A C T A   U N I V E R S I T A T I S   L O D Z I E N S I S
FOLIA MATHEMATICA 3, 1989

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SOME NEW THEOREMS ON THE MATRICES THAT ARISE IN THE ANALYSIS OF NETWORKS

The paper contains some new results which exhibit the relationships between certain types of matrices with which one deals in the analysis of nonlinear dc networks.

Just recently, searching for a relatively simple method which would allow to localize the solutions of the general dc equation of nonlinear networks some new pure algebraical relations between certain very well known types of matrices have been established by the author. The subject of this paper is to report briefly on the most interesting and elegant results. There is no main idea here, the physical interpretations and potential applications are not important, all the theorems are elementary and clear at the first glance. Also the methods and tools applied below and conceptually simple.

The first section explains the basic notation and terminology.

I. NOTATION AND TERMINOLOGY

For an arbitrary positive integer n, the set of indices 1, ..., n will be denoted by $\mathcal{n}$. A square matrix is a real function on $\mathcal{p} \times \mathcal{p}$ where $\mathcal{p}$ is some index set. If $\mathcal{m} \subset \mathcal{n}$ and if **A** is a matrix on $\mathcal{n} \times \mathcal{n}$ we denote by $\mathbf{A}(\mathcal{m})$ the principal submatrix of **A** corresponding to $\mathcal{m}$. A matrix $\mathbf{A} = [a_{ij}]$ is said to be reducible if there exists a nonvoid $\mathcal{p} \subset \mathcal{n}$, $\mathcal{p} \neq \mathcal{n}$, such that $a_{ij} = 0$ for $i \in \mathcal{p}$ and $j \in \mathcal{n} \setminus \mathcal{p}$. A matrix is irreducible if it is not reducible. A matrix **A** is said to be nonnegative (positive) or

$A \geqslant (>) 0$ if $a_{ij} \geqslant (>) 0$ for all its elements. If A and B are two matrices of the same size we write $B \geqslant (>) A$ for $B - A \geqslant (>) 0$. We denote the $n \times n$ identity matrix by either I_n or, when the dimension is unimportant or is clear from the context simply by I. An $n \times n$ block-diagonal (block-triangular) matrix is said to possess the irreducible-block-diagonal (irreducible-block-triangular) form if each of its blocks on the main diagonal is an irreducible square submatrix. The phrase "diagonal nonnegative positive matrix" D is used to describe a square matrix with only nonnegative (positive) elements on the main diagonal and with all other elements having the value zero.

An $n \times n$ matrix A is said to satisfy a weak (strong) row-sum dominance condition if $a_{ii} \geqslant (>) \sum_{j \neq 1} |a_{ij}|$ for each $i \in \mathcal{n}$. Similarly, A is weakly (strongly) column-sum dominant if and only if its elements satisfy $a_{ii} \geqslant (>) \sum_{j \neq 1} |a_{ji}|$ for $i \in \mathcal{n}$. The class of all weakly row-sum (column-sum) dominant $n \times n$ matrices is denoted by $\mathcal{D}_r$ ($\mathcal{D}_c$).

In [2] and [3] M. Fiedler and V. Pták have introduced the classes of matrices denoted by $\mathcal{P}$ ($\mathcal{P}_o$) and defined by any one of several equivalent properties:

(i) All principal minors of A are positive (nonnegative).

(ii) For each vector $x \neq 0$ there exists an index $k \in \mathcal{n}$ such that $x_k y_k > (\geqslant) 0$, where $y = Ax$.

(iii) For each vector $x \neq 0$ there exists a diagonal matrix $D_x > (\geqslant) 0$ such that $\langle Ax, D_x x\rangle > 0$ ($\langle x, D_x x\rangle > 0$ and $\langle Ax, D_x x\rangle \geqslant 0$). ($\langle x, y\rangle = \sum_{i \in \mathcal{n}} x_i y_i$ denotes here the scalar product of the vectors $x = [x_1, \ldots, x_n]^T$ and $y = [y_1, \ldots, y_n]^T$).

(iv) Every real eigenvalue of A as well as of each principal submatrix of A is positive (nonnegative).

I. W. Sandberg and A. N. Willson have proved in [6] that another property can be added to the list of equivalent properties, namely:

(v) A belongs to $\mathcal{P}_o$ if and only if $\det(A + D) \neq 0$ for every diagonal positive matrix D.

In part IV we formulate and prove yet another, topological, property defining the class $\mathcal{P}_o$.

For every square matrix $A = [a_{ij}]$ of order $n \geqslant 2$ let $P(A) = [p_{ij}]$ be a matrix of elements defined as follows:

$$p_{ij} = \begin{cases} \sum_{k \neq i} |a_{ik}| & \text{when } i = j \\ - \ |a_{ji}| & \text{when } i \neq j \end{cases} \qquad \text{for } i, j \in \mathcal{n}$$

We denote by S_o the class of all positive semidefinite n x n matrices, i.e., satisfying the condition $\langle \mathbf{x}, \mathbf{A}\mathbf{x} \rangle \geqslant 0$ for all n--vectors $\mathbf{x}$.

II. RELATIONSHIP BETWEEN DOMINANCE CONDITIONS AND POSITIVE SEMIDEFINITENESS

The following theorem plays the central role in this section.

Theorem 2.1. Let $n \geqslant 2$. Then the classes $\mathcal{D}_r$, $\mathcal{D}_c$ and S_o fulfill the conditions:

(i) each of sets $\mathcal{D}_r \setminus S_o$, $\mathcal{D}_c \setminus S_o$ and $S_o \setminus (\mathcal{D}_r \cup \mathcal{D}_c)$ is nonempty;

(ii) $\mathcal{D}_r \cap \mathcal{D}_c \subset S_o$;

(iii) for every $\mathbf{A} \in \mathcal{D}_r(\mathcal{D}_c)$ there exists a nonzero nonnegative diagonal matrix $\mathbf{D}$ such that $\mathbf{DA} \in S_o$ $(\mathbf{AD} \in S_o)$.

P r o o f. It is easy to show the nontrivial matrices of order $n \geqslant 2$ which belong to $\mathcal{D}_r \setminus S_o$, $\mathcal{D}_c \setminus S_o$ and $S_o \setminus (\mathcal{D}_r \cup \mathcal{D}_c)$, respectively. For example

$$\mathbf{A}_1 = \left[\begin{array}{cc|c} 1 & 1 & \mathbf{0} \\ 6 & 8 & \\ \hline & \mathbf{0} & \mathbf{I} \end{array}\right] \in \mathcal{D}_r \setminus S_o; \ \mathbf{A}_1^T \in \mathcal{D}_c \setminus S_o; \ \mathbf{A}_2 = \left[\begin{array}{cc|c} 3 & 1 & \mathbf{0} \\ 5 & 4 & \\ \hline & \mathbf{0} & \mathbf{I} \end{array}\right] \in S_o \setminus (\mathcal{D}_r \cup \mathcal{D}_c)$$

where $\mathbf{I}$ denotes the (n-2) x (n-2) identity matrix.

The inclusion (ii) is an immediate consequence of the following proposition.

Lemma 2.1.1. Let $\mathbf{A}$ be an n x n matrix whose elements a_{ij} satisfy the condition $a_{ii} \geqslant \frac{1}{2} \sum_{j \neq i} (|a_{ij}| + |a_{ji}|)$ $(i \in \mathcal{n})$. Then $\mathbf{A} \in S_o$.

P r o o f. With an arbitrary chosen vector $\mathbf{x} = [x_1, \ldots, x_n]^T$, consider the scalar product

$$\langle \mathbf{x}, \mathbf{A}\mathbf{x} \rangle = \sum_{i,j} a_{ij} x_i x_j = \sum_i a_{ii} x_i^2 + \frac{1}{2} \sum_{\substack{i,j \\ i \neq j}} (a_{ij} + a_{ji}) x_i x_j =$$

$$= \sum_i \left[a_{ii} - \frac{1}{2} \sum_{j=i} (|a_{ij}| + |a_{ji}|) \right] x_i^2 + \frac{1}{2} \sum_i \sum_{j \neq i} \big(|a_{ij}| +$$

$$+ |a_{ji}| \big) x_i^2 + \frac{1}{2} \sum_{\substack{i,j \\ i \neq j}} (a_{ij} + a_{ji}) x_i x_j = \sum_i \Big[a_{ii} -$$

$$- \frac{1}{2} \sum_{j \neq i} (|a_{ij}| + |a_{ji}|) \Big] x_i^2 + \frac{1}{4} \sum_{\substack{i,j \\ i \neq j}} (|a_{ij}| + |a_{ji}|)(x_i^2 + x_j^2) + \frac{1}{2} \sum_{\substack{i,j \\ i \neq j}} (a_{ij} + a_{ji})\, x_i x_j = \sum_i \Big[a_{ii} - \frac{1}{2} \sum_{j \neq i} (|a_{ij}| + |a_{ji}|) \Big] x_i^2 + \frac{1}{4} \sum_{\substack{i,j \\ i \neq j}} (|a_{ij}| + |a_{ji}|)(|x_i| - |x_j|)^2 + \frac{1}{2} \sum_{\substack{i,j \\ i \neq j}} \Big[(|a_{ij}| + |a_{ji}|)\, |x_i x_j| + (a_{ij} + a_{ji})\, x_i x_j \Big].$$

According to the assumption concerning the elements of $\mathbf{A}$ we have $\langle \mathbf{x}, \mathbf{A}\mathbf{x} \rangle \geqslant 0$, i.e., $\mathbf{A}$ belongs to S_o.

To prove the last part we need.

Lemma 2.1.2. For every matrix $\mathbf{A} = [a_{ij}]$ of order $n \geqslant 2$ there exists a nonzero nonnegative vector $\mathbf{d} = [d_1, \ldots, d_n]^T$ such that

$$\mathbf{P}(\mathbf{A})\mathbf{d} \geqslant \mathbf{0} \tag{2.1}$$

P r o o f. Assume that $\mathbf{P}(\mathbf{A})\mathbf{d}^o \geqslant \mathbf{0}$ for the matrix $\mathbf{P}(\mathbf{A})$ and a vector $\mathbf{d}^o = [d_1^o, \ldots, d_n^o]^T$, i.e., $\mathbf{P}(\mathbf{A})\mathbf{d}^o = \mathbf{c}$ with $\mathbf{c} = [c_1, \ldots, c_n]^T \geqslant \mathbf{0}$. Adding all equations of this system we find that $0 = \sum_{i \in \mathcal{N}} c_i$, hence immediately $\mathbf{c} = 0$. Therefore, we can suppose

$$\mathbf{P}(\mathbf{A})\mathbf{d}^o = \mathbf{0} \tag{2.2}$$

instead of the corresponding inequality. The necessary and sufficient condition for (2.2) to have a nonzero solution is that the matrix $\mathbf{P}(\mathbf{A})$ is singular. As it is easy to show that $\det \mathbf{P}(\mathbf{A}) = 0$, one can assume $\mathbf{d}^o$ to be nonzero. When $\mathbf{d}^o$ is nonnegative, the theorem is proved. Thus, suppose the first r coordinates of $\mathbf{d}^o$ are negative, i.e.

$$\begin{aligned} d_i^o &< 0 \quad \text{for } i \leqslant r \\ d_i^o &\geqslant 0 \quad \text{for } 1 > r \end{aligned} \qquad \text{where } 1 \leqslant r \leqslant n$$

(by an appropriate interchange of rows and columns of $\mathbf{P}(\mathbf{A})$, we can do it without loss of generality). If $r = n$, then the vector $-\mathbf{d}^o > \mathbf{0}$ satisfies the desired conditions. Let $r < n$. Adding the first r equations of (2.2), we have

$$\sum_{i=1}^{r} \sum_{j=r+1}^{h} d_i^o \, |a_{ij}| - \sum_{i=r+1}^{n} \sum_{j=1}^{r} d_i^o \, |a_{ij}| = 0,$$

whence $a_{ij} = 0$ for $1 \leqslant i \leqslant r$ and $r+1 \leqslant j \leqslant n$, but then the vector $\tilde{\mathbf{d}}^o = [\tilde{d}_1^o, \ldots, \tilde{d}_n^o]^T$ defined as follows

$$\tilde{d}_i^o = -d_i^o \quad \text{for } i \leqslant r,$$
$$\tilde{d}_i^o = 0 \quad \text{for } i > r$$

is the desired one. In this way, we proved that the proposition is always true (see the more general *theorem 5.9* in [2]).

Now the *theorem 2.1* (iii) is easy to prove. Namely, we will show that if $\mathbf{d} = [d_1, \ldots, d_n]^T$ satisfies the conditions of *lemma 2.1.2*, then one can assume $\mathbf{D} = \operatorname{diag}[d_1, \ldots, d_n]$. Consider the scalar product

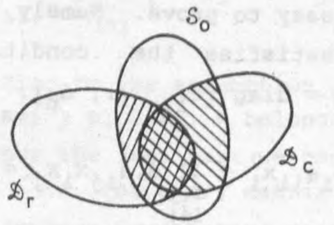
$$\begin{aligned}
\langle \mathbf{x}, \mathbf{DAx} \rangle &= \sum_{i,j} d_i a_{ij} x_i x_j = \sum_i d_i a_{ii} x_i^2 + \sum_{\substack{i,j\\ i\neq j}} d_i a_{ij} x_i x_j = \\
&= \sum_i d_i (a_{ii} - \sum_{j\neq i} |a_{ij}|) x_i^2 + \sum_i \sum_{j\neq i} d_i\, |a_{ij}|\, x_i^2 + \\
&\quad + \sum_{\substack{i,j\\ i\neq j}} d_i a_{ij} x_i x_j = \sum_i d_i\, (a_{ii} - \sum_{j\neq i} |a_{ij}|)\, x_i^2 + \\
&\quad + \sum_{\substack{i,j\\ i\neq j}} d_i\, |a_{ij}|\, x_i^2 + \sum_{\substack{i,j\\ i\neq j}} d_i\, (|a_{ij} x_i x_j| + a_{ij} x_i x_j) + \\
&\quad - \sum_{\substack{i,j\\ i\neq j}} d_i\, |a_{ij} x_i x_j| = \sum_i d_i\, (a_{ii} - \sum_{j\neq i} |a_{ij}|)\, x_i^2 + \\
&\quad + \sum_{\substack{i,j\\ i\neq j}} d_i\, (|a_{ij} x_i x_j| + a_{ij} x_i x_j) + \sum_{\substack{i,j\\ i\neq j}} d_i\, |a_{ij}|\, x_i^2 - \\
&\quad - \frac{1}{2} \sum_{\substack{i,j\\ i\neq j}} d_i\, |a_{ij}|\, [x_i^2 + x_j^2 - (|x_i| - |x_j|)^2] = \\
&= \sum_i d_i\, (a_{ii} - \sum_{j\neq i} |a_{ij}|)\, x_i^2 + \sum_{\substack{i,j\\ i\neq j}} d_i\, (|a_{ij} x_i x_j| + \\
&\quad + a_{ij} x_i x_j) + \frac{1}{2} \sum_{\substack{i,j\\ i\neq j}} d_i\, |a_{ij}| (|x_i| - |x_j|)^2 + \\
&\quad + \frac{1}{2} \sum_i x_i^2 (d_i \sum_{j\neq i} |a_{ij}| - \sum_{j\neq i} d_j\, |a_{ji}|).
\end{aligned}$$

Since (2.2), the last term of this expression is équal to zero so that finally $\langle \mathbf{x}, \mathbf{DAx} \rangle \geqslant 0$.

Now suppose $\mathbf{A} \in \mathcal{D}_c$, then obviously $\mathbf{A}^T \in \mathcal{D}_r$ and $\mathbf{D}^T\mathbf{A}^T \in S_o$ for an appropriate choice of $\mathbf{D}$. But then also $\mathbf{AD} = (\mathbf{D}^T\mathbf{A}^T)^T \in S_o$.

This completes the proof of *theorem 2.1.*

The most interesting and important is the case (iii), also from the point of view of practical applications. It is easily seen that the multiplication by the diagonal matrix $\mathbf{D}$ establishes an elementary parallelism inside the class of the dominance matrices, namely, transformes the sets $\mathcal{D}_r$ and $\mathcal{D}_c$ into $\mathcal{D}_r \cap S_o$ and $\mathcal{D}_c \cap S_o$, respectively.



$$\mathcal{D}_r \xrightarrow[\text{on the left}]{\text{multip. by } \mathbf{D}} \mathcal{D}_r \cap S_o$$

$$\mathcal{D}_c \xrightarrow[\text{on the right}]{\text{multip. by } \mathbf{D}} \mathcal{D}_c \cap S_o$$

Fig. 1. A symbolic configuration of the sets $\mathcal{D}_r$, $\mathcal{D}_c$ and S_o

The solution of (2.1) is only one of many possibilities to get a matrix $\mathbf{D}$ such that $\mathbf{DA} \in S_o$. Unfortunately, I do not know another way to determine, effectively for given $\mathbf{A}$, a matrix $\mathbf{D}$ which could have the mentioned property. Indeed, the necessary and sufficient conditions for $\mathbf{D}$ to satisfy the condition $\mathbf{DA} \in S_o$ can be specified (see *theorem 2.3)* but their practical usefulness is small.

Having established a practical method for getting the vector $\mathbf{d} \geqslant \mathbf{0}$ (and simultaneously the matrix $\mathbf{D}$) a natural and important question to arise is: How many components of $\mathbf{d}$ can be equal to zero and when?

Theorem 2.2. Let $\mathbf{A} \in \mathcal{D}_r$ be of order $n \geqslant 2$. There exists an n-vector $\mathbf{d} > \mathbf{0}$ satisfying (2.2) if and only if the matrix $\mathbf{A}$ can, via an appropriate interchange of rows and a corresponding interchange of columns, be brought to the irreducible-block-diagonal form.

P r o o f. (if part) Suppose that it is possible to bring the matrix $\mathbf{A}$ to the required representation. Then, which is easy to verify, the matrix $\mathbf{P(A)}$ is also block-diagonal. It may happen that one of its blocks is of order 1, then, of course, it must contain 0 because the corresponding part of $\mathbf{A}$ possesses a diagonal structure. Let $\mathbf{B} = [b_{ij}]$ denote one of such irreducible blocks of $\mathbf{A}$ and $\mathbf{P}(\mathbf{B})$ the corresponding block of $\mathbf{P}(\mathbf{A})$. By what has already been proved, there exists a nonzero nonnegative b--vector $\mathbf{f} = [f_1, \ldots, f_b]^T$ such that $\mathbf{P}(\mathbf{B})\mathbf{f} = 0$. When $b = 1$, then $\mathbf{P}(\mathbf{B}) = [0]$ and we can take $\mathbf{f} > 0$. Let $\mathbf{B}$ be of order $b \geqslant 2$. Without loss of generality, one can suppose $f_i = 0$ for $i \leqslant r$ and $f_i > 0$ for $i > r$, where $1 \leqslant r < b$. Adding the last b-r equations of the system $\mathbf{P}(\mathbf{B})\mathbf{f} = \mathbf{0}$ we have $\sum_{i=r+1}^{b} \sum_{j=1}^{r} f_i |b_{ij}| = 0$ whence $b_{ij} = 0$ for $i > r$, $j \leqslant r$, i.e., the matrix $\mathbf{B}$ would be reducible. But that is impossible. By choosing an appropriate vector $\mathbf{f} > \mathbf{0}$ for each of blocks one constructes the whole desired vector $\mathbf{d}$.

(only if part) We will prove this part by induction on the order of $\mathbf{A}$. If $n = 2$, the equation (2.2) is of the form $|a_{12}|d_1 - |a_{21}|d_2 = 0$ hence a_{12} and a_{21} are both either equal or not equal to zero, i.e., the theorem holds. Suppose our proposition is true for all natural numbers $\leqslant n-1$ and consider a matrix $\mathbf{A}$ having order $n \geqslant 3$. If $\mathbf{A}$ is irreducible, then the proposition holds for chosen n. Thus, assume $\mathbf{A}$ to be reducible. Without loss of generality, let us take $a_{ij} = 0$ for $i > r$, $j \leqslant r$, where $1 \leqslant r < n$. Adding the first r rows of equation (2.2), we obtain $\sum_{i=1}^{r} \sum_{j=r+1}^{n} d_i |a_{ij}| = 0$, hence $a_{ij} = 0$ also for $i \leqslant r$, $j > r$, i.e., the matrix $\mathbf{A}$ is of the block-diagonal form with the first of blocks of order $r < n$ and the second one of order $n - r < n$. By the induction hypothesis, each of these blocks can be brought to the required representation and, consequently, the same is true for the whole matrix $\mathbf{A}$ and the theorem holds for n.

On the base of the condition (iii) it is possible to construct a new class of matrices which is a little larger than $\mathcal{S}_r$ and S_o. We denote it by $\mathcal{S}_o$ and define by any one of the following equivalent properties.

Theorem 2.3. The following three properties of an n x n matrix $\mathbf{A}$ are equivalent:

(i) for each of the principal submatrices $\mathbf{A}(\mathcal{M})$ of $\mathbf{A}$ there exists a nonzero nonnegative diagonal matrix $\mathbf{D}_{\mathcal{M}}$ such that $\mathbf{D}_{\mathcal{M}}\mathbf{A}(\mathcal{M}) \in S_o$;

(ii) for each of the principal submatrices $\mathbf{A}(\mathcal{M})$ of $\mathbf{A}$ there exists a nonzero nonnegative diagonal matrix $\mathbf{D}_{\mathcal{M}}$ such that $\mathbf{A}^T(\mathcal{M})\mathbf{D}_{\mathcal{M}} + \mathbf{D}_{\mathcal{M}}\mathbf{A}(\mathcal{M}) \in \mathcal{P}_o$;

(iii) all blocks on the main diagonal in one of the irreduible-block-triangular forms of A may be represented as the product DS of a positive diagonal matrix D and a positive semidefinite matrix S.

P r o o f. To establish equivalence of (i) and (ii) we formulate the following useful proposition (see [4], problem 1210b).

Lemma 2.3.1. For a quadratic form f to be nonnegative it is necessary and sufficient that all principal minors of its matrix be nonnegative.

Now, suppose $A(\mathcal{m})$ is one of the principal submatrices of A and $D_{\mathcal{m}}$ a nonzero nonnegative diagonal matrix and consider the scalar product $\langle x, D_{\mathcal{m}}A(\mathcal{m})x\rangle = x^T D_{\mathcal{m}}A(\mathcal{m})x$. This expression is a quadratic form having the matrix $\frac{1}{2}((D_{\mathcal{m}}A(\mathcal{m}))^T + D_{\mathcal{m}}A(\mathcal{m}))$ or equivalently $\frac{1}{2}(A^T(\mathcal{m})D_{\mathcal{m}} + D_{\mathcal{m}}A(\mathcal{m}))$. Making use of the above lemma and of the definition of the class Φ_o we conclude that $\langle x, D_{\mathcal{m}}A(\mathcal{m})x\rangle \geqslant 0$ for every x (i.e., also $D_{\mathcal{m}}A(\mathcal{m}) \in S_o$) if and only if $A^T(\mathcal{m})\, D_{\mathcal{m}} + D_{\mathcal{m}}A(\mathcal{m}) \in \Phi_o$.

Lemma 2.3.2. Every reducible matrix A of order $n \geqslant 2$ can, via an appropriate interchange of rows and a corresponding interchange of columns, be brought to the irreducible-block-triangular form.

P r o o f. First, assume that we prove the theorem on an additional assumption that the zero blocks in the block-triangular form of A lie above the principal diagonal. Obviously, our theorem holds for the second-order matrix A. Suppose the truth for all matrices of order $\leqslant n-1$ and focus attention on an arbitrarily chosen reducible matrix A of order $n \geqslant 3$. By definition, there exists a nonvoid set $\mathcal{p} \subset \mathcal{n}$ and $\mathcal{p} \neq \mathcal{n}$ such that $a_{ij} = 0$ for $i \in \mathcal{p}$ and $j \in \mathcal{n}\setminus\mathcal{p}$. Let us carry all rows whose elements have the first of indices belonging to $\mathcal{p}$ to the first positions of A successively interchanging each of them with all abovelying rows and then perform the corresponding symmetrical interchange of columns. Finally, we obtain a zero matrix in the upper right-hand corner of A, i.e., A is of the following block-triangular form

$$\left[\begin{array}{c|c} A_1 & 0 \\ \hline B & A_2 \end{array}\right]$$

where each of matrices A_1 and A_2 is of order $< n$ ($\mathcal{p}$ is nonvoid) and can by assumption, be brought to the required form (the place of the zero part of A remains unchanged). Hence the proposition holds for n and the proof is complete.

We can suppose there are exactly r diagonal blocks $1 \leqslant r \leqslant n$ denoted by $A_1, A_2, \ldots, A_r$, respectively:

$$A = \left[\begin{array}{c|c} A_1 & 0 \\ \hline & \ddots \\ \hline & A_r \end{array}\right].$$

(if (iii) then (i)) Suppose the $n \times n$ matrix A has the mentioned property and let $A_i = D_i S_i (i = 1, \ldots, r;\ 1 \leqslant r \leqslant n)$ denote its diagonal blocks with their corresponding decomposition. Let $D = \operatorname{diag}[D_1, \ldots, D_r] = \operatorname{diag}[d_1, \ldots, d_n]$. We will show that then (i) is true. It is immediate for $r = 1$ because then $A = D_1 S_1$ and $D_{\mathcal{M}} = D_1^{-1}(\mathcal{M})$ is the desired matrix. Thus, take $r \geqslant 2$ and choose an arbitrary submatrix $A(\mathcal{M})$ of A with $\mathcal{M} = \{k_1, \ldots, k_s\}$, $1 \leqslant k_1 < \ldots < k_s \leqslant n$. Let the element $a_{k_s k_s}$ belong to A_j and suppose $k_u, k_{u+1}, \ldots, k_s$ are these and only these indices g from $\mathcal{M}$ for which the element a_{gg} belongs to A_j. $A(\mathcal{M})$ is also of the block-triangular form with the submatrix $A(\{k_u, \ldots, k_s\})$ located on the main diagonal in the lower right-hand corner of $A(\mathcal{M})$ (this submatrix can be reducible). Then $D_{\mathcal{M}} = \operatorname{diag}[0, \ldots, 0, d_{k_u}, \ldots, d_{k_s}]$ having zeros on the first $u - 1$ positions of the main diagonal is such a matrix that $D_{\mathcal{M}} A(\mathcal{M}) \in S_o$.

(if (i) then (iii)) Suppose A is brought to an arbitrary of its irreducible-block-triangular forms and let A fulfill (i). For A of the first order, (iii) holds. Thus, assume A to be of order $\geqslant 2$. Choose an arbitrary diagonal block $B = [b_{ij}]$ of order b. For nontriviality, let $b \geqslant 2$. B is a principal submatrix of A. Therefore (i) holds for B, i.e., there exists a nonzero nonnegative diagonal matrix $D = \operatorname{diag}[d_1, \ldots, d_b]$ such that $C = DB \in S_o$. Suppose, without loss of generality, that the first $1 \leqslant r \leqslant b-1$ diagonal elements of D are zero. But then in $C = [c_{ij}]$ there are only zeros in the first r rows. We will show that only zeros must lie in the first r columns as well. The matrix $C \in S_o$, i.e., for every vector x the scalar product $\langle x, Cx\rangle$ is nonnegative. Let $b_{ij} \neq 0$ (i.e., also $c_{ij} \neq 0$) for a pair (i,j) chosen so that $i > r$ and $j \leqslant r$ and take such a vector x whose the jth component is equal to $(2c_{ii} + d_i)/c_{ij}$, the ith one equal to -1 and all other elements are zero. Then

$$\langle \mathbf{x}, \mathbf{C}\mathbf{x}\rangle = (-1)\left(\frac{2c_{ii}+d_i}{c_{ij}}\cdot c_{ij} + (-1)\, c_{ii}\right) = -(c_{ii}+d_i) < 0$$

because $d_i > 0$ for $i > r$ and $c_{ii} \geqslant 0$ as a diagonal element of a positive semidefinite matrix $\mathbf{C}$. Hence, it must be $b_{ij} = 0$ for all $i > r$ and $j \leqslant r$, which is a contradiction because $\mathbf{B}$ is irreducible. Therefore, $\mathbf{D}$ must be positive and the required representation $\mathbf{B} = \mathbf{D}^{-1}\mathbf{C}$ is possible.

Note here, it is very difficult to verify if the given matrix $\mathbf{A}$ with the diagonal blocks of order greater than 3 belongs to $\mathcal{S}_o$.

III. CLASS $\mathcal{P}$ AND DOMINANCE CONDITIONS

The theorem stated below establishes the relationship between the dominance conditions and properties defining the class $\mathcal{P}$.

Theorem 3.1. An irreducible weakly row-sum (column-sum) dominant matrix $\mathbf{A}$ of order $n \geqslant 2$ does not belong to the class $\mathcal{P}$ if and only if it satisfies the condition

$$\mathbf{P}(\mathbf{A}) = \mathbf{D}\mathbf{A}^T\mathbf{D} \qquad (\mathbf{P}(\mathbf{A}^T) = \mathbf{D}\mathbf{A}\mathbf{D}) \tag{3.1}$$

where $\mathbf{D}$ is a diagonal matrix having each of diagonal elements equal to either 1 or -1. If this condition is satisfied, then all principal submatrices of $\mathbf{A}$ of order less than n belong to $\mathcal{P}$.

P r o o f. (if part) Trivial because of the singularity of $\mathbf{A}$ resulting from (3.1).

(only if part) This part of the proof is based on the following lemma.

Lemma 3.1.1. An irreducible weakly row-sum dominant matrix $\mathbf{B} = [b_{ij}]$ of order $n \geqslant 2$ is singular if and only if

$$\mathbf{P}(\mathbf{B}) = \mathbf{D}\mathbf{B}^T\mathbf{D}$$

for a diagonal matrix $\mathbf{D}$ with diagonal elements equal to 1 and -1.

P r o o f. The "if" part is again obvious that is why we will prove that the conditions imposed on $\mathbf{B}$ imply the required representation. Suppose that an irreducible weakly row-sum dominant matrix $\mathbf{B} = [b_{ij}]$ of order n greater or equal to 2 is singular. Then there exists a nonzero vector $\mathbf{y} = [y_1, \ldots, y_n]^T$ such that $\mathbf{B}\mathbf{y} = \mathbf{0}$. Without loss of generality, we can assume $|y_1| = \max_{i \in \underline{n}} |y_i|$ and let $|y_i| = |y_1|$ for $i \leqslant r$, where $1 \leqslant r \leqslant n$. Of course $|y_i| > 0$. First, let $r < n$, then for each $i \leqslant r$:

$$0 = y_i(b_{ii}y_i + \sum_{j\neq i} b_{ij}y_j) = (b_{ii} - \sum_{j\neq i} |b_{ij}|)\, y_i^2 +$$

$$+ \sum_{j\neq i} |b_{ij}|\, |y_i|\, (|y_i| + y_j \operatorname{sgn}(b_{ij}y_i)).$$

That is possible only if $b_{ii} = \sum_{j\neq i} |b_{ij}|$ and $|b_{ij}|\,(|y_i| + y_j \operatorname{sgn}(b_{ij}y_i)) = 0$ for $j\neq i$. Let us now consider the indices $j > r$, then $|y_j| < |y_i|$ for an arbitrarily chosen index $i \leqslant r$, i.e., $|y_i| + y_j \operatorname{sgn}(b_{ij}y_i)$ cannot be equal to zero and as an immediate implication $b_{ij} = 0$ for each $i \leqslant r$ and $j > r$, which is a contradiction because $\mathbb{B}$ is irreducible. Hence $r = n$, i.e., all the components of $\mathbf{y}$ are nonzero and of the same absolute value, for example, let the components of indices belonging to $\emptyset \neq \mathcal{M} \subset \mathcal{N}$ be equal to 1 and all other equal to -1. Then for an arbitrary index $i \in \mathcal{N}$ we have $\sum_{j\in \mathcal{M}} b_{ij} - \sum_{j\in \mathcal{N}\setminus\mathcal{M}} b_{ij} = 0$, hence

$$0 = (b_{ii} - \sum_{j\neq i} |b_{ij}|) + \sum_{j\in \mathcal{M}\setminus\{i\}} (b_{ij} + |b_{ij}|) + $$

$$+ \sum_{j\in \mathcal{N}\setminus\mathcal{M}} (|b_{ij}| - b_{ij}) \quad \text{when } i \in \mathcal{M}$$

and

$$0 = (\sum_{j\neq i} |b_{ij}| - b_{ii}) + \sum_{j\in \mathcal{M}} (b_{ij} - |b_{ij}|) + \sum_{j\in(\mathcal{N}\setminus\mathcal{M})\setminus\{i\}} (-b_{ij} -$$

$$- |b_{ij}|) \quad \text{when } i \in \mathcal{N}\setminus\mathcal{M}$$

Consequently, $b_{ii} = \sum_{j\neq i} |b_{ij}|$ for all i taken from $\mathcal{N}$ and either

$$b_{ij} = \begin{cases} -|b_{ij}| & \text{for } j \in \mathcal{M}\setminus\{i\} \\ |b_{ij}| & \text{for } j \in \mathcal{N}\setminus\mathcal{M} \end{cases} \quad \text{when } i \in \mathcal{M} \quad \text{or } b_{ij} =$$

$$= \begin{cases} |b_{ij}| & \text{for } j \in \mathcal{M} \\ -|b_{ij}| & \text{for } j \in (\mathcal{N}\setminus\mathcal{M})\setminus\{i\} \end{cases}$$

when $i \in \mathcal{N}\setminus\mathcal{M}$. Consider the diagonal matrix $\mathbb{D} = \operatorname{diag}[d_1, \ldots, d_n]$ of elements defined as follows: $d_i = 1$ for $i \in \mathcal{M}$ and $d_i = -1$ for $i \in \mathcal{N}\setminus\mathcal{M}$. It is easily seen that the matrix $\mathbb{D}$ defined in such a way satisfies the formula $\mathbb{P}(\mathbb{B}) = \mathbb{D}\mathbb{B}^T\mathbb{D}$.

The above proposition enables us to prove the main result. Since $\mathbb{A}$ taken from $\mathcal{D}_r$ does not belong to $\mathcal{P}$ but, of course, still belongs to $\mathcal{P}_o$, there exists a principal submatrix $\mathbb{A}(\mathcal{M})$ of $\mathbb{A}$

which is singular. We will show that $\mathfrak{M} = \mathfrak{N}$. By an appropriate interchange of rows and a corresponding interchange of columns we can assume that $\mathfrak{M} = \{1, \ldots, m\}$ and that $A(\mathfrak{M})$ has already been brought to the irreducible-block-triangular form (see *theorem 2.3)* having $A_1(\mathfrak{M}), \ldots, A_t(\mathfrak{M})$ as its diagonal blocks. But in this case, $\det A(\mathfrak{M}) = \prod_{k=1}^{t} \det A_k(\mathfrak{M})$ and there exists an index $j \leqslant t$ such that $A_j(\mathfrak{M})$ is singular. Obviously, $A_j(\mathfrak{M})$ is also weakly row-sum dominant and irreducible so that, by lemma, $P(A_j(\mathfrak{M})) = D_{\mathfrak{M}} A_j^T(\mathfrak{M}) D_{\mathfrak{M}}$. If $m < n$ this formula implies the conditions $a_{ij} = 0$ for all $i \leqslant m$ and $j > m$ which, however, are impossible. Thus $m = n$ and the decomposition (3.1) follows from our lemma. Clearly, each of principal submatrices $A(\mathfrak{M}) \neq A$, because of its nonsingularity, must belong to the class $\mathcal{P}$.

Now let $A \in \mathcal{D}_c$ not belong to $\mathcal{P}$, then $A^T \in \mathcal{D}_r$ and $\notin \mathcal{P}$ and the first of formulas (3.1) implies the second one.

This theorem shows the case in which a weakly dominant matrix does not belong to the class $\mathcal{P}$ is rather very specific and, by given formulas, can be easily identified.

IV. NEW PROPERTY OF $\mathcal{P}_o$-CLASS

To prove the main result of this section, an additional terminology is needed. Let, for n, $\mathcal{F}^n$ denote the collection of mappings of the n-dimensional Euclidean space E^n onto itself defined as follows: $F(\cdot) \in \mathcal{F}^n$ if and only if there exist, for $i \in \mathfrak{N}$, strictly monotone increasing functions $f_i(\cdot)$ mapping E^1 onto E^1 such that for each $x = [x_1, \ldots, x_n]^T \in E^n$, $F(x) = [f_1(x_1), \ldots, f_n(x_n)]^T$. This class of functions plays a central role in the analysis of nonlinear electrical networks (see only the first works [7] and [5]).

The mentioned new property of the class $\mathcal{P}_o$ can be formulated as follows.

Theorem 4.1. An n x n matrix A belongs to $\mathcal{P}_o$ if and only if for every $L(\cdot), H(\cdot) \in \mathcal{F}^n$ the set

$$\mathcal{W} = \{x \in E^n : -L(x) \leqslant Ax \leqslant -H(x)\}$$

is bounded.

P r o o f. (if part) Suppose $\mathbf{A}$ does not belong to $\mathcal{P}_o$, then there exists a diagonal positive matrix $\mathbf{D}$ such that $\det(\mathbf{A} + \mathbf{D}) = 0$ (see property (v) in part I and the set

$$\mathcal{W} = \{\mathbf{x} \in \mathbf{E}^n : -\mathbf{Dx} \leqslant \mathbf{Ax} \leqslant -\mathbf{Dx}\}$$

is unbounded although $\mathbf{L}(\mathbf{x}) = \mathbf{H}(\mathbf{x}) = \mathbf{Dx} \in \mathcal{F}^n$. Hence $\mathbf{A} \in \mathcal{P}_o$.

(only if part) We apply induction on n. The proof is obvious for $n = 1$. Suppose the truth for $n-1 \geqslant 1$ and consider the set $\mathcal{W}$ where $\mathbf{L}(\mathbf{x}) = [l_1(x_1), \ldots, l_n(x_n)]^T$, $\mathbf{H}(\mathbf{x}) = [h_1(x_1), \ldots, h_n(x_n)]^T$, $n \geqslant 2$. Choose arbitrarily $\mathbf{x} \in \mathcal{W}$. Since $\mathbf{A} \in \mathcal{P}_o$, there exists an index $k \in \mathcal{N}$ such that $x_k(\mathbf{Ax})_k \geqslant 0$ (property (ii), § 1), so that if $x_k \geqslant 0$, then $x_k h_k(x_k) \leqslant 0$ and automatically $\min(0, h_k^{-1}(0)) \leqslant x_k \leqslant \max(0, h_k^{-1}(0))$; if $x_k \leqslant 0$, then $x_k l_k(x_k) \leqslant 0$ and as above $\min(0, l_k^{-1}(0)) \leqslant x_k \leqslant \max(0, l_k^{-1}(0))$. Let $\alpha_k = \min(0, l_k^{-1}(0), h_k^{-1}(0))$ and $\beta_k = \max(0, l_k^{-1}(0), h_k^{-1}(0))$, then always $\alpha_k \leqslant x_k \leqslant \beta_k$. The kth component of x is thus bounded. Now consider the set

$$\bar{\mathcal{W}} = \{\bar{\mathbf{x}} \in \mathbf{E}^{n-1} : -\bar{\mathbf{L}}(\bar{\mathbf{x}}) \leqslant \mathbf{A}(\mathcal{N}\setminus\{k\})\, \bar{\mathbf{x}} \leqslant -\bar{\mathbf{H}}(\bar{\mathbf{x}})\}$$

in which $\bar{\mathbf{x}} = [x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n]^T$, $\bar{\mathbf{L}}(\cdot) = [\bar{l}_1(\cdot), \ldots, \bar{l}_{k-1}(\cdot), \bar{l}_{k+1}(\cdot), \ldots, \bar{l}_n(\cdot)]^T$ with $\bar{l}_j(x_j) = -l_j(x_j) - |a_{jk}| \max(\alpha_k \operatorname{sgn} a_{jk}, \beta_k \operatorname{sgn} a_{jk})$ and $\mathbf{H}(\cdot) = [\bar{h}_1(\cdot), \ldots, \bar{h}_{k-1}(\cdot), \bar{h}_{k+1}(\cdot), \ldots, \bar{h}_n(\cdot)]^T$ with $\bar{h}_j(x_j) = -h_j(x_j) - |a_{jk}| \min(\alpha_k \operatorname{sgn} a_{jk}, \beta_k \operatorname{sgn} a_{jk})$. Of course, $\bar{\mathbf{L}}(\cdot), \bar{\mathbf{H}}(\cdot) \in \mathcal{F}^{n-1}$ and $\mathbf{A}(\mathcal{N}\setminus\{k\}) \in \mathcal{P}_o$ hence, by the induction hypothesis, there exist α_j, β_j, $j \in \mathcal{N}\setminus\{k\}$ such that $\alpha_j \leqslant x_j \leqslant \beta_j$ and, finally, $\mathcal{W}$ is enclosed by the parallelepiped $\bigtimes_{j \in \mathcal{N}} \langle \alpha_j ; \beta_j \rangle$ and therefore bounded.

The presented material has some practical applications. For the details, the reader is referred to [1].

ACKNOWLEDGMENT

This paper is a little part of the author's work supported by Gesellschaft für Strahlen und Umweltforschung mbH Munich, West Germany during his practical training in August, September and October 1986.

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PEWNE NOWE TWIERDZENIA O MACIERZACH WYSTĘPUJĄCYCH W ANALIZIE OBWODÓW

Praca zawiera pewne nowe rezultaty ujawniające związki zachodzące pomiędzy niektórymi klasami macierzy występującymi w analizie nieliniowych obwodów dc.