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ON A CONNECTION BETWEEN VECTOR FIELDS AND LOCAL ONE-PARAMETER GROUPS OF LOCAL TRANSFORMATIONS IN DIFFERENTIAL SPACES

In this paper we consider the problem of a correspondence between vector fields and local one-parameter groups of local transformations for a differential space (M, C), where C is a differential structure on M generated by a finite number of functions. The obtained results generalizes the theorem given in [1].

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Throughout this paper we shall use notation and terminology introduced in [1].

We begin with the proof of the following fact.

Lemma. Let $f_1, f_2, \ldots, f_n$ be real functions on a set M, and let C be the smallest differential structure on M containing $f_1, f_2, \ldots, f_n$. Let $G_1 = \{U_\alpha, \varepsilon_\alpha, \varphi_t^{(\alpha)}\}_{\alpha\in A}$ and $G_1' = \{V_\beta, \eta_\beta, \psi_t^{(\beta)}\}_{\beta\in B}$ be two local one-parameter groups of local transformations, which have the same vector field as their infinitesimal transformation. Then for any $\alpha \in A$, $\beta \in B$ and any $p_0 \in U_\alpha \cap V_\beta$, there exists a neighbourhood $U \subset U_\alpha \cap V_\beta$ of p_0 and a positive number $\varepsilon < \min(\varepsilon_\alpha, \eta_\beta)$ such that

$$(1) \qquad (f_i \circ \varphi_t^{\alpha})(p) = (f_i \circ \psi_t^{(\beta)})(p) \text{ for } p \in U,\ t \in I \qquad i = 1, 2, \ldots, n$$

P r o o f. Let X be the common infinitesimal transformation of G_1 and G_1'. Let us fix $\alpha \in A$, $\beta \in B$ and $p_0 \in U_\alpha \cap V_\beta$. The functions $X(f_i)$, $i = 1, 2, \ldots, n$, are smooth. Thus there exists a neighbourhood U_1 of p_0 contained in $U_\alpha \cap V_\beta$ and functions $g_i \in C^\infty(R^n)$, $i = 1, 2, \ldots, n$, such that

$$(2) \qquad X_p(f_i) = (g_i(f_1, f_2, \ldots, f_n))(p)$$

$$\text{for } p \in U_1,\ i = 1, \ldots, n$$

Let $\overline{\varphi}^{(\alpha)}$ and $\overline{\Psi}^{(\beta)}$ be the smooth mappings $(t, p) \mapsto \varphi_t^{(\alpha)}(p)$ from $I_{\varepsilon_\alpha} \times (U_\alpha, C(U_\alpha))$ into (M, C) and $(t, p) \mapsto \psi_t^{(\beta)}(p)$ from $I_{\eta_\beta} \times (V_\beta, C(V_\beta))$ into (M, C), respectively. From the continuity of the mappings $\overline{\varphi}^{(\alpha)}$ and $\overline{\Psi}^{(\beta)}$ it follows now that there is a neighbourhood U_2 of p_0 and a positive number $\varepsilon_0 < \min(\varepsilon_\alpha, \eta_\beta)$ such that

$$\varphi_t^{(\alpha)}(U_2) \cup \psi_t^{(\beta)}(U_2) \subset U_1 \quad \text{for } |t| < \varepsilon_0$$

According to the definition of a local one-parameter group of local transformations, for $p \in U_\alpha \cap V_\beta$, $|t| < \varepsilon_0$ we have

$$(3) \qquad X(\varphi_t^{(\alpha)}(p))(f_i) = \frac{d}{dt}(f_i \circ \varphi_t^{(\alpha)})(p) \qquad i = 1, 2, \ldots, n$$

$$X(\psi_t^{(\beta)}(p))(f_i) = \frac{d}{dt}(f_i \circ \psi_t^{(\beta)})(p)$$

Next, by (2), for $p \in U_2$, $|t| < \varepsilon_0$, we get

$$(4) \qquad X(\varphi_t^{(\alpha)}(p))(f_i) = (g_i(f_1, f_2, \ldots, f_n))(\varphi_t^{(\alpha)}(p))$$

$$X(\psi_t^{(\beta)}(p))(f_i) = (g_i(f_1, f_2, \ldots, f_n))(\psi_t^{(\beta)}(p))$$

$$i = 1, 2, \ldots, n$$

The mapping $f \circ \overline{\varphi}^{(\alpha)}$ belongs to $C(U_\alpha) \times C^\infty(I_{\varepsilon_\alpha})$, and the mapping $f \circ \overline{\Psi}^{(\beta)}$ – to $C(V_\beta) \times C^\infty(I_{\eta_\beta})$. Therefore there exists a neighbourhood $U_3 \subset U_2$ of p_0, a positive number $\varepsilon < \varepsilon_0$ and mappings F_i, $F_i \in C^\infty(R^{n+1})$, $i = 1, 2, \ldots, n$, such that for $p \in U_3$ and $|t| < \varepsilon$ we have

$$(5) \qquad (f_i \circ \overline{\varphi}^{(\alpha)})(p, t) = F_i^\varphi(f_1(p), \ldots, f_n(p), t)$$

$$i = 1, \ldots, n$$

$$(f_i \circ \bar{\varphi}^{\beta}(p, t) = F_i^{\psi}(f_1(p), \ldots, f_n(p), t)$$

Thus, if $p \in U_3 = U$ and $|t|<\varepsilon$, then from (3) and (4) we obtain

(6)
$$\frac{d}{dt}F_i^{\varphi}(f_1(p), \ldots f_n(p), t) =$$
$$= g_i((f_1, \ldots, f_n), \bar{\varphi}^{(\alpha)}(p, t))$$
$$F_i^{\varphi}(f_1(p), \ldots f_n(p), 0) = f_i(p)$$

$$\frac{d}{dt} F_i^{\psi}(f_1(p), \ldots, f_n(p), t) =$$
$$= g_i((f_1, \ldots, f_n), \bar{\psi}^{(\beta)}(p, t))$$
$$F_i^{\psi}(f_1(p), \ldots, f_n(p), 0) = f_i(p)$$

for $i = 1, 2, \ldots, n$. Let us define vector functions

$$F^{\varphi} = (F_1^{\varphi}, \ldots, F_n^{\varphi}), \ F^{\psi} = (F_1^{\psi}, \ldots, F_n^{\psi})$$
$$g = (g_1, \ldots, g_n), \quad f = (f_1, \ldots, f_n)$$

Then by (6) we have

$$\frac{d}{dt} F^{\varphi}(f(p), t) = g \circ F^{\varphi}(f(p), t), \quad F^{\varphi}(f(p), 0) = f(p)$$

$$\frac{d}{dt} F^{\psi}(f(p), t) = g \circ F^{\psi}(f(p), t), \quad F^{\psi}(f(p), 0) = f(p)$$

From the uniqueness theorem for solutions of differential equations (see, for example, [2], § 8.4) we get

$$F^{\varphi}(f_1(p), \ldots, f_n(p), t) = F^{\psi}(f_1(p), \ldots, f_n(p), t)$$
$$\text{for } p \in U, \quad |t| < \varepsilon$$

Hence, by (5), we obtain (1).

<u>Theorem.</u> Let (M, C) be a Hausdorff differential space. Let us suppose that there are functions $f_1, \ldots, f_n$ in C such that C coincides with the smallest differential structure on M which contains

$f_1, \ldots, f_n$. If two local one-parameter groups of local transformations on (M, C) have the same vector field as their infinitesimal transformation, then they are equivalent.

P r o o f. Suppose that the groups $G_1 = \{U_\alpha, \varepsilon_\alpha, \varphi_t^{(\alpha)}\}_{\alpha \in A}$ and $G_1' = \{V_\beta, \eta_\beta, \psi_t^{(\beta)}\}_{\beta \in B}$ have the same vector field X as their infinitesimal transformation. We shall show that G_1 and G_1' are equivalent. To do this, choose any $\alpha \in A$, $\beta \in B$ and $p_0 \in U_\alpha \cap V_\beta$. Then, according to Lemma, there is a neighbourhood $U \subset U_\alpha \cap V_\beta$ of p and a positive $\varepsilon < \min(\varepsilon_\alpha, \eta_\beta)$, such that

$$(f_i \circ \varphi_t^{(\alpha)})(p) = (f_i \circ \psi_t^{(\beta)})(p)$$

for $p \in U$, $|t| < \varepsilon$, $i = 1, 2, \ldots, n$. Since, however, the functions $f_1, f_2, \ldots, f_n$ separate the points of M, this means that

$$\varphi_t^{(\alpha)}(p) = \psi_t^{(\beta)}(p) \quad \text{for } p \in U, \quad |t| < \varepsilon$$

which completes the proof.

Without the assumption that the topology of (M, C) is Hausdorff the above theorem fails. Suitable examples one can find in [1].

References

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O ZWIĄZKU POMIĘDZY POLAMI WEKTOROWYMI I LOKALNYMI JEDNOPARAMETROWYMI GRUPAMI PRZEKSZTAŁCEŃ LOKALNYCH NA PRZESTRZENIACH RÓŻNICZKOWYCH

W pracy rozważany jest problem wzajemnej odpowiedniości pomiędzy polami wektorowymi i lokalnymi jednoparametrowymi grupami przekształceń lokalnych w kategorii przestrzeni różniczkowych generowanych przez skończoną ilość funkcji.