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SOME REMARKS ABOUT HOMEOMORPHISMS PRESERVING (f)-POROSITY

This paper gives a necessary and sufficient condition for homeomorphism h to preserve (f)-porosity for every $f \in G$.

The notion of a point of (f)-porosity was defined by Zajíček [1]. We shall give his definition in the form which is suitable for our purposes.

Let G denotes the set of all real functions, which are increasing and continuous on some interval $(0, A)$.

Definition 1. Let $M \subset R$, $x \in R$. We say that x is a point of (f)-porosity of M if and only if

$$\limsup_{r \to 0^+} \frac{f(\gamma(x, r, M))}{r} > 0$$

where $\gamma(x, r, M)$ is the least upper bound of the set $\{a > 0;$ for some $z \in R$, $K(z, a) \subset K(x, r)$ and $K(z, a) \cap M = \emptyset\}$. $K(x, r)$ denotes the open ball with the centre $x \in R$ and the radius $r > 0$.

Definition 2. We shall say that a homeomorphism $h : R \xrightarrow[\text{onto}]{} R$ preserves points of (f)-porosity if and only if for every set $M \subset R$, and for every $x_0 \in R$, which is a point of (f)-porosity of M, a point $h(x_0)$ is a point of (f)-porosity of $h(M)$.

Remark 1. Let $a = \lim_{x \to 0^+} f(x)$, where $f \in G$. We shall suppose that $a = 0$. If $a > 0$ then $f(x) > a$ for every $x \in (0, A)$ and every homeomorphism preserves points of (f)-porosity.

Theorem 1. A point x_o is a point of (f)-porosity of a set M, where $f \in G$ if and only if there exists a sequence $\{(a_k, b_k)\}_{k\in N}$ of open intervals which are mutually disjoint and disjoint with M such that $b_k \searrow x_o$ and $\lim\limits_{k\to\infty} \frac{f\left(\frac{b_k - a_k}{2}\right)}{b_k - x_o} > 0$ or $a_k \nearrow x_o$ and $\lim\limits_{k\to\infty} \frac{f\left(\frac{b_k - a_k}{2}\right)}{x_o - a_k} > 0$.

P r o o f. Necessity.

From the assumption it follows that there exists a sequence $\{r_n\}_{n\in N}$ such that $r_n \to 0_+$ and $\lim\limits_{n\to\infty} \frac{f\,(\gamma(x_o, r_n, M))}{r_n} > 0$.

Let $\gamma_n = \gamma(x_o, r_n, M)$.

We shall suppose that intervals (a_k, b_k) lie to the right from x_o. Let $\varepsilon_n = \frac{r_n}{n}$, then there exists δ_n such that $0 < \delta_n < \gamma_n$ and $f(\gamma_n) - \varepsilon_n < f\,(\gamma_n - \delta_n)$.

Using the mathematical induction we choose a sequence $\{n_k\}$ and a sequence $\{(a_k, b_k)\}_{k\in N}$ of intervals such that

$$x_o < x_o + r_{n_{k+1}} < a_k < b_k < x_o + r_{n_k}$$

$$(a_k, b_k) \cap M = \emptyset \text{ and } \gamma_{n_k} - \delta_{n_k} \leqslant \frac{b_k - a_k}{2}$$

Hence

$$\frac{f(\gamma_{n_k}) - \frac{r_{n_k}}{n_k}}{r_{n_k}} \leqslant \frac{f\left(\frac{b_k - a_k}{2}\right)}{r_{n_k}} < \frac{f\left(\frac{b_k - a_k}{2}\right)}{b_k - x_o}$$

If it is impossible to construct such a sequence, then it is easy to see that we are able to construct a sequence with required properties lying to the left from x_o.

S u f f i c i e n c y. Let a sequence $\{(a_k, b_k)\}_{k\in N}$ be such that $\lim\limits_{k\to\infty} \frac{f\left(\frac{b_k - a_k}{2}\right)}{b_k - x_o} = \alpha > 0$, and $b_k \searrow x_o$. Put $r_k = b_k - x_o$, then $\gamma_k \geqslant$ $\geqslant \frac{b_k - a_k}{2}$ and $f(\gamma_k) \geqslant f\left(\frac{b_k - a_k}{2}\right)$ because $f \in G$.

From the assumption we obtain

$$\limsup_{k\to\infty} \frac{f(r_k)}{r_k} \geq \alpha$$

From now we shall suppose that all homeomorphisms under considerations are increasing.

Theorem 2. If a homeomorphism h has at every point only finite derived numbers and inverse homeomorphism h^{-1} fulfills the Lipschitz condition with constant one then h preserves points of (f)-porosity for every f belonging to class G.

P r o o f. Let $M \subset R$, and $x_0 \in R$ be a point of (f)-porosity of a set M. In virtue of th 1 there exists a sequence $\{(a_n, b_n)\}_{n\in N}$ of open intervals such that $(a_i, b_i) \cap (a_j, b_j) = \emptyset$ for $i \neq j$, $(a_n, b_n) \cap M = \emptyset$, $b_n \searrow x_0$ and $\lim_{n\to\infty} \frac{f\left(\frac{b_n - a_n}{2}\right)}{b_n - x_0} = \alpha > 0$, or analogous sequence of intervals convergent to a point x_0 from the left side. From the assumption it follows that there exists $d > 0$, such that

$$\frac{h(b_n) - h(x_0)}{b_n - x_0} \leq d \quad \text{for } n \in N$$

Hence

$$\frac{f\left(\frac{h(b_n) - h(a_n)}{2}\right)}{h(b_n) - h(x_0)} \geq \frac{f\left(\frac{b_n - a_n}{2}\right)}{d(b_n - x_0)}$$

and

$$(h(a_n), h(b_n)) \cap h(M) = \emptyset$$

From the last inequality and theorem 1 it follows that $h(x_0)$ is a point (f)-porosity of the set h(M) for every function $f \in G$.

Theorem 3. If h is a homeomorphism preserving points of (f)-porosity for every function $f \in G$ then h^{-1} fulfills the Lipschitz condition with constant one.

P r o o f. We shall suppose that h^{-1} does not fulfill the Lipschitz condition with constant one. Then there exist points x_0 and y_0 such that $h(y_0) - h(x_0) < y_0 - x_0$ (*) and we shall say that the interval $[x_o, y_o]$ fulfills condition (*). The midpoint of $[x_o, y_o]$ divides $[x_o, y_o]$ onto two closed subintervals at least one of which fulfills condition (*). Consider two cases. If both subintervals fulfill (*), then we stop. If only one subinterval fulfills (*), then we divide this interval onto two parts in analogous way. Continuing in this way we obtain at some step neighbouring intervals $[c_1, a_1]$, $[a_1, b_1]$ of equal length which both fulfill condition (*). We suppose that above assumption is false. Let $[x_i, y_i]$ denotes this subinterval of $[x_{i-1}, y_{i-1}]$ which fulfills the condition (*), $[z_i, t_i]$ denotes remaining subinterval of $[x_i, y_i]$ for $i = 1, 2, \ldots$

Then

$$[x_o, y_o] = \bigcup_{i=1}^{\infty}[z_i, t_i] \cup \bigcap_{i=1}^{\infty}[x_i, y_i]$$

where $(z_i, t_i) \cap (z_j, t_j) = \emptyset$ $i \neq j$ and $\bigcap_{i=1}^{\infty}[x_i, y_i]$ is a singleton.

Hence

$$h(y_o) - h(x_o) = \sum_{i=1}^{\infty} (h(t_i) - h(z_i)) \geqslant$$

$$\geqslant \sum_{i=1}^{\infty} (t_i - z_i) = y_o - x_o$$

We obtained the contradiction.

We have already the interval $[a_1, b_1]$. The interval $[a_2, b_2]$ we obtain in analogous way if instead the interval $[x_o, y_o]$ we consider the interval $[c_1, a_1]$.

Continuing in this way we obtain the sequence of intervals $\{[a_n, b_n]\}_{n\in N}$ such that $(a_i, b_i) \cap (a_j, b_j) = \emptyset$ and $b_n \searrow \hat{x}$, where $\hat{x} \in [x_o, y_o]$.

Put $\alpha_n = \dfrac{b_n - a_n}{2}$ and $\bar{h}(\alpha_n) = \dfrac{h(b_n) - h(a_n)}{2}$ of course $\bar{h}(\alpha_n) < < \alpha_n$ for every $n \in N$.

We can suppose that $\hat{x} = 0$ and $h(0) = 0$, because the translation of the graph has no influence for preserving (f)-porosity. Using the mathematical induction we choose a sequence $\{k_n\}_{n\in N}$ of natural number.

Let $k_1 = \min\{n \in N;\ h(b_n) < 1\}$, $f(\alpha_{k_1}) = \max(b_{k_1}, \quad h(b_{k_1}))$,

$f(\bar{h}(\alpha_{k_1})) = h^2(b_{k_1})$. If we have $k_1, k_2, \ldots, k_n$, then we choose k_{n+1} such that $\alpha_{k_{n+1}} < \bar{h}(\alpha_{k_n})$ and $\max(b_{k_{n+1}}, h(b_{k_{n+1}})) < h^2(b_{k_n})$. We put $f(\alpha_{k_{n+1}}) = \max(b_{k_{n+1}}, h(b_{k_{n+1}}))$, $f(\bar{h}(\alpha_{k_{n+1}})) = h^2(b_{k_{n+1}})$. Let f be a linear function on intervals $[\bar{h}(\alpha_{k_n}), \alpha_{k_n})]$ and $[\alpha_{k_{n+1}}, \bar{h}(\alpha_{k_n})]$ for $n \in N$. From the inequality $\frac{f(\alpha_{k_n})}{b_{k_n}} \geqslant \frac{b_{k_n}}{b_{k_n}}$ it follows that zero is a point of (f)-porosity of the set $M = \bigcup_{n\in N} [b_{k_{n+1}}, a_{k_n}]$.

Since $\frac{f(\bar{h}(\alpha_{k_n}))}{h(b_{k_n})} = h(b_{k_n})$, so $h(0)$ is not a point of (f)-porosity of $f(M)$. We obtained the contradiction which ends the proof.

Theorem 4. If a homeomorphism h preserves points of (f)-porosity for every function $f \in G$ then all derived numbers of h at every point are finite.

P r o o f. We shall suppose that there exist $x_0 \in R$ and the sequence $\bar{b}_n \searrow x_0$ such that $\lim \frac{h(\bar{b}_n) - h(x_0)}{b_n - x_0} = +\infty$. Then there exists the sequence of the intervals $\{(a_n, \bar{b}_{k_n})\}_{n\in N}$ mutually disjoint such that $\bar{h}(\alpha_{k_n}) > \alpha_{k_n}$, where $\alpha_{k_n} = \frac{\bar{b}_{k_n} - a_n}{2}$, $\bar{h}(\alpha_{k_n}) = \frac{h(\bar{b}_{k_n}) - h(a_n)}{2}$. Let $b_n = \bar{b}_{k_n}$. We choose a sequence $\{k_n\}_{n\in N}$ of natural numbers. Let $k_1 = 1$. We denote $f(\alpha_1) = b_1 - x_0$, $f(\bar{h}(\alpha_1)) = 2(b_1 - x_0)$.

If we have $k_1, k_2, \ldots, k_n$ then we choose k_{n+1} such that $\bar{h}(\alpha_{k_{n+1}}) < \alpha_{k_n}$ and $(b_{k_{n+1}} - x_0)(1 + \frac{1}{k_{n+1}}) < b_{k_n} - x_0$. Put $f(\bar{h}(\alpha_{k_{n+1}})) = (1 + \frac{1}{k_{n+1}})(b_{k_{n+1}} - x_0)$ and $f(\alpha_{k_{n+1}}) = b_{k_{n+1}} - x_0$. Let f be a linear function on intervals $[\alpha_{k_n}, \bar{h}(\alpha_{k_n})]$ and $[\bar{h}(\alpha_{k_{n+1}}), \alpha_{k_n}]$ for $n \in N$. Of course $f \in G$.

From the equality $\frac{f(\alpha_{k_n})}{b_{k_n} - x_0} = 1$ it follows that x_0 is a point of (f)-porosity of $M = \bigcup_{n\in N} [b_{k_{n+1}}, a_{k_n}]$. From $\frac{f(\bar{h}(\alpha_{k_n}))}{h(b_{k_n}) - h(x_0)} = \frac{(1 + \frac{1}{k_n})(b_{k_n} - x_0)}{h(b_{k_n}) - h(x_0)}$ it follows that $h(x_0)$ is not a point of (f)-porosity of $h(M)$.

Remark 2. A homeomorphism h preserves points of (f)-porosity for every $f \in G$ if and only if the inverse homeomorphism h^{-1} fulfills the Lipschitz condition with constant one and all derived numbers of function h at every point are finite.

References

[1] L. Z a j i č e k, Sets of δ-porosity and sets of δ-porosity(ϱ), Čas pro Pěstováni Mat. Roč 1976, 101, p. 350-358.

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UWAGI DOTYCZĄCE HOMEOMORFIZMÓW
ZACHOWUJĄCYCH PUNKTY (f)-POROWATOŚCI

W pracy tej jest podany warunek konieczny i dostateczny na to, aby homeomorfizm h zachował punkty (f)-porowatości dla dowolnej funkcji f należącej do klasy G.