A C T A   U N I V E R S I T A T I S   L O D Z I E N S I S
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A NOTE ON THE RECURRENCE THEOREM

This note contains a simple proof of a generalization of the recurrence theorem from T a a m's paper ([3] *Theorem 2*).

Let X be a nonempty set, S - a σ-algebra of subsets of X, and I - a σ-ideal included in S. In this note, T is always a measurable transformation of X into itself (i.e. $T^{-1}E \in S$ if $E \in S$). T is called dissipative if there is a set $E \in S-I$ such that E, $T^{-1}E$, $T^{-2}E$, ... are pairwise disjoint; in the contrary case, T is called conservative. T is called compressible if there exists a set $E \in S$ such that $E \subset T^{-1}E$ and $T^{-1}E-E \in S-I$; in the contrary case, T is called incompressible. T is called recurrent if, for each set $E \in S$, I-almonst every point of E returns to E under the action of T (i.e. $\{x \in E;\ \forall (n \in N)\ T^n x \notin E\} \in I$); strongly recurrent if, for each set $E \in S$, I-almost every point of E returns to E infinitely many times under the action of T (i.e. $\{x \in E;\ \exists (k \in N)\ \forall (n \geqslant k)\ T^n x \notin E\} \in I$).

In this note, we shall prove the following theorem:

Theorem. Whenever T has one of the following properties:

(1) conservativity,

(2) incompressibility,

(3) recurrence,

(4) strong recurrence,

T, T^2, T^3, ... have all these properties.

In the case when I is the σ-ideal of null sets of a measure space (X, S, m), this theorem was proved in [3]. The proof presented there can be applied also in the general case. However, we shall give a simpler proof.

P r o o f. The equivalence of conditions (1) - (4) for the transformation T was proved in [4]. Another proof can be obtained from [2] *theorem 17.2* ((1) $\Rightarrow$ (4)) and [1] p. 11 ((3) $\Rightarrow$ (2) and (2) $\Rightarrow$ (1)).

To end the proof of theorem, we shall show that if T is strongly recurrent, then T^n is conservative for each $n \in N$.

Suppose T^n is dissipative for some positive intiger n. There is an $F \in S-I$ such that the collection $\mathcal{A}_0 = \{F, T^{-n}F, T^{-2n}F, \ldots\}$ consists of pairwise disjoint sets. Hence, for each $k \in \{0, 1, \ldots, n-1\}$, the collection $\mathcal{A}_k = \{T^{-k}F, T^{-k-n}F, T^{-k-2n}F, \ldots\}$ consists of pairwise disjoint sets. Thus the intersection of more than n sets of the form $T^{-j}F$ is empty $(j \in \{0, 1, 2, \ldots\})$. Therefore, no point $x \in F$ belongs to infinitely many sets of the form $T^{-j}F$ and, consequently, no $x \in F$ returns infinitely many times. Hence T is not strongly recurrent.

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UWAGA O TWIERDZENIU REKURENCYJNYM

Praca zawiera prosty dowód uogólnienia twierdzenia rekurencyjnego z pracy T a a m a ([3] *twierdzenie 2*).