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FUNCTIONS WITH FIBRES LARGE ON EACH NONVOID OPEN SET

Let X be an infinite set. For ideals $I, J \subseteq P(X)$ and a family $F \subseteq P(X)$, we give conditions guaranteeing the existence of an $f : X \to X$ which is constant on $X \setminus C$ for some $C \in J$ and fulfils the condition: $(*)$ $f^{-1}[\{x\}] \cap V \notin I$ for any $x \in X$ and $V \in F$. The result and its proof are related to the investigations made by H.I. Miller and W. Poreda. In the case when X forms a perfect Polish space and F consists of all nonvoid open sets, we study ideals I admitting an $f : X \to X$ which satisfies $(*)$ and is Borel measurable.

1. Introduction

Carathéodory showed in 5 that there exists a Lebesgue measurable function $f : \mathbb{R} \to \mathbb{R}$ such that $f^{-1}[E] \cap U$ has positive measure for each set E of positive measure and each nondegenerate interval U. A modified version employing the Baire category was obtained by H. Miller in [7]. He proved the existence of a Lebesgue measurable $f : \mathbb{R} \to \mathbb{R}$ such that $f^{-1}[E] \cap U$ is of the second category for each set E of second category and each nondegenerate interval U. He even obtained (in ZFC) a stronger result where E in $f^{-1}[E] \cap U$ is replaced by $\{x\}$ (for any $x \in \mathbb{R}$). The same was shown in [11] in a different way (Continuum Hypothesis used there can be removed which was observed by K.P.S. Bhashara Rao in [3]). In Section 2 we prove a more general result with the help of a mixed method joining the tricks from [7] and [11]. In particular, we get a simple proof in ZFC, good for the measure and category cases. Since there is no uncountable disjoint family of measurable sets of positive measure (this is the so-called countable chain condition, abbr. ccc), there is no Lebesgue measurable $f : \mathbb{R} \to \mathbb{R}$ such that $f^{-1}[\{x\}]$ has positive measure for each $X \in \mathbb{R}$. The analogous observation can be done for the category case. However, there are natural examples of ideals J (which do not satisfy ccc) admitting a Borel measurable $f : \mathbb{R} \to \mathbb{R}$ whose all fibres are large (i.e. not in I). That property, called (M), was introduced in [2] for ideals of subsets of a perfect Polish space. In Section 2 of the present paper, we study a stronger property, called (M^*), which requires the fibres of f to be large on each nonvoid open set.

In general, we consider ideals I of subsets of an infinite set X and always assume that $X \notin I$. A subfamily H of I is called a base of I if each $A \in I$ is contained in some $B \in H$. We say that two ideals I and J are orthogonal if there are $B \in I$ and $C \in J$ such that $B \cup C = X$.

2. Remarks on Miller's result

Recall the following theorem due to Abian and Miller (see [1] and [7]) which generalizes the result of [12].

Theorem 2.1. *Let X be a set of infinite cardinality κ. Let A be a family of at most κ subsets of X, each having cardinality κ. Denote by $\Delta(A)$ the family of all $D \subseteq X$ such that $U \cap D \neq \emptyset$ for each $U \in A$. Then, for each cardinal $\lambda \leq \kappa$, the set X can be expressed as the union of λ pairwise disjoint sets belonging to $\Delta(A)$.*

Theorem 2.2. *Assume that I and J are orthogonal ideals of subsets of a set X of cardinality κ. Let I have a base H of size $\leq \kappa$ and let $F \subseteq P(X)$ be a given family of size $\leq \kappa$ such that $|V \setminus E| = \kappa$ for any $V \in F$ and $E \in H$. Then, for each $x_0 \in X$, there are a set $C \in J$ and a function $f : X \to X$ such that $f(x) = x_0$ for each $x \in X \setminus C$,* and

(*) $f^{-1}[\{x\}] \cap V \notin I$ *for any* $x \in X$ *and* $V \in F$.

Proof. Put $A = \{V \setminus E : V \in F \text{ and } E \in H\}$ and apply Theorem 1.1 to it. Then X can be expressed as the union of a disjoint family $\Delta^* \subseteq \Delta(A)$ of size κ. Let $X = B \cup C$ where $B \in I$, $C \in J$ and $B \cap C = \emptyset$. Choose any bijection $h : \Delta^* \to X$ and define $f : X \to X$ as follows. If $x \in B$, put $f(x) = x_0$, and if $x \notin B$, choose a unique $D_x \in \Delta^*$ such that $x \in D_x$ and put $f(x) = h(D_x)$. Then, obviously, $f(x) = x_0$ for $x \in X \setminus C$. If $x \in X$, then

$$f^{-1}[\{x\}] = \begin{cases} h^{-1}(x) \setminus B & \text{for } x \neq x_0, \\ h^{-1}(x) \cup B & \text{for } x = x_0. \end{cases}$$

Consider any $V \in F$. Observe that $V \cap D \notin I$ for each $D \in \Delta(A)$. Indeed, if $V \cap D \in I$ for some $D \in \Delta(A)$, we choose $E \in H$ such that $V \cap D \subseteq E$. We infer that $V \setminus E \in A$ and $(V \setminus E) \cap D = \emptyset$, which contradicts the assumption $D \in \Delta(A)$. Now, taking $D = h^{-1}(x)$, we have $h^{-1}(x) \cap V \notin I$. Since $B \in I$, we get $f^{-1}[\{x\}] \cap V \notin I$.

In particular, let $X = \mathbb{R}$ and let I (resp. J) be the ideal of all Lebesgue null sets (resp. meager sets) in $\mathbb{R}$. It is well kown that the family H of all G_δ null sets (resp. F_σ meager sets) forms a base of I (resp. J), its cardinality equals $c = |\mathbb{R}|$, and $|V \setminus E| = c$ for any open $V \neq \emptyset$ and $E \in H$. Moreover, I and J are orthogonal (see [10]). Thus from Theorem 2.2 we derive

Corollary 2.3. (a) *There is an* $f : \mathbb{R} \to \mathbb{R}$ *such that* $\{x \in \mathbb{R} : f(x) \neq 0\}$ *is meager (thus* f *has the Baire property) and* $f^{-1}[\{x\}] \cap V$ *has positive outer measure for any* $x \in \mathbb{R}$ *and open* $V \neq \emptyset$.

(b) *(see [7], [11]). There is an* $f : \mathbb{R} \to \mathbb{R}$ *such that* $\{x \in \mathbb{R} : f(x) \neq 0\}$ *is a null set (thus* f *is Lebesgue measurable) and* $f^{-1}[\{x\}] \cap V$ *is of the second category for any* $x \in \mathbb{R}$ *and open* $V \neq \emptyset$.

Another interesting pair of orthogonal ideals to which Theorem 2.2 can be applied is described in [9], Proposition 5.

3. Property (M^*)

Now, we add the requirement of the Borel measurability of f to condition $(*)$ formulated in Theorem 2.2. Let X be a perfect Polish space and I - an ideal of subsets of X. We say (cf. [2]) that I has property (M) (resp. property (M^*)) if there is a Borel measurable function $f : X \to X$ such that $f^{-1}[\{x\}] \notin I$ for each $x \in X$ (resp. $f^{-1}[\{x\}] \cap V \notin I$ for any $x \in X$ and open $V \neq \emptyset$). We then say that f realizes (M) (resp. (M^*)) for I. Obviously, (M^*) implies (M). We shall show that the converse is false (Example 3.5).

Remarks. (a) If I and J are ideals of subsets of X such that $I \subseteq J$ and J has (M) (resp. (M^*)), then I has (M) (resp. (M^*)).

(b) Since any two perfect Polish space are Borel isomorphic (see [8], 1 G4), we may replace $f : X \to X$ in the definition of (M) and (M^*) by $f : X \to Y$ for a suitable perfect Polish Y.

In [2], several examples of ideals with property (M) are given. Our aim is to find nontrivial ideals with property (M^*).

It was noticed in [4], Ex. 1.3, p. 4, that there exists a Borel function f from $(0,1)$ into $(0,1)$ such that $f^{-1}[\{x\}]$ is dense for each $x \in (0,1)$ (this was treated as a strong version of the Darboux property). The same can be inferred from [2], Th. 3.4, p. 44, where another method leads to a Borel mapping from a perfect Polish space X onto the Cantor space, with all fibres dense in X. In fact, the existence of such a mapping implies that the ideal of all nowhere dense sets in X has property (M^*). Our next example of an ideal with property (M^*) is also derived from [2]. It turns out that the respective proof for (M) given in [2] (generalizing Mauldin's construction from [6]) works for (M^*), but some parts require a more detailed analysis which will be done below.

Theorem 3.2 (cf.[2], Th.3.3, p. 42). *Let I be a σ-ideal of subsets of a perfect Polish space X. Assume that I contains all singletons, does not contain nonempty open sets and has a base consisting of G_δ sets. Then the σ-ideal J of all sets that can be covered by F_σ sets from I has property (M^*).*

A nonempty closed set $F \subseteq X$ will be called I-perfect if $F \cap V \neq \emptyset$ implies $F \cap V \notin I$ for any open $V \subseteq X$.

Let us explain some notation. Let $\omega = \{0, 1, 2, \dots\}$. By $2^{<\omega}$ and 2^{ω} we denote, respectively, the sets of all finite and infinite sequences of zeros and ones. The empty sequence (which also belongs to $2^{<\omega}$) will be written as $\langle\,\rangle$. By $s0$ and $s1$ we denote the respective extensions of $s \in 2^{<\omega}$. For $z \in 2^{\omega}$ and $n \in \omega$, put $z|n = \langle z(0), z(1), \dots, z(n-1)\rangle$. The set 2^{ω}, endowed with the product topology, is called the Cantor space. It forms a perfect Polish space.

The following lemma results immediately from the contruction given in [2], pp. 42-43.

Lemma 3.3. *Under the assumptions of Theorem 3.2, there is a family* $\{C_s^n : s \in 2^{<\omega}, n \in \omega\}$ *of I-perfect sets with the properties:*

(1) *for each nonempty open* $V \subseteq X$*, there is an* $n \in \omega$ *such that* $C^n_{\langle\,\rangle} \subseteq V$;

(2) *for any* $s \in 2^{<\omega}$, $n \in \omega$ *and a nonempty* V *relatively open in* C_s^n*, there is an* $m \in \omega$ *such that* $C^m_{s0} \cup C^m_{s1} \subseteq V$;

(3) *for any* $s \in 2^{<\omega}$ *and* $m \in \omega$*, the condition* $C^m_{s0} \cap C^m_{s1} = \emptyset$ *holds and there is an* $n \in \omega$ *such that* $C^m_{s0} \cup C^m_{s1} \subseteq C^n_s$.

Lemma 3.4. *Under the assumptions of Theorem 3.2, if a family* $\{C_s^n : s \in 2^{<\omega}, n \in \omega\}$ *fulfils conditions (1)-(2) of Lemma 3.3, then, for any* $z \in 2^{\omega}$*, a set* $H \in J$ *and a nonempty open* $V \subseteq X$*, there exists a sequence* $\langle n_i : i \in \omega\rangle$ *of nonnegative integers such that*

$$\emptyset \neq \bigcap_{i \in \omega} C^{n_i}_{z|i} \subseteq V \setminus H.$$

Proof. Since $H \in J$, there is a sequence of closed sets $F_n \in I$ such that $H \subseteq \bigcup_{n\in\omega} F_n$. The set $V \setminus F_0$ is open and nonempty (in fact, $V \setminus F_0 \notin I$ since $V \notin I$ and $F_0 \in I$). By (1), pick $n_0 \in \omega$ so that $C^{n_0}_{\langle\,\rangle} \subseteq V \setminus F_0$. For any $i \in \omega$, having n_i chosen, pick $n_{i+1} \in \omega$ so that $C^{n_{i+1}}_{z|i+1} \subseteq C^{n_i}_{z|i} \setminus F_{i+1}$ (we use (2)); here $C^{n_i}_{z|i} \setminus F_{i+1}$ is nonempty (in fact, it does not belong to I) and relatively open in $C^{n_i}_{z|i}$. From the classical Cantor theorem we get $C = \bigcap_{i\in\omega} C^{n_i}_{z|i} \neq \emptyset$. Of course, C is disjoint from $\bigcup_{n=1}^{\infty} F_n$ and, consequently, from H.

Proof of Theorem 3.2. We use the sets C_s^n from Lemma 2.3. Put $C_s = \bigcup_{n\in\omega} C_s^n$ for $s \in 2^{<\omega}$. Then we have

(a) $C_{s0} \cap C_{s1} = \emptyset$ for all $s \in 2^{<\omega}$,
(b) $C_{s0} \cup C_{s1} \subseteq C_s$ for all $s \in 2^{<\omega}$,

which follows from (3). Define $B = \bigcap_{n\in\omega} \bigcup_{z\in 2^\omega} C_{z|n}$. It is not hard to prove (see [2]) that:

(c) B is a Borel set;
(d) for each $x \in B$, there is a unique $h(x) \in 2^\omega$ such that $x \in \bigcap_{n\in\omega} C_{h(x)|n}$;
(e) the function $h : B \to 2^\omega$ defined in (d) is Borel measurable;
(f) $h^{-1}[\{z\}] = \bigcap_{n\in\omega} C_{z|n}$ for each $z \in 2^\omega$.

Let $g : X \to 2^\omega$ be a fixed Borel measurable extension of h. By (f), we have $g^{-1}[\{z\}] \supseteq \bigcap_{n\in\omega} C_{z|n}$. Consider any nonempty open $V \subseteq X$. It suffices to show that $V \cap \bigcap_{n\in\omega} C_{z|n} \notin J$. Suppose that $V \cap \bigcap_{n\in\omega} C_{z|n} = H \in J$. According to Lemma 3.4, there is a sequence $\langle n_i : i \in \omega\rangle$ for which $\emptyset \neq \bigcap_{i\in\omega} C_{z|i}^{n_i} \subseteq V \setminus H$. On the other hand

$$V \cap \bigcap_{n\in\omega} C_{z|i}^{n_i} \subseteq V \cap \bigcap_{i\in\omega} C_{z|i} = H,$$

a contradiction.

By Theorem 3.2, the ideal of sets that can be covered by F_σ Lebesgue null sets has (M^*).

Example 3.5. Let I consist of all sets $A \subseteq \mathbb{R}$ such that $A \cap (-\infty, 0)$ is of Lebesgue measure zero and $A \cap [0,\infty)$ is contained in an F_σ set of measure zero. Then I forms a σ-ideal of subsets of $\mathbb{R}$. Observe that I has property (M). Indeed, the family $I_+ = \{A \in I : A \subseteq [0,\infty)\}$ is a σ-ideal of subsets of $X = [0,\infty)$, which fulfils the assumptions of Theorem 3.2. Hence it has property (M^*) and, consequently, property (M) (in X). Let $f_+ : X \to \mathbb{R}$ realize property (M) for I_+. If we extend f_+ to a Borel $f : \mathbb{R} \to \mathbb{R}$, then f realizes property (M) for I. On the other hand, I has not (M^*). Indeed, suppose that $g : \mathbb{R} \to \mathbb{R}$ realizes (M^*) for I. Then $\{g^{-1}[\{y\}] \cap (-\infty, 0) : y \in \mathbb{R}\}$ forms an uncountable disjoint family of Borel sets with positive measure, which is impossible.

In the above example, I is not translation-invariant, i.e. the condition

$$A + x \in I \quad \text{for any} \quad A \in I \quad \text{and} \quad x \in \mathbb{R},$$

where $A + x = \{a + x : a \in A\}$, is not fulfilled. So, it would be interesting to find an example omitting that fault.

Let us note that Example 3.5 essentially uses the fact that property (M) (unlike (M^*)) need not be hereditary with respect to open sets. To be more precise, let us say that an ideal I has property (M') if $I \cap P(V)$ has (M) (in V) for any nonvoid open $V \subseteq X$. Obviously, $(M^*) \Rightarrow (M') \Rightarrow (M)$. Our example shows, in fact, that $(M) \Rightarrow (M')$ is false. This suggests the question whether $(M') \Rightarrow (M^*)$ must hold.

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FUNKCJE O DUŻYCH WŁÓKNACH NA KAŻDYM NIEPUSTYM ZBIORZE OTWARTYM

Niech X będzie zbiorem nieskończonym. Dla pewnych ideałów $I, J \subseteq P(X)$ i rodziny $F \subseteq P(X)$ uzyskano warunki dostateczne istnienia funkcji $f : X \to X$ stałej na $X \setminus C$ dla pewnego $C \in J$ oraz spełniającej warunek: $(*)$ $f^{-1}[\{x\}] \cap V \notin I$ dla dowolnych $x \in X$ i $V \in F$. Wynik i jego dowód wiążą się z wcześniejszymi badaniami H. Millera i W. Poredy. W przypadku gdy X jest doskonałą przestrzenią polską oraz F składa się z niepustych zbiorów otwartych, badamy ideały I, dla których istnieje borelowska funkcja $f : X \to X$ spełniająca $(*)$.

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