(r) > 0$ for $r \in (0, 1)$ and the equality $\operatorname{sgn} u(r^{**}_{-1}) = \operatorname{sgn} v(r^{**}_{-1})$, it results that $r_1 > r^{**}_{-1}$ and $\omega(r) > 0$ for $0 \le r < r_1$, $\omega(r) < 0$ for $r_1 < r < 1$. Analogous considerations in the remaining cases conclude the proof of the lemma.

Lemma 5. The zero r_2 of the polynomial $v(r)$ satisfies the condition $r_2 < r_1^*$ or $r_2 > r_{-1}^{**}$ if and only if $y(A, B) > 0$; $r_{-1}^* < r_2 < r_{-1}^{**}$ when $y(A, B) < 0$ where

$$y(A, B) = B^3 + k_1(A)B^2 + k_2(A)B + k_3(A), \tag{4.7}$$

$$\begin{aligned} k_1(A) &= 2A^2 + 5A + 2,\\ k_2(A) &= -(A^4 - 4A^3 - 9A^2 - 4A + 1),\\ k_3(A) &= -A(3A^4 + 6A^3 + 7A^2 + 6A + 3). \end{aligned} \tag{4.8}$$

P r o o f. If $\Phi_o(s; -1)$ defined by (2.10) attains its minimum equal to zero for $r = r_2$, then there must be $\Phi_o(c + \varrho; -1) = 0$, that is, for $r = r_2$

$$(c_1 + c_3)(c + \varrho) + (c_1 + c_5)(c + \varrho)^{-1} - (c_4 + c_2) = 0.$$

Hence

$$\frac{c_1 + c_5}{c_1 + c_3} + (c + \varrho)^2 - \frac{c_4 + c_2}{c_1 + c_3}(c + \varrho) = 0, \quad r = r_2. \tag{4.9}$$

On the other hand, the numbers r_{-1}^*, r_{-1}^{**} are roots of the equation $s_{-1}^2 = (c + \varrho)^2$ i.e. of the equation

$$\frac{c_1 + c_5}{c_1 + c_3} - (c + \varrho)^2 = \frac{rg(r; -1, 1)}{(c_1 + c_3)(1 + Br)^2(1 - r^2)}, \; r = r_2 \tag{4.10}$$

(cf. (2.16)).

Equation (4.9) and (4.10), in view of (2.5), imply

$$Ar_2^3 - (1 + 2A)r_2^2 - (A + 2)r_2 + 1 = \frac{r_2 g(r_2; -1, 1)}{1 + Br_2}. \tag{4.11}$$

Since $v(r_2) = 0$, therefore

$$A^2r_2^2 + (A + B)r_2 + 1 = 0. \tag{4.12}$$

From (4.11) and (4.12), we obtain

$$A^2r_2^2 - A(1 + A)^2r_2 - A(2A + B + 2) = \frac{Ag(r_2; -1, 1)}{1 + Br_2}. \tag{4.13}$$

By (4.12) and (4.13), we get

$$-(A^3 + 2A^2 + 2A + B)r_2 - (2A^2 + 2A + 1 + AB) = \frac{Ag(r_2; -1, 1)}{1 + Br^2}. \tag{4.14}$$

The polynomial $g(r; -1, 1)$ takes negative values for $r \in (0, r^*_{-1}) \cup (r^{**}_{-1}, 1)$ and for $(A, B) \in D_1$, and $g(r^*_{-1}; -1, 1) = g(r^{**}_{-1}; -1, 1) = 0$. Therefore $r_2 < r^*_{-1}$ or $r_2 > r^{**}_{-1}$ when the left-hand side of equality (4.14) is positive, whereas $r_2 \in (r^*_{-1}, r^{**}_{-1})$ when it is negative.

Denote $\hat{r} = -(2A^2 + 2A + 1 + AB)/(A^3 + 2A^2 + 2A + B)$. We have $0 < \hat{r} < 1$ for $(A, B) \in D_1$. Hence $r_2 < r^*_{-1}$ or $r_2 > r^{**}_{-1}$ if $r_2 > \hat{r}$. This implies that $v(\hat{r}) > 0$, i.e. $A^2\hat{r}^2 + (A + B)\hat{r} + 1 > 0$. Hence, after some transformations, we obtain that $y(A, B) > 0$ where $y(A, B)$ is defined by (4.7) and (4.8). The analogous examination leads to the fact that $r_2 \in (r^*_{-1}, r^{**}_{-1})$ when $y(A, B) < 0$.

Lemma 6. For each $A \in (-1; -0{,}8)$. the equation $y(A, B) = 0$ with the unknown B has exactly one solution $B = B^*(A)$ in the interval $(-1, B_1(A))$ where $B_1(A) = -(A^2 + A + 1)$.

P r o o f. Let $A \in (-1; -0{,}8)$. By (4.7) and (4.8), we have $y(A, -1) = (A + 1)x(A)$ where $x(A) = -(3A^4 + 2A^3 + 9A^2 + 4A - 2)$. Since $x'(A) > 0$ for $A \in (-1; -0{,}8)$, therefore $x(A)$ increases in this interval. But $x(-1) < 0$ and $x(-0{,}8) < 0$. Thus $x(A) < 0$ for $A \in (-1; -0{,}8)$. Thereby,

$$y(A, -1) < 0 \quad \text{for} \quad A \in (-1; -0{,}8), \tag{4.15}$$

After simple calculations we get

$$y(A, B_1(A)) = 2(A + 1)^2[A^4 + 1 - 2A(A^2 + 1)] > 0. \tag{4.16}$$

Differentiating the function $y(A, B)$ twice with respect to B we have $y''_{BB}(A, B) < 0$, thus $y'_B(A, B)$ decreases in $(-1, B_1(A))$. But for $A \in (-1; -0{,}8)$, $y'_B(A, -1) > 0$ and $y'_B(A, B_1(A)) > 0$. Therefore $Y(A, B)$ increases in the interval $(-1, B_1(A))$ which ends the proof by (4.15) and (4.16).

Corollary. If $(A, B) \in E_2$, then $y(A, B) < 0$ if and only if $B < B^*(A)$, and $y(A, B) > 0$ if and only if $B > B^*(A)$.

Denote by A^* the root of the equation $B^*(A) = B_o(A; -1)$ in the interval (A_1, A_o) (cf. (2.15)), and let

$$E_{21} = \{(A, B) : (-1 < A \le A^*, \ -1 \le B < B^*(A)) \cup \cup (A^* < A < A_o, \ -1 \le B < B_o(A; -1))\},$$

$$E_{22} = \{(A, B) : (-1 < A \le A_1, \ B^*(A) < B < B_1(A)) \cup \cup (A_1 < A < A^*, \ B^*(A) < B < B_o(A, -1))\}.$$

Obviously, $E_{21} \cup E_{22} = E_2$.

In virtue of *lemmas 4-6*, we get

Theorem 4. The radius of convexity of the family $\Sigma^*(A, B)$ is defined by r. c. $\Sigma^*(A, B) = r_o$ where

$$r_o = \begin{cases} r_1 & \text{if } (A, B) \in E_1 \in E_{22} \cup D_2 \cup \overline{D_3}, \\ r_2 & \text{if } (A, B) \in E_{21}, \end{cases} \tag{4.17}$$

r_1 and r_2 are defined by formulae (4.5) and (4.6).

The equality $r_o = r_1$ holds for the function F^{**}, $r_o = r_2$ - for the function F^*, where

$$-\frac{zF^{**'}(z)}{F^{**}(z)} = P^{**}_{-1}(z), \qquad -\frac{zF^{*'}(z)}{F^*(z)} = P^*_1(z),$$

P^{**}_{-1} and P^*_1 are given by (2.33) and (2.34).

Adopting in (4.17) $A = 1$, $B = \frac{1}{M} - 1$, $M \geq 1$, one obtains the r. c. $\Sigma^*(M)$ [15], whereas for $A = 1 - 2\alpha$, $B = -1$, $\alpha \in [0, 1)$ - the r. c. Σ^*_α [16], and hence, when $\alpha = 0$, the result of R o b e r t s o n [14]:

$$\text{r. c. } \Sigma^* = 3^{-1/2}.$$

V. MEROMORPHIC CLOSE-TO-CONVEX FUNCTIONS

Let $J(A, B; M, N)$, $-1 < A \leq 1$, $-1 \leq B < A$, $-1 < M \leq 1$, $-1 \leq N < M$, denote the family of all functions of the form

$$f(z) = \frac{1}{z} + b_o + b_1 z + b_2 z^2 + \ldots \tag{5.1}$$

holomorphic in the ring Q and such that

$$-\frac{zf'(z)}{F(z)} = P(z), \qquad z \in Q, \tag{5.2}$$

for some functions $P \in \wp(A, B)$ and some function $F \in \Sigma^*(M, N)$.

The class $J(1 - 2\lambda, -1; 1 - 2\sigma, -1)$, $\lambda, \sigma \in [0, 1)$, was studied by Libera [7], whereas in [8] there are results concerning the class $J(1, -1; 1, -1)$ of all meromorphic close-to-convex functions of form (5.1).

Between the coefficients of the function $f \in J(A, B; M, N)$ and those of the function $F \in \Sigma^*(M, N)$ of form (1.2) the relation defined in the following theorem takes place:

Theorem 5. If $f \in J(A, B; M, N)$, then

$$(n|b_n| + |a_n|)^2 \le (A - B)^2 + 4\,\frac{M - N}{\sqrt{2(1-MN)}}, \quad n = 0, 1, 2, \ldots \tag{5.3}$$

P r o o f. It follows from (5.2) that

$$-\frac{zf'(z)}{F(z)} = \frac{1 + A\omega(z)}{1 + B\omega(z)}$$

for some function ω holomorphic in K and satisfying the conditions $\omega(0) = 0$, $|\omega(z)| < 1$ for $z \in K$; then, in view of (5.1) and (1.2),

$$-\sum_{k=0}^{\infty} (kb_k + a_k)z^{k+1} = \left[A - B + \sum_{k=0}^{\infty} (Aa_k + kBb_k)z^{k+1}\right]\omega(z).$$

Hence, applying Clunie's method [1], we shall get

$$\sum_{k=0}^{n} |kb_k + a_k|^2 \le (A - B)^2 + \sum_{k=0}^{n-1} |Aa_k + kBb_k|^2.$$

Consequently,

$$|nb_n + a_n|^2 \le (A - B)^2 - \sum_{k=0}^{n-1} \left[|kb_k + a_k|^2 - |Aa_k + kBb_k|^2\right].$$

Since $|kb_k + a_k|^2 - |Aa_k + kBb_k|^2 = k^2(1 - B^2)|b_k|^2 + (1 - A^2)|a_k|^2 + 2k(1 - AB)\,\mathrm{re}\, a_k\overline{b_k}$, therefore

$$|nb_n + a_n|^2 \le (A - B)^2 + 2(1 - AB) \sum_{k=1}^{n-1} k|a_kb_k|.$$

But $(n|b_n| + |a_n|)^2 = |nb_n + a_n|^2 + 2n|a_nb_n| - 2n\,\mathrm{re}\, a_n\overline{b_n}$; thus

$$(n|b_n| + |a_n|)^2 \le (A - B)^2 + 2(1 - AB) \sum_{k=1}^{n-1} k|a_kb_k| + 4n|a_nb_n| =$$

$$= (A - B)^2 + 4 \sum_{k=1}^{n} k|a_kb_k| - 2(1 + AB) \sum_{k=1}^{n-1} k|a_kb_k| \le (A - B)^2 +$$

$$+ 4 \sum_{k=1}^{n} k|a_kb_k|. \tag{5.4}$$

Each function f of form (5.1), analytic and univalent in the ring Q, satisfies, by the area theorem, the inequality $\sum_{k=1}^{\infty} k|b_k|^2 \le 1$, and thereby, $\sum_{k=1}^{n} k|b_k|^2 \le 1$. So, from Schwarz's inequality

$$\sum_{k=1}^{n} k|a_kb_k| \le \left(\sum_{k=1}^{n} k|a_k|^2\right)^{1/2} \left(\sum_{k=1}^{n} k|b_k|^2\right)^{1/2},$$

we obtain

$$\sum_{k=1}^{n} k|a_k b_k| \le \left(\sum_{k=1}^{n} k|a_k|^2\right)^{1/2}. \tag{5.5}$$

The coefficients a_k, $k = 0, 1, 2, \ldots,$ of expansion (1.2) of the function F from the class $\Sigma^*(M, N)$ have the following property (cf. [4]):

$$\sum_{k=0}^{\infty} [(1 - N^2)k^2 + 2(1 - MN)k + 1 - M^2] |a_k|^2 \le (M - N)^2.$$

Consequently,

$$\left(\sum_{k=1}^{n} k|a_k|^2\right)^{1/2} \le \frac{M - N}{\sqrt{2(1 - MN)}}. \tag{5.6}$$

So, (5.4)-(5.6) imply (5.3).

Adopting in (5.3) $A = 1 - 2\lambda$, $M = 1 - 2\sigma$, $B = N = -1$, $\lambda, \sigma \in [0, 1)$, we obtain the result of L i b e r a [7], whereas when $A = M = 1$, $B = N = -1$ - that of L i b e r a and R o b e r t s o n [8].

For a fixed r, $0 < r < 1$, let us denote $w(\theta) = f(re^{i\theta})$, $0 \le \theta < 2\pi$. We shall investigate the behaviour of the angle $\Psi(\theta)$ of inclination of the tangent at the point $w(\theta)$ to the image Γ_r of the circle $C_r = \{z : |z| = r\}$ under the mapping by means of a function f from the class $J(A, B; M, N)$. We have

$$\psi(\Theta) = \Theta + \frac{\pi}{2} + \arg f'(re^{i\Theta})$$

and, for $\Theta_1 < \Theta_2$, $\Theta_1, \Theta_2 \in [0, 2\pi)$,

$$\psi(\Theta_2) - \psi(\Theta_1) = \Theta_2 + \arg f'(re^{i\Theta_2}) - \Theta_1 - \arg f'(re^{i\Theta_1}). \tag{5.7}$$

Since

$$\Theta + \arg f'(re^{i\Theta}) = \Theta + \mathrm{re}\,\{-i \ln f'(re^{i\Theta}),$$

therefore

$$\frac{\partial}{\partial\Theta} [\Theta + \arg f'(re^{i\Theta})] = \mathrm{re}\,\{1 + re^{i\Theta} \frac{f''(re^{i\Theta})}{f'(re^{i\Theta})}\}$$

and

$$\int_{\Theta_1}^{\Theta_2} \frac{\partial}{\partial\Theta} [\Theta + \arg f'(re^{i\Theta})]\, d\Theta = \int_{\Theta_1}^{\Theta_2} \mathrm{re}\,\{1 + re^{i\Theta} \frac{f''(re^{i\Theta})}{f'(re^{i\Theta})}\}\, d\Theta.$$

On the other hand, by (5.7),

$$\int_{\Theta_1}^{\Theta_2} \frac{\partial}{\partial\Theta}\,[\Theta + \arg f'(re^{i\Theta})]\,d\Theta = \psi(\Theta_2) - \psi(\Theta_1).$$

Consequently,

$$\psi(\Theta_2) - \psi(\Theta_1) = \int_{\Theta_1}^{\Theta_2} \mathrm{re}\{1 + re^{i\Theta}\,\frac{f''(re^{i\Theta})}{f'(re^{i\Theta})}\}\,d\Theta.$$

Thus, the integral on the right-hand side of the last equality characterizes the increment of the angle of inclination of the tangent to the curve Γ_r between the points $w(\Theta_2)$ and $w(\Theta_1)$ for $\Theta_2 > \Theta_1$.

Theorem 6. If $f \in J(A, B; M, N)$ and $0 < r < 1$, then, for $\Theta_1 < \Theta_2$, $\Theta_1, \Theta_2 \in [0, 2\pi)$,

$$\int_{\Theta_1}^{\Theta_2} \mathrm{re}\{1 + re^{i\Theta}\,\frac{f''(re^{i\Theta})}{f'(re^{i\Theta})}\}\,d\Theta \le \pi - \frac{1 - Mr}{1 - Nr}\,(\Theta_2 - \Theta_1) + \tag{5.8}$$

$$- 2 \arc\cos \frac{(A - B)r}{1 - ABr^2}.$$

P r o o f. From definition condition (5.2) of the function f of the class $J(A, B; M, N)$ we have

$$\mathrm{re}\,\{1 + \frac{zf''(z)}{f'(z)}\} = \mathrm{re}\,\frac{zF'(z)}{F(z)} + \mathrm{re}\,\frac{zP'(z)}{P(z)}, \tag{5.9}$$

where $F \in \Sigma^*(M, N)$, $P \in \wp(A, B)$. Putting $z = re^{i\Theta}$, $0 < r < 1$, $\Theta \in [0, 2\pi)$, into (5.9) and integrating with respect to Θ in the interval $[\Theta_1, \Theta_2]$, $\Theta_1 < \Theta_2$, we shall get

$$\int_{\Theta_1}^{\Theta_2} \mathrm{re}\,\{1 + re^{i\Theta}\,\frac{f''(re^{i\Theta})}{f'(re^{i\Theta})}\}\,d\Theta = \int_{\Theta_1}^{\Theta_2} \mathrm{re}\,\{re^{i\Theta}\,\frac{F'(re^{i\Theta})}{F(re^{i\Theta})}\}\,d\Theta +$$

$$+ \int_{\Theta_1}^{\Theta_2} \mathrm{re}\,\{re^{i\Theta}\,\frac{P'(re^{i\Theta})}{P(re^{i\Theta})}\}\,d\Theta. \tag{5.10}$$

From (2.2), (2.3) and definition condition (1.3) of the function F of the class $\Sigma^*(M, N)$ it follows that

$$- \min_{F\in\Sigma^*(M, N)} \int_{\Theta_1}^{\Theta_2} \mathrm{re}\,\{re^{i\Theta}\,\frac{F'(re^{i\Theta})}{F(re^{i\Theta})}\}\,d\Theta = - \frac{1 - Mr}{1 - Nr}\,(\Theta_2 - \Theta_1) \tag{5.11}$$

We shall estimate the second integral on the right-hand side of (5.10). For the purpose, note that

$$\frac{\partial}{\partial\Theta} \arg P(re^{i\Theta}) = \frac{\partial}{\partial\Theta} \mathrm{re}\{-i \ln P(re^{i\Theta})\} = \mathrm{re}\,\{re^{i\Theta} \frac{P'(re^{i\Theta})}{P(re^{i\Theta})}\};$$

thus

$$\int_{\Theta_1}^{\Theta_2} \mathrm{re}\,\{re^{i\Theta} \frac{P'(re^{i\Theta})}{P(re^{i\Theta})}\}\, d\Theta = \arg P(re^{i\Theta_2}) - \arg P(re^{i\Theta_1}).$$

Hence

$$\max_{P\in\wp(A,\,B)} \left|\int_{\Theta_1}^{\Theta_2} \mathrm{re}\,\{re^{i\Theta} \frac{P'(re^{i\Theta})}{P(re^{i\Theta})}\}\, d\Theta\right| = \max_{P\in\wp(A,\,B)} |\arg P(re^{i\Theta_2}) - \arg P(re^{i\Theta_1})|. \tag{5.12}$$

By (2.2) and (2.3),

$$\max_{P\in\wp(A,\,B)} \arg P(re^{i\Theta}) = \arc \sin \frac{(A-B)r}{1-ABr^2},$$

so, in view of (5.13),

$$\max_{P\in\wp(A,\,B)} \left|\int_{\Theta_1}^{\Theta_2} \mathrm{re}\,\{re^{i\Theta} \frac{P'(re^{i\Theta})}{P(re^{i\Theta})}\}\, d\Theta\right| \le \max_{P\in\wp(A,\,B)} |\arg P(re^{i\Theta})| +$$

$$- \min_{P\in\wp(A,\,B)} |\arg P(re^{i\Theta})| \le 2 \arc \sin \frac{(A-B)r}{1-ABr^2} =$$

$$= \pi - 2 \arc \cos \frac{(A-B)r}{1-ABr^2}.$$

Hence and from (5.11), in view of (5.10), we get (5.8).

In special cases of values of the parameters A, B, M, N, from *theorem 6* one obtains the earlier results (cf. [8] and [7]).

Let us still observe that the addends on the right-hand side of inequality (5.8), depending on r, are negative and increasing, so they both may be omitted or, in particular, one of them.

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WŁASNOŚCI EKSTREMALNE FUNKCJI GWIAŹDZISTYCH W PIERŚCIENIU $0 < |z| < 1$

Niech $\wp(A, B)$, $-1 \leq B < A \leq 1$, oznacza rodzinę funkcji P, $P(0) = 1$, holomorficznych w kole $K = \{ z : |z| < 1 \}$ i takich, że

$$P(z) = \frac{1 + A\,\omega(z)}{1 + B\,\omega(z)}$$

dla pewnej funkcji ω, $\omega(0) = 0$, $|\omega(z)| < 1$, holomorficznej w K. Następnie, niech $\Sigma^*(A, B)$ będzie rodziną funkcji postaci

$$F(z) = \frac{1}{z} + a_o + a_1 z + a_2 z^2 + \ldots$$

holomorficznych w pierścieniu $Q = \{z : 0 < |z| < 1\}$ i takich, że $-zF'(z)/F(z) \in \wp(A, B)$ dla $z \in Q$.

W pracy oszacowano funkcjonały: re $\{P(z) - zP'(z)/P(z)\}$, $P \in \wp(A, B)$ i $z \in K$, $|F(z)|$, $|F'(z)|$, gdy $F \in \Sigma^*(A, B)$ i $z \in Q$ oraz wyznaczono promień wypukłości rodziny $\Sigma^*(A, B)$. Na koniec udowodniono dwie własności pewnej klasy funkcji meromorficznych prawie wypukłych generowanej funkcjami klas $\wp(A, B)$ i $\Sigma^*(A, B)$.