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ESTIMATION OF THE FUNCTIONAL $|a_3 - \alpha | a_2^2|$ IN THE CLASS S OF HOLOMORPHIC AND UNIVALENT FUNCTIONS FOR α COMPLEX

In this paper there has been investigated a maximal value of the functional $|a_3 - \alpha a_2^2|$ in the well-known class S of functions holomorphic and univalent in the unit disc, where α is an arbitrary complex parameter. In the investigations carried out here use is made of the variational method and, in particular, of the differential-functional equation of A, C. Schaeffer and D. C. Spencer.

1. INTRODUCTION

In 1952 Goluzin [1] for the first time introduced to the literature the functional

(1) $|a_3 - \alpha a_2^2|$,

defined in the class S of functions holomorphic and univalent in the disc $K = \{z : |z| < 1\}$ of the form

(2)
$$f(z) = z + a_2 z^2 + a_3 z^3 + \dots$$

On the basis of Löwner's theory, he determined the upper bound of functional (1) for each fixed $\alpha \in < 0,1$). For any real α , this functional was estimated by J e n k i n s [3] in 1960. In the subsequent years, a lot of mathematicians took up their investigations of functional (1) for α real in other

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classes of univalent functions. In particular, in the family of functions of form (2), holomorphic, univalent and bounded in the disc K, the upper bound of functional (1) was obtained, among others, by J = k = 0 where k = 12 for any real α .

In the present paper there has been taken up a problem of determining the upper bound of functional (1) defined in the family S for any $\alpha \in C$ (C - the open plane) on the basis of the variational method.

For our purposes, it will be most convenient to make use of the differential-functional equation for extremal functions in the class S, obtained by Schaeffer and Spencer [6].

2. EQUATION FOR EXTREMAL FUNCTIONS

Let us consider a functional

$$H(f) = re (a_3 - \alpha a_2^2),$$

defined in the class S, where a is any complex number. The family S is compact, and functional (3) - continuous, so there exists a function of the class S for which it attains its upper bound. In the sequel, such functions will be called extremal ones.

Functional (3) satisfies the assumptions of the Schaeffer--Spencer theorem [6], therefore each extremal function of form (2) satisfies the following equation _

(4)
$$\left[\frac{z f'(z)}{f(z)}\right]^2 \frac{1+2 \cdot (1-\alpha)a_2 f(z)}{f^2(z)} =$$

$$=\frac{z^4 + 2(1-\bar{\alpha})\bar{a}_2 z^3 + 2B_0 z^2 + 2(1-\alpha)a_2 z + 1}{z^2}, \quad z \in \mathbb{K}$$

where (5)

(3)

$$B_0 = a_3 - \alpha a$$

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with that $B_0 > 0$, and the right-hand side of equation (4) is non-negative on the circle |z| = 1 and possesses on it at least one double root.

Since $B_0 > 0$, from (5) it follows that, for each extremal function satisfying equation (4),

$$re(a_3 - \alpha a_2^2) = a_3 - \alpha a_2^2$$
.

Putting

(6)
$$u = 2(1 - \alpha)a_{2}$$

in (4), we get

(7)
$$\left[\frac{z \cdot f'(z)}{f(z)}\right]^2 \frac{1 + uf(u)}{f^2(z)} = \frac{z^4 + \bar{u}z^3 + 2B_0 z^2 + uz + 1}{z^2}, \quad z \in K.$$

Consequently, the determination of the upper bound of functional (3) for any fixed $a \in C$ is reduced to the finding of suitable functions that satisfy equation (7). It is worth recalling here that equation (7) is only a necessary condition for the function f(z) to be an extremal one.

Our virtual considerations will be preceded by a few remarks. Concerning general properties of equation (7) and extremal functions.

Let us put

(8)

 $N(z) = \frac{z^4 + \bar{u}z^3 + 2B_0 z^2 + uz + 1}{z^2}$

It follows from general properties of equation (4) that function (8) is factorized in the following way

(9)
$$N(z) = \frac{(z - e^{i\psi})^2 (z^2 - te^{-i\psi} z + e^{-2i\psi})}{z^2}$$

where $\Psi \in (-\pi, \pi >, t \ge 2.$

By comparing (8) and (9), we obtain the relations

(10)

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 $\begin{cases} re \ u = -(t + 2) \ \cos \Psi, \\ im \ u = -(t - 2) \ \sin \Psi, \end{cases}$

and

$$B_0 = t + \cos 2\Psi$$

Let α be any fixed complex number, and f - an extremal function of form (2). Put

(12)'
$$g(z) = -f(-z) = z - a_2 z^2 + a_2 z^3 + \dots$$

Note that $g(z) \in S$ and H(f) = H(g). So, it f is an extremal function, then function (12) has the same property. Let us next notice that, if f is an extremal function, then the functional $re(a_3 - \overline{\alpha} a_2^2)$ attains its maximum for the function $h(z) = \overline{f(\overline{z})}$, the respective extremal values being equal to each other.

It follows from the above that while considering equation (7) we may confine ourselves to u's such that re u ≤ 0 and im u ≤ 0 , which, in view of (10), is equivalent to the condition $\Psi \in \langle 0, \frac{\pi}{2} \rangle$.

Finally, taking into account all the factorizations of function (8) as well as the cases $u \neq 0$ and u = 0, it is easy to prove that equation (7) can only be of the form

(a)
$$\left[\frac{z f'(z)}{f(z)}\right]^2 \frac{1 + uf(z)}{f^2(z)} = \frac{(z - z_0)^2(z - z_1)(z - z_2)}{z^2}, \quad u \neq 0,$$

or

(b)
$$\left[\frac{z \ f'(z)}{f(z)}\right]^2 \frac{1 + uf(z)}{f^2(z)} = \frac{(z - z_0)^2 (z - \overline{z}_0)^2}{z^2}, \quad u \neq 0, \quad z_0 \neq z_0,$$

or

(c)
$$\left[\frac{z f'(z)}{f(z)}\right]^2 \frac{1 + uf(z)}{f^2(z)} = \frac{(z - z_0)^4}{z^2}, \quad u \neq 0,$$

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$$\left[\frac{z \ f'(z)}{f(z)}\right]^2 \frac{1}{f^2(z)} = \frac{(z - z_0)^2 (z - \overline{z}_0)^2}{z^2}, \quad z_0 \neq z_0,$$

where $z_0 = e^{i\Psi}$, $z_1 = \rho e^{i\Psi}$, $z_2 = \frac{1}{\overline{z}_1}$, $0 < \rho < 1$, $\psi \in < 0$,

 $\frac{\pi}{2}$ >, $\varphi \in \mathbb{R}$ (\mathbb{R} - the set of real numbers).

or

(d)

In sections 3, 4, 5 and 6 of the paper we shall be successively concerned with a thorough analysis of all solutions to equations of form (a), (b), (c) and (d).

3. EQUATION OF FORM (a)

Let us first consider the case when equation (7) is of form (a). By comparing the right-hand sides of (7) and (a), we get

(13)
$$z_1 = \rho \bar{z}_0, \quad z_2 = \frac{1}{\rho} \bar{z}_0,$$

(14)
$$u = -2e^{-i\psi} - (\varrho + \frac{1}{\varrho})e^{i\psi}$$

$$B_{o} = \rho + \frac{1}{\rho} + \cos 2\Psi,$$

Note that relations (14) and (15) are identical with (10) and (11) for $t = \rho + \frac{1}{\rho}$.

From formulae (5) and (15) it can be seen that the value of the expression $(a_3 - \alpha a_2^2)$ for an extremal function satisfying an equation of form (a) is determined by two real parameters Ψ and ρ . Hence it appears that, in order to determine the upper bound of functional (3), one has to find some relationships between Ψ , ρ and α . Our further considerations will concern the seeking for these relationships.

In virture of (13), equation (a) is equivalent to

16)
$$\left[\frac{zf'(z)}{f(z)}\right]^2 \frac{1 + uf(z)}{f^2(z)} = \frac{(1 - \overline{z}_0 z)^2 (1 - \rho z_0 z)^2 \frac{1 - \frac{1}{\rho} z_0 z}{1 - \rho z_0 z}}{z^2}, z \in \frac{1 - \frac{1}{\rho} z_0 z}{z}$$

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In this equation we shall introduce new variables. For the purpose, let us notice that the homography

(17)
$$\omega(z) = \frac{1 - \frac{1}{\rho} z_0 z}{1 - \rho z_0 z}$$

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transforms the simply connected domain $K' = K - \langle \rho e^{-i\Psi}, e^{-i\Psi} \rangle$ onto the simply connected domain

" = {
$$\omega$$
 : $|\omega| < \frac{1}{\varrho}$ - (- $\frac{1}{\varrho}$, 0 >,

with that

(18)
$$\omega(0) = 1$$
, $\lim_{Z \to Z} \omega(z) = 0$,
 $z \in K'$

Define in K" a mapping

(19)
$$s = \sqrt{\omega}, \quad \sqrt{1} = 1.$$

Let us further notice that, if a function w = f(z) satisfies equation (a), i.e. (16), then D' = f(K') is a simply connected domain containing zero, and

(20)
$$\lim_{\substack{z \to z_1 \\ z \in K'}} f(z) = -\frac{1}{u}.$$

Since D' is a domain not containing the points $(-\frac{1}{u})$ and ∞ , there exists in it a unique branch of the root

(21)
$$T(w) = \sqrt{1 + uw}, \quad \sqrt{1} = 1.$$

Moreover,

(22)
$$\lim_{w \to -\frac{1}{u}} \sqrt{1 - uw} = 0.$$

From the above considerations follows

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Remark 1. By a suitable superposition of the functions expressed by formulae (17), (19), (21) and of w = f(z), $z \in K'$, *We can define in the domain re s > 0 and $|s| < \frac{1}{\sqrt{\rho}}$ a function T = T(s). The function is holomorphic and univalent in this domain; what is more,

$$T(1) = 1$$
, $\lim_{s \to 0} T(s) = 0$.

For $z \in K'$, equation (16) is written in the form

(23)
$$\frac{z f'(z)}{f(z)} \frac{\sqrt{1 + u f(z)}}{f(z)} = \frac{(1 - \overline{z}_0 z)(1 - \rho z_0 z)}{z} \sqrt{\frac{1 - \frac{1}{\rho} z_0 z}{1 - \rho z_0 z}}$$

where we have taken those branches of roots which assume the value 1 for z = 0 and w = 0, respectively.

By introducing in equation (23) the new variables τ and s, defined by (21) and (19), and making use of *Remark 1*, after some transformations we obtain

(24)
$$\frac{\tau^2}{(1-\tau^2)^2} \cdot \frac{d\tau}{ds} = -\frac{1}{u} \cdot \frac{(1-\varrho^2)^2 (s^2-\varrho) s^2 (1-\varrho z_0^2)}{z_0 (s^2-1)^2 (\varrho^2 s^2-1)^2}$$

where

$$q = s^{2}(z_{0}) = \frac{1 - \frac{1}{p} z_{0}^{2}}{1 - p z_{0}^{2}}.$$

Integrating equation (24), we get

(25)
$$\log \frac{1-\tau}{1+\tau} - \frac{1}{\tau-1} - \frac{1}{\tau+1} = -\frac{1}{uz_0} (1-\varrho z_0^2) \left[A_1 \log \frac{1-s}{1+s} + A_2 \log \frac{1-\rho s}{1+\varrho s} - A_3 \left(\frac{1}{s-1} + \frac{1}{s+1} \right) - A_4 \frac{1}{\varrho} \left(\frac{1}{\varrho s-1} + \frac{1}{\varrho s+1} \right) \right] + C,$$

$$A_{1} = \frac{3 + \rho^{2} - (1 + 3\rho^{2})q}{1 - \rho^{2}}, \qquad A_{2} = \frac{q\rho^{2}(3 + \rho^{2}) - (1 + 3\rho^{2})}{\rho(1 - \rho^{2})},$$
$$A_{2} = 1 - q, \qquad A_{4} = 1 - \rho^{2}q,$$

while C is a constant. In (25) we have taken those branches of logarithms which assume the value zero at the points s = 0 and $\tau = 0$, respectively.

We shall now determine the constant C. For the purpose, note that when $z \rightarrow z_1$, then, by (20) as well as (22), (18) and (19), we get $s \rightarrow 0$ and $\tau \rightarrow 0$. So, passing in (25) to the limit with $s \rightarrow 0$ and $\tau \rightarrow 0$, we obtain

$$C = 0.$$

In view of the above , and formula (14), after some transformations equation (25) will take the form

(26)
$$\log\left(\frac{1-\tau}{1+\tau}\cdot\frac{1+s}{1-s}\right) - \frac{1}{\tau-1} - \frac{1}{\tau+1} = B_1 \log\frac{1-\rho s}{1+\rho s} + B_2 \left(\frac{1}{s-1} + \frac{1}{s+1}\right) + B_3 \left(\frac{1}{\rho s-1} + \frac{1}{\rho s+1}\right),$$

where

$$B_{1} = -\frac{\rho + \frac{1}{\rho} + 2z_{0}^{2}}{2 + (\rho + \frac{1}{\rho})z_{0}^{2}}, \qquad B_{2} = \frac{(\rho - \frac{1}{\rho})z_{0}^{2}}{2 + (\rho + \frac{1}{\rho})z_{0}^{2}},$$
$$B_{3} = \frac{\rho - \frac{1}{\rho}}{2 + (\rho + \frac{1}{\rho})z_{0}^{2}}.$$

Note further that from (17), (19) and (21) we have

(27)
$$-\frac{1}{u}(1-\tau^2) = f\left(\frac{\varrho}{z_0} \cdot \frac{s^2-1}{\varrho^2 s^2-1}\right)$$

By Remark 1, we may expand the function $\tau = \tau(s)$ in a pow-

where

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er series in some neighbourhood of the point s = 1. The coefficients of this series will be obtained from relation (27) by succesive differentiation of both sides with respect to s. Taking this into account, let us expand each particular function by equation (26) in a power series, also in a neighbourhood of the point s = 1. By comparing the terms not containing the variable s, we get

28)
$$\log \frac{2 + (\varrho + \frac{1}{\varrho})z_0^2}{(\frac{1}{\varrho} - \varrho)z_0^2} + \frac{\varrho + \frac{1}{\varrho} + 2z_0^2}{2 + (\varrho + \frac{1}{\varrho})z_0^2} \log \frac{1 - \varrho}{1 + \varrho} =$$

$$\frac{2(a_2 z_0 + 2)}{2 + (\rho + \frac{1}{\rho}) z_0^2}.$$

From (14), in view of (6), we have

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(29)
$$a_2 = \frac{2 + (\rho + \frac{1}{\rho})z_0^2}{-2(1 - \alpha) \cdot z_0}$$

In virtue of (29), after some transformations equation (28) will take the form

(30)

$$\left[2\overline{z}_{0} + (\rho + \frac{1}{\rho})z_{0}\right] \log \frac{2 + (\rho + \frac{1}{\rho})z_{0}^{2}}{(\frac{1}{\rho} - \rho)z_{0}^{2}} + \left[(\rho + \frac{1}{\rho})\overline{z}_{0} + 2z_{0}\right] \log \frac{1 - \rho}{1 + \rho} = 4\overline{z}_{0} - \frac{2\overline{z}_{0} + (\rho + \frac{1}{\rho})z_{0}}{1 - \alpha}.$$

From (30), by isolating the real part and the imaginary one, ... We obtain a system of equations

31)
$$(t-2)\left[\operatorname{arc} tg \frac{2 \sin 2\psi}{t+2 \cos 2\psi} + \frac{\operatorname{im} (1-\alpha)}{|1-\alpha|^2} \right] \sin \psi + (t+2)$$

$$\frac{160}{\left[\log \frac{|t + 2\cos 2\Psi - 2i\sin 2\Psi|}{t + 2} - \frac{4}{t + 2} + \frac{re(1 - \alpha)}{11 - \alpha l^2}\right]\cos\Psi}{(32)} = 0,$$

$$(32) \quad (t - 2) \left[\log \frac{|t + 2\cos 2\Psi - 2i\sin 2\Psi|}{t - 2} + \frac{4}{t - 2} + \frac{4}{t - 2} + \frac{re(1 - \alpha)}{11 - \alpha l^2}\right]\sin\Psi + \frac{4}{t - 2} + \frac{re(1 - \alpha)}{11 - \alpha l^2}\left[\sin\Psi + \frac{4}{t - 2}\right] \sin\Psi + \frac{1}{t - 2} + \frac{re(1 - \alpha)}{t - 2} + \frac{re(1 - \alpha)}{t - 2} + \frac{re(1 - \alpha)}{t - 2}\right] \cos\Psi = 0,$$

where

$$t = \varrho + \frac{1}{\varrho} > 2, \quad \Psi \in \langle 0, \frac{\pi}{2} \rangle, \quad 0 \leq \text{arc tg} \frac{2 \sin 2\Psi}{t + 2 \cos 2\Psi} < \frac{\pi}{2}$$

To sum up, we have proved

Lemma 1. For each extremal function satisfying the equation of form (a),

(33)
$$H(f) = t + \cos 2\Psi$$
,

where \$\Psi\$ and t are roots of system of equation (31), (32).
 Of course, there arises a question: for what a's does sy stem of equations (31), (32) possess a solution and is this so lution the only one?

In order to answer the question, we shall first prove

Lemma 2. The jacobian of system (31), (32), calculated with respect to the unknown quantities Ψ and t, is different from zero at each point (re α , im α ; Ψ , t) satisfying this system. P r o o f. The jacobian of system (31), (32) is expressed by the formula

(34)
$$J = (t - 2 \cos 2\psi) \left\{ \left[\arctan tg \frac{2 \sin 2\psi}{t + 2 \cos 2\psi} + \frac{\operatorname{im} (1 - \alpha)^2}{(1 - \alpha)^2} \right] + \left[\operatorname{Log} \frac{|t + 2 \cos 2\psi - 2i \sin 2\psi|}{t + 2} + \frac{\operatorname{re} (1 - \alpha)}{(1 - \alpha)^2} \right] \right\}$$

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$$\frac{|t + 2\cos 2\Psi - 2i\sin 2\Psi|}{t - 2} + \frac{re(1 - \alpha)}{|1 - \alpha|^2}$$

It is easy to check that, for $\Psi = 0$ or $\Psi = \frac{\pi}{2}$, Lemma 2 is true. Assume in the further part of the proof that $\Psi \in (0, \frac{\pi}{2})$. Transforming equations (31) and (32) in a suitable way and next, dividing them, after simple calculations we get

(35)
$$\left[\arctan tg \frac{2 \sin 2\Psi}{t + 2 \cos 2\Psi} + \frac{im(1 - \alpha)}{11 - \alpha!^2} \right]^2 + S_1 \cdot S_2 =$$

$$= \frac{4}{t^2 - 4} [(t - 2)s_2 - (t + 2)s_1 + 4],$$

where

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$$S_{1} = Log \frac{|t + 2 \cos 2\Psi - 2i \sin 2\Psi|}{t + 2} + \frac{re(1 - a)}{|1 - a|^{2}},$$

$$S_{2} = Log \frac{|t + 2 \cos 2\Psi - 2i \sin 2\Psi|}{t - 2} + \frac{re(1 - a)}{|1 - a|^{2}}.$$

Let us now consider any fixed a such that re(1 - a) < 0. Suppose that at some point satisfying system (31), (32), such that re(1 - a) < 0, jacobian (34) is equal to zero. It can be easily verified that this is possible only when

(36)
$$S_1 < 0$$
 and $S_2 > 0$.

In virtue of (35), jacobian (34) is written down in the form

(37)
$$J = \frac{4(t - 2\cos 2\Psi)}{t^2 - 4} \left[(t - 2)s_2 - (t + 2)s_1 + 4 \right].$$

It follows from (36) and (37) that J > 0, which contradicts Our supposition. In a similar way we show that the jacobian is different from zero for all points satisfying system (31), (32), such that $re(1 - \alpha) \ge 0$. Lemma 2 has thus been proved.

Remark 2. The jacobian of system (31), (32), calculated with

respect to the unknown quantities Ψ and ϱ , is different from zero at each point (re d, im d, Ψ , ϱ) satisfying this system.

Indeed, this follows at once from Lemma 2 and the fact that $t = \rho + \frac{1}{\rho}$ and thus, $t' = 1 - \frac{1}{\rho^2} < 0$ for $\rho \in (0, 1)$. In the investigations carried out so far we have limited ourselves to considering a from one of the half-planes im a ≤ 0 or im $\alpha \ge 0$. Now, let a be any point of the complex plane C; then the parameter $\Psi \in \langle -\frac{\pi}{2}, \frac{\pi}{2} \rangle$. It is easy to verify that, for the range of variability of Ψ thus extended, system (31), (32) remains unchanged, and only

 $-\frac{\pi}{2} < \arctan tg \frac{2 \sin 2\Psi}{t + 2 \cos 2\Psi} < \frac{\pi}{2}.$

Thereby, Lemma 2 and Remark 2 remain true.

Note that by putting $\Psi = \frac{+\pi}{2}$ in system (31), (32), we obtain t = 4 rea - 2, im a = 0. Hence follows

Remark 3. For each fixed $\alpha > 1$, system (31), (32) is satisfied by the pairs $(-\frac{\pi}{2}, 4\alpha - 2)$, $(\frac{\pi}{2}, 4\alpha - 2)$. From (33) we then have

re
$$(a_3 - \alpha a_2^2) = 4\alpha - 3$$
,

which is in accordance with the result obtained by Jenkins [3]. The extremal function is the Koebe function. Denote

(38)
$$B = \{(\Psi, \varrho) : -\frac{\pi}{2} < \Psi < \frac{\pi}{2} \text{ and } 0 < \varrho < 1\}.$$

System of equations (31), (32) is equivalent to (30), and hence follows

Remark 4. To each pair $(\Psi, \varrho) \in B$ there corresponds exactly one number α such that system of equations (31), (32) is satisfied. This number is defined by the formula

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+
$$(\rho + \frac{1}{R})e^{2i\Psi}$$

a = 1 +

 $\left[2 + (\varrho + \frac{1}{\varrho})e^{2i\Psi}\right] \operatorname{Log} \frac{2 + (\varrho + \frac{1}{\varrho})e^{2i\Psi}}{(\frac{1}{\varrho} - \varrho)e^{2i\Psi}} + (\varrho + \frac{1}{\varrho} + 2e^{2i\Psi}) \operatorname{Log} \frac{1 - \varrho}{1 + \varrho} - 4$

In our further considerations we shall determine the set of values of mapping (39) for $(\Psi, \rho) \in B$ and prove that it is a homeomorphic mapping. Consequently, in virtue of *Remark* 4, we shall tell for what α 's system (31), (32) possesses a solution and show that it is the only solution.

We shall first demonstrate that mapping (39) is locally homeomorphic in the set B. To that end, it is enough to prove

Lemma 3. The jacobian of mapping (39) is different from zero at each point of the set B.

Proof. Let us introduce the following notations

$$1 = 5 + i\eta$$
, $5 = 5(\Psi, \rho)$, $\eta = \eta(\Psi, \rho)$,

where $\xi = \xi(\Psi, \rho)$ stands for the real part, while $\eta = \eta(\Psi, \rho)$ - the imaginary part of the right-hand side of equation (39). Denote by ω and Θ the left-hand sides of equations (31) and (32), respectively. So, everywhere on the set B we shall obtain

> $\omega(\xi, \eta, \Psi, \rho) = 0,$ $\Theta(\xi, \eta, \Psi, \rho) = 0.$

Differentiating the above equations with respect to Ψ and ρ , and next, multiplying them and subtracting in a suitable way, we shall finally obtain the equality

 $(\omega_{\varrho}^{*}\Theta_{\psi}^{*}-\omega_{\psi}^{*}\Theta_{0}^{*})\ +\ (\omega_{\xi}^{*}\Theta_{\eta}^{*}-\omega_{\eta}^{*}\Theta_{\xi}^{*})\ \cdot\ (\xi_{\psi}^{*}\eta_{\varrho}^{*}\ -\xi_{\varrho}^{*}\eta_{\psi}^{*})\ =\ 0\,.$

Since the first addend is, by Remark 2, different from zero in the set B, therefore

 $\xi_{\psi}\eta_{e}^{*} - \xi_{e}^{*}\eta_{\psi}^{*} \neq 0$ for $(\psi, e) \in B$,

Which gives the verity of Lemma 3.

We shall now construct some simply connected set G and then prove that it is identical with the set of values of mapping (39) for $(\Psi, g) \in B$.

To this effect, note first that when $\Psi = \frac{\pi}{2}$ or $\Psi = -\frac{\pi}{2}$ with a fixed $\rho \in (0,1)$, from (39) we get

(40)
$$\alpha + \frac{1}{4}(\varrho + \frac{1}{\varrho} + 2).$$

Since $\rho \in (0,1)$, therefore from (40) it can be seen that a tends to a real number greater than one.

Next, note that by passing in (39) to the limit with $\Psi = \frac{\pi}{2}$. $\pm \frac{\pi}{2}$ and $\rho = 1$ at the same time, we shall get that $\alpha = 1$.

Let now $\rho \to 1$ with a fixed $\Psi \in (-\frac{\pi}{2}, \frac{\pi}{2})$. From (39) we then obtain that a tends to a^* , where a^* is defined by the formulae

$$\frac{re(1 - \alpha^{*})}{|1 - \alpha^{*}|^{2}} = 1 - Log \cos \Psi,$$

(41)

$$\frac{\mathrm{im}(1-\alpha^{*})}{|1-\alpha^{*}|^{2}} = \mathrm{tg}\,\Psi - \Psi.$$

Assume now that $\Psi \in \langle 0, \frac{\mathfrak{N}}{2} \rangle$. From (41) we have

(42)
$$[(1-x)^{2} + y^{2}] \cdot \sqrt{1 - e^{2} \left[1 - \frac{1-x}{(1-x)^{2} + y^{2}} \right]} - [(1-x)^{2} + y^{2}]$$
$$- [(1-x)^{2} + y^{2}]$$
$$1 - \frac{1-x}{2} \qquad 1 - \frac{1-x}{2}$$

$$(1 - x)^2 + y^2$$
 arc cos e $(1 - x)^2 + y^2$.

$$1 - \frac{1 - x}{(1 - x)^2 + y^2} = 0.$$

where $x = re a^*$, $y = im a^*$. It is easy to prove

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Remark 5. Equation (42) defines exactly one continuous function y = y(x) for $x \in < 0,1$). The graph of the function contains the point (0,0) and, besides this point, it is contained in the half-plane $y = \operatorname{im} \alpha^{*} < 0$ and in the disc

$$|a^* - \frac{1}{2}| < \frac{1}{2}$$

With that $\lim_{x \to 1^-} y(x) = 0$.

Note that, for $\Psi \in (-\frac{\pi}{2}, 0 >$, system of equations (41) is satisfied by α^{\pm} 's such that $x \in < 0,1$) and y = -y(x), where the function y(x) is defined by equation (42).

Passing in (39) to the limit with $g \to 0$ when $\Psi \in (-\frac{\pi}{2}, \frac{\pi}{2})$ is fixed, we get that $\alpha \to \infty$.

We shall now proceed to constructing the set G announced before. For the purpose, denote by E a set consisting of the segment < 0.1 > of the axis Ox and closed segments, parallel to the axis Oy, such that their terminal points are defined by the conditions $y = \frac{1}{2}y(x)$, $0 \le x < 1$. The set E thus constructed is a connected set [5].

Let us put

43)
$$G = C - (E \cup \{(x,y) : x \ge 1 \text{ and } y = 0\}).$$

G is an open set. It can be easily verified that any two points of the set G can be linked together by a polygonal line contained in this set. Hence it appears that G is connected. Since its complement to the plane C is connected too, therefore G is a simply connected set.

We shall now prove

Lemma 4. The set of values of mapping (39), for $(\Psi, \varrho) \in \mathbb{B}$, is identical with the set G.

Proof. It easily follows from the construction of the Set G and from Lemma 3 that

 $\alpha(B) \subset G$ and $\delta G \subset \delta \alpha(B)$,

Where δ denotes the boundary of the respective set, while $\alpha(B)$ is the image of the set B under mapping (39).

The sets $\alpha(B)$ and G are open ones, and G is simply connected, so, in order to demonstrate that $G = \alpha(B)$, it suffices to prove the inclusion $\delta \alpha(B) \subset \delta G$.

To this end, suppose that $\delta \alpha(B) - \delta G \neq \emptyset_{\psi}$ i.e. $\delta \alpha(B) - \alpha(\delta B) \neq$ $\neq \emptyset$. Consequently, there exists some α_0 such that $\alpha_0 \in \delta \alpha(B)$ and $a_0 \notin \alpha(\delta B)$. Since $a_0 \in \delta \alpha(B)$, there exist a sequence $a_n \in \alpha(B)$ such that $\alpha_n - \alpha_0$ and a sequence $(\Psi_n, \rho_n) \in B$ such that $\alpha_n = \alpha(\Psi_n, \rho_n)$. From the sequence (Ψ_n, ρ_n) one may extract a subsequence (Ψ_{n_n}, ρ_n) convergent to some point (Ψ, ϱ) . It may be the case that Rn,) $(\Psi, \rho) \in \delta B$ or $(\Psi, \rho) \in B$. If $(\Psi, \rho) \in \delta B$, then $\alpha_0 \in \alpha(\delta B)$, which is a contradiction. On the other hand, if $(\Psi, \varrho) \in B$, then from Lemma 3 it follows that there exists a neighbourhood O of the point (Ψ, ρ) such that α_{10} is a homeomorphism. Hence we have that $\alpha(0)$ is an open set contained in $\alpha(B)$. And so, the point $a(\Psi, \rho) = a_0$ would belong to a(B), together with its certain neighbourhood, thus it would not be a boundary point of this set, which yields a contradiction. Consequently, we have shown that $\delta \alpha(B) \subset \delta G$, and thereby, that $\alpha(B) = G$. The proof of Lemma 4 has thus been concluded.

Finally, we shall demonstrate that mapping (39) is a homeomorphism on the set B.

For the purpose, we shall first prove

Lemma 5. Let $\alpha(\Psi, \rho)$ stand for the mapping defined by formula (39). For each point $(\Psi_0, \rho_0) \in B$ and each curve T with parametric description

 $\begin{cases} x = x(s), \\ y = y(s), \end{cases} s_1 \leqslant s \leqslant s_2,$

where $x = re \alpha$, $y = im \alpha'$, issuing from the point $\alpha_0 = \alpha(\Psi_0, \varphi_0)$ and running in G, there exists exactly one curve L with parametric description

 $\begin{cases} \psi = \psi(s), \\ \varrho = \varrho(s), \end{cases} \quad s_1 \leqslant s \leqslant s_2, \end{cases}$

issuing from the point (Ψ_0, ρ_0) , running in B and such that along it we always have

$$\alpha(\psi(s), \rho(s)) = \alpha(s),$$

where a(s) = x(s) + iy(s).

Proof. Note first that from the local holomorphy of mapping (39) (see Lemma 3) it follows that, if $\hat{\alpha} = (\hat{x}, \hat{y})$ is some point of the curve T, corresponding to a parameter \hat{s} (i.e. $\hat{x} = \hat{x}(\hat{s}), \hat{y} = (\hat{s})$), whereas $(\hat{\psi}, \hat{\rho}) = [\Psi(\hat{s}), \rho(\hat{s})]$ belonging to B is a point such that $\alpha(\Psi(\hat{s}), \rho(\hat{s})) = \hat{\alpha}$, then in some neighbourhood of the parameter s there is exactly one curve \hat{L} with description

$$\begin{cases} \psi = \hat{\psi}(s), & \\ \varrho = \hat{\varrho}(s), \end{cases}$$
 running in B,

along which we always have $\alpha(\hat{\Psi}(s), \hat{\rho}(s)) = \alpha(s)$, under the condition $\hat{\Psi}(\hat{s}) = \Psi(\hat{s}), \hat{\rho}(\hat{s}) = \rho(\hat{s}).$

It is easy to show that, if we are given two curves L and L with descriptions, respectively,

$$\begin{cases} \psi = \dot{\psi}(s), \\ & \text{and} \end{cases} \begin{cases} \psi = \ddot{\psi}(s), \\ & \text{for } \lambda \leq s \leq \mu, \\ \varrho = \dot{\varrho}(s), \end{cases}$$

where $s_1 \leq \lambda < \mu \leq s_2$, running in B, along which we always have $\alpha(\dot{\Psi}(s), \dot{\alpha}(s)) = \alpha(s)$ and $\alpha(\ddot{\Psi}(s), \ddot{\alpha}(s)) = \alpha(s)$, then the coincidence of the initial points of these curves involves their coincidence in the whole run.

From the considerations carried out it follows that there exist a greatest such interval $\tilde{I} = \langle s_1, \tilde{s} \rangle$ and in it a respective curve \tilde{L} with description

$$\begin{cases} \Psi = \widetilde{\Psi}(s), \\ \varrho = \widetilde{\varrho}(s), \end{cases}$$

running in B and such that $\alpha(\tilde{\Psi}(s), \tilde{\varrho}(s)) = \alpha(s)$ for $s \in \tilde{I}$. Making use of the continuity and local holomorphy of mapping (39), in a simple way one can extend the curve \tilde{L} to the curve L required in the lemma, which ends the proof.

Corollary 1. Mapping (39) is a homeomorphism for $(\Psi, \varrho) \in B$. Proof. Indeed, from Lemmas 3 and 5 it follows that Mapping (39) is a convering [4]. Since the set G of its values

for $(\Psi, \rho) \in B$ is a simply connected set, therefore the mapping is a homeomorphic one [4].

To sum up, we have proved.

Lemma 6. If, for $\alpha \in G$, an extremal function satisfies the differential-functional equation of form (a), then the maximum of the functional considered is defined by formula (33), where Ψ and $t = \rho + \frac{1}{\rho}$ are the only solutions to system of equations (31), (32) (t > 2 and $\Psi \in (-\frac{\pi}{2}, \frac{\pi}{2})$). Moreover, for $\alpha \notin G$, there is no extremal function satisfying the equation of form (a).

4. EQUATION OF FORM (b)

We shall now examine the case when equation (7) is of form (b), that is,

(44)
$$\left[\frac{zf'(z)}{f(z)}\right]^2 \frac{1 + uf(z)}{f^2(z)} = \frac{(z - z_0)^2 (z - \overline{z}_0)^2}{z^2}, \quad z \in K,$$

where $z_0 = e^{i\Psi}$, $\Psi \in (0, \frac{\pi}{2} >, z_0 \neq \overline{z}_0, u \neq 0$.

By comparing the right-hand sides of equations (44) and (7), we obtain the relations

$$(45) u = -4 \cos \Psi,$$

(46)
$$B_{a} = 1 + 2 \cos^{2} \Psi$$
.

Transforming equation (44) in a suitable manner and then integrating its both sides, we get

(47)
$$\frac{\sqrt{1 + uf(z)}}{f(z)} = \frac{u}{2} \log \frac{uf(z)}{z(1 + \sqrt{1 + uf(z)})^2} = \frac{1}{z} - z - C,$$

$$z \in K,$$

where a suitable branch of logarithm has been chosen, and C stands for a constant.

Expanding the left-hand side of equation (47) in a power

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series with centre at the point z = 0 and comparing the coefficients at the same powers of z, we obtain the relations

(48)
$$C = a_2 - \frac{u}{2} (1 + \log \frac{4}{u}),$$

(49)
$$a_3 - a_2^2 - \frac{1}{8}u^2 + \frac{1}{2}ua_2 = 1$$

where $\log \frac{4}{u}$ is determined by the choice of the branch of logarithm in (47). Relation (49) is equivalent to (46).

It is known that there exists a point $z = e^{iX}$, $x \in \mathbb{R}$, such that $f(e^{iX}) = -\frac{1}{u}$. Taking this into account in equation (47), we get

$$(50)$$
 re C = 0

Since $u \neq 0$, therefore, dividing both sides of (48) by u and taking account of (6) and (50), we obtain

$$|a_2| = \frac{2}{|1-\alpha|}e^{-re}$$

By making another use of notation (6) and employing relation (45), it can be easily computed that

$$arg a_2 = Arg \frac{1}{1-\alpha} + k\pi, \quad k = 0, \pm 1, \dots$$

Since in the case under consideration $u \neq 0$, therefore $\Psi \in (0, \frac{\pi}{2})$. Hence and from (45) it follows that u is a negative real number, which, in consequence, gives

(51)
$$a_2 = \frac{-2}{11 - \alpha 1} e^{-re \frac{\alpha}{1 - \alpha} + i \operatorname{Arg} \frac{1}{1 - \alpha}}$$

Taking account of (51) in (6), we have

$$u = -4e^{-re \frac{\alpha}{1-\alpha}},$$

Which, along with (45), yields the equation

 $e^{-re \frac{\alpha}{1-\alpha}} = \cos \Psi.$

It can be easily checked that equation (52) possesses one root $\Psi \in (0, \frac{\pi}{2})$ whenever $|\alpha - \frac{1}{2}| < \frac{1}{2}$. What is more, for $|\alpha - \frac{1}{2}| > \frac{1}{2}$, equation (52) possesses no solutions in the interval $(0, \frac{\pi}{2})$.

Taking account of (52) in (46) and of notation (5), we eventually obtain

53)
$$a_3 - \alpha a_2^2 = 1 + 2e^{-re\frac{2\alpha}{1-\alpha}}$$

We have thus proved

Lemma 7. If, for a's satisfying the condition $|\alpha - \frac{1}{2}| < \frac{1}{2}$, there exists an extremal function which satisfies the equation of form (b), then the maximum of the functional considered is expressed by formula (53). For a's such that $|\alpha - \frac{1}{2}| > \frac{1}{2}$, there is no extremal function satisfying this equation.

5. EQUATION OF FORM (c)

We shall investigate the case when equation (7) is of form (c), that is,

(54)
$$\left[\frac{zf'(z)}{f(z)}\right]^2 \frac{1 + uf(z)}{f^2(z)} = \frac{(z - z_0)^4}{z^2}, \quad z \in K$$

where $z_0 = e^{i\Psi}$, $\Psi \in \langle 0, \frac{\pi}{2} \rangle$, $u \neq 0$. From the comparison of the right-hand sides of equations (7) and (54)-and from the fact that $B_0 > 0$ it follows that

$$(55) - 1, z_0 = 1,$$

and in consequence,

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u = -4, $B_0 = 3$.

(56)

an 21.

On account of (55) and (56), equation (54) is equivalent to

$$\frac{zf'(z)}{f(z)} \frac{\sqrt{1-4f(z)}}{f(z)} = \frac{(1-z)^2}{z}, \quad z \in \mathbb{K}, \quad \sqrt{1} = 1.$$

After this last equality has been integrated, we obtain

(57)
$$-\frac{\sqrt{1-4f(z)}}{f(z)} - 2 \log \frac{-4f(z)}{z(1+\sqrt{1-4f(z)})^2}$$
$$= -\frac{1}{z} + z + C,$$

where a suitable branch of logarithm has been chosen, and C denotes a constant.

Expanding each particular addend of equation (57) in a power series with centre at the point z = 0 and comparing the terms not containing the variable z, we get

(58)
$$C = 2 + a_0 - 2 \log(-1)$$
,

where log(-1) is determined by the choice of the branch of logarithm in (57).

On the other hand, also from equation (57) we easily obtain that

(59) re C = 0.

From (58) and (59) we have

(60) re $a_2 = -2$.

Since in the class S the estimate $|a_2| \leq 2$ is well known [6], from (60) it follows that

(61) $a_2 = -2.$

In view of notation (6), from (56) and (61) we have

 $\alpha = 0.$

Consequently, the following is true.

Lemma 8. If, for $\alpha = 0$, an extremal function satisfies the equation of form (c), then H(f) = 3. For $\alpha \neq 0$, extremal functions do not satisfy this equation.

6. EQUATION OF FORM (d)

We shall now examine the last case, namely, when equation (7) is of form (d), that is,

(62)
$$\left[\frac{zf'(z)}{f(z)}\right]^2 \frac{1}{f^2(z)} = \frac{(z-z_0)^2(z-\bar{z}_0)^2}{z^2}, \quad z \in \mathbb{K},$$

where

 $z_{o}=e^{i\Psi}, \quad z_{o}\neq \bar{z}_{o}, \quad \Psi\in \{0, \frac{\pi}{2}\}.$

Putting u = 0 in equation (7) and comparing the right-hand side of this equation with that of (62), we get that $\Psi = \frac{\pi}{2}$. In consequence of this, equation (62) will take the form

$$\left[\frac{zf'(z)}{f(z)}\right]^2 \frac{r}{f^2(z)} = \frac{(z^2+1)^2}{z^2},$$

whence, since f(0) = 0, we have

(63)

$$\frac{zf'(z)}{f(z)} \cdot \frac{1}{f(z)} = \frac{z^2 + 1}{z^2} \, .$$

Integrating equation (63), we obtain

(64)
$$f(z) = \frac{z}{1 - Cz - z^2}, z \in I$$

where C is a constant.

It is easy to check that function (64) belongs to the class S if and only if

(65) re C = $0, -2 \leq im C \leq 2.$

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On the other hand, from (64) it follows that

(66)

$$C = -a_2, a_3 - a_2^2 = 1.$$

Since in the case under consideration u = 0, therefore a = 1or, $a_2 = 0$. In view of this fact and from relations (64), (65) and (66) follows

Lemma 9. If the function f(z) is an extremal function and satisfies the equation of form (d), then it is of form (64) under conditions (65), and

(67)
$$H(f) = 1.$$

If $\alpha = 1$, equality (67) holds for each function of form (64), whereas if $\alpha \neq 1$, then only for the function

$$f(z) = \frac{z}{1-z^2}$$
.

7. THE FUNDAMENTAL THEOREM

So far, we have considered all possible forms of equation (7) for the extremal functions for which the functional H(f)attains its upper bound. At present, we shall formulate and prove a theorem giving a final piece of information on the maximum of the functional $|a_3 - \alpha a_2^2|$.

We introduce the following notations:

$$E_1 = \{(x,y) : 0 < x < 1 \text{ and } y(x) \leq y \leq -y(x)\}$$
$$E_2 = \{(x,y) : x > 1 \text{ and } y = 0\},$$

$$E_3 = \{(x,y) : (x - \frac{1}{2})^2 + y^2 \ge \frac{1}{4}\} - (E_2 \cap \{(0,0), (1,0)\}\},\$$

where $x = re \alpha$, $y = min \alpha$, while the function y(x) is defined by equation (42).

Theorem. For any function f of the family S, the estimate

(68)

1a2

$\int \frac{-re}{1+2e} \frac{2\alpha}{1-\alpha}$	for	α ∈ E ₁ ,
1 1	for	$\alpha = 1$,
3	for	α = 0,
4a - 3	for	a ∈ E ₂ ,
[t + cos 2'Ψ	for	α ∈ E ₃

is true, where Ψ and t are the only roots of system of equations (31), (32) (t > 2 and $\Psi \in (-\frac{\pi}{2}, \frac{\pi}{2})$). Estimate (68) is sharp.

Proof. Let us first notice that, together with the function f(z), to the class S also belongs the function $e^{-i\theta}$. $f(e^{i\theta}z)$, $\theta \in \mathbf{R}$. In consequence, as can be easily verified, the maximum of the functional $|a_3 - \alpha a_2^{-2}|$ coincides in this class of functions with that of the functional $H(f) = re(a_3 - \alpha a_2^{-2})$. So it is sufficient to confine oneself to investigating the latter.

Note that both

 $1 + 2e^{-re} \frac{2a}{1-a} > 1$

for $\alpha \in E_1$ and $t + \cos 2\Psi > 1$ for $\alpha \in E_3$ as well as $4\alpha - 3 > 1$ for $\alpha \in E_2$. Hence, and from the fact that the upper bound of the functional H(f) is a continuous function of the variable α , it is easy to prove that the estimate

 $H(f) \leq 1$, $f \in S$,

resulting from Lemma 9, holds true only for $\alpha = 1$. The extremal functions in this case are functions of form (64) which satisfy equation (d).

Let us further notice that the set E_3 is disjoint from the set E_2 and from the disc $|\alpha - \frac{1}{2}| < \frac{1}{2}$. Hence and from remark 5 it follows that E_3 is disjoint from the set E_1 , as well, and does not contain the points $\alpha = 0$ and $\alpha = 1$. Besides, $E_3 \subset G$, where the set G is defined by formula (43). From these considerations and from Lemmas 6, 7, 8 it follows that, for $\alpha \in E_3$, the estimate

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(69) $H(f) \leq t + \cos 2\Psi$, $f \in S$, holds true, where t and Ψ are the only roots of system

equations (31), (32), and this estimate is sharp. In the same manner, let us note that, since the sets E_1 and G (see (43)) are disjoint and $E_1 \subset \{\alpha : |\alpha - \frac{1}{2}| < \frac{1}{2}\}$, there-

fore, in virtue of the considerations carried out in the successive sections of the paper, for $\alpha \in E_1$, we have

(70)
$$H(f) \leq 1 + 2e^{-re \frac{2\alpha}{1-\alpha}}, \quad f \in S,$$

and the estimate is sharp. It is worth noting that extimate (70) is compatible with that obtained by Goluzin [1] for $a \in (0,1) \subset E_1$.

For $\alpha \in E_2$, by Remark 3, we obtain the well-known estimate

$$H(f) \leq 4\alpha - 3$$
.

The extremal function is the Koebe function

$$f(z) = \frac{z}{(1 \pm iz)^2}$$

From the considerations carried out and from Lemma ϑ also follows the well-known estimate $H(f) \leq 3$ for $\alpha = 0$. The extremal function is also the Koebe function

$$f(z) = \frac{z}{(1 + z)^2}$$
.

The theorem has thus been proved.

It is worth pointing out that the estimate of the functional under consideration for $\alpha < 0$, [3], is a special case of estimate (69). For $\alpha < 0$, the solution to system of equations (31), (32) is the pair $(\Psi, t) = (0, 2 - 4\alpha)$.

The theorem we have just proved gives no information on the maximum of the functional for α 's belonging to the intersection of the sets G((see (43)) and $\{\alpha : |\alpha - \frac{1}{2}| < \frac{1}{2}\}$. From the considetations carried out in sections 3 and 4 it follows that, for these α 's, either estimate (69) or (70) can hold true.

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To finish with, it is still worth stating that the Koebe function can satisfy differential-functional equation (7) only when α is real. And so, it is an extremal function only for $\alpha \leq 0$ and $\alpha \geq 1$.

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OSZACOWANIE FUNKCJONAŁU la₃ - α a²₂l W KLASIE S FUNKCJI HOLOMORFICZNYCH I JEDNOKROTNYCH DLA ZESPOLONYCH LICZB α

W niniejszej pracy badamy maksimum funkcjonału $|a_3 - \alpha a_2^2|$ w znanej klasie S funkcji holomorficznych i jednolistnych w kole jednostkowym, gdzie α jest dowolnym parametrem zespolonym. W przeprowadzonych badaniach korzystaliśmy z metody wariacyjnej, a w szczególności z równania różniczkowo-funkcyjnego A. C. Schaeffera i D. C. Spencera. W konsekwencji wyróżniono w płaszczyźnie parametra α , w sposób efektywny, sześć rozłącznych zbiorów i podano w nich odpo-

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wiednio sześć dokładnych wartości ekstremalnych funkcjonału. Wśród nich, jako szczególne przypadki, wynik J. Jenkinsa. Wartości ekstremalne wyrażaja się przez parametr a prostymi wzorami analitycznymi; jedynie w jednym ze zbiorów występuje nie rozstrzygnięta alternatywa, czyli maksimum dwóch wyrażeń analitycznych.