A C T A   U N I V E R S I T A T I S   L O D Z I E N S I S
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ON THE DIAMETERS OF SETS IN TOPOLOGICAL LINEAR SPACES WITH THE UNIFORM CONVERGENCE ON COMPACTA

In this note there has been examined a diameter of a compact set in a linear space X with a metric

$$d(x,y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{p_n(x-y)}{1+p_n(x-y)}, \quad x, y \in X,$$

where $\{p_n\}_{n \in N}$ stands for a family of seminorms defined on X, separating points of this space.

1. Let X be a linear space and let P be any family of seminorms separating points of X. If we assume that P is countable then X is metrizable and the function

$$d(x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{p_n(x-y)}{1+p_n(x-y)}, \quad x, y \in X \tag{1}$$

when $P = \{p_n\}_{n=1}^{\infty}$, is a metric agreeing with the topology of X. Then the following theorem can be proved:

Theorem 1. Let K be a compact and nonempty subset of X. If there exist points $\tilde{e}_1, \tilde{e}_2 \in E_{\overline{co}\,K}$ for which

$$p_n(e_1 - e_2) \leqslant p_n(\tilde{e}_1 - \tilde{e}_2) \tag{2}$$

for every seminorm $p_n \in P$ and every $e_1, e_2 \in E_{\overline{co}\,K}$ then

$$\varrho(k) = \varrho(E_{\overline{co}\,K}) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{p_n(\tilde{e}_1 - \tilde{e}_2)}{1 + p_n(\tilde{e}_1 - \tilde{e}_2)}$$

when ϱ means a diameter.

P r o o f. It follows from our assumptions about K that $\overline{co}\,K$ is compact so $E_{\overline{co}\,K} \subset K$. Then we have

$$\varrho(E_{\overline{co}\,K}) \leqslant \varrho(K)$$

Now we are going to prove that the inequality $\varrho(E_{\overline{co}\,K}) \leqslant \varrho(K)$ holds. Let us consider the family of sets

$$V(p_n, m) = \{x \in X : p_n(x) < \frac{1}{m}\}, \quad n, m \in N.$$

Let x_1, x_2 be any elements of K. Then because of the Krein-Milman theorem for each set $V(p_n, m)$ there exist such points y_1, y_2 that the following conditions are fulfilled:

$$y_1 = \sum_{l=1}^{k} \lambda_l e'_l, \quad y_2 = \sum_{r=1}^{s} \mu_r e''_r$$

where $e'_l, e''_r \in R_{\overline{co}\,K}$ for $l = 1, \ldots, k$, $r = 1, \ldots, s$,

$$\sum_{l=1}^{k} \lambda_l = \sum_{r=1}^{s} \mu_r = 1, \quad \lambda_l, \mu_r \geqslant 0$$

and $x_1 - y_1 \in V(p_n, m)$, $x_2 - y_2 \in V(p_n, m)$

From this and from (2) we obtain

$$p_n(y_1 - y_2) = p_n\left(\sum_{l=1}^{k} \sum_{r=1}^{s} \lambda_l \mu_r (e'_l - e''_r)\right) \leqslant p_n(\tilde{e}_1 - \tilde{e}_2),$$

so

$$p_n(x_1 - x_2) < \frac{2}{m} + p_n(\tilde{e}_1 - \tilde{e}_2).$$

Because $V(p_n, m)$ are arbitrary we conclude that $p_n(x_1 - x_2) \leqslant$

$\leqslant p_n(\tilde{e}_1 - \tilde{e}_2)$ for each n. It is now clear that

$$d(x_1,x_2) \leqslant d(\tilde{e}_1,\tilde{e}_2) = \sup_{\tilde{e}_1,\tilde{e}_2 \in E_{\overline{co}\,K}} d(e_1,e_2) = \varrho(E_{\overline{co}\,K})$$

and finally

$$\varrho(K) \leqslant \varrho(E_{\overline{co}\,K}).$$

This completes the proof.

The assumption about the existence of points $\tilde{e}_1,\tilde{e}_2$ is necessary as it follows from the example below.

Example. Let X be the set of all continuous mappings $f : (0,1> \to \mathbb{R}$. The countable family of seminorms is defined on X by the following formulae

$$p_n(f) = \max_{x \in I_n} |f(x)|, \quad n \in N,$$

where $I_n = < \frac{1}{n+1}, \frac{1}{n} >$. This family separates points of X so we can define a metric by formula (1). The Frechet space is obtained in this way with the topology of the uniform convergence on compacta on $(0,1>$.

Suppose that $A = \{f_1,f_2,f_3\} \subset X$, where

$$f_k(x) = \begin{cases} 0 & x \in (0,\frac{1}{k+1}) \cup (\frac{1}{k},1> \\ 2k^2[(k+1)\,x - 1] & x \in <\frac{1}{k+1}, \frac{2k+1}{2k(k+1)}> \qquad k = 1,2,3 \\ -2k(k+1)(kx-1) & x \in <\frac{2k+1}{2k(k+1)}, \frac{1}{k}> \end{cases}$$

Let $K = \overline{co}\,A$. Then K is compact on X and $E_{\overline{co}\,K} = E_k = A \cup \{f_0\}$, where $f_0(x) = 0$ for $x \in (0,1>$.

It is easy to see that $\varrho(E_{\overline{co}\,K}) = d(f_1,f_2) = \frac{5}{12}$ but $d(f_1, \frac{1}{2}(f_2 + f_3)) = \frac{9}{20}$. Hence

$$\varrho(K) > \varrho(E_{\overline{co}\,K}).$$

For each pair of different points $f_l, f_k \in E_{\overline{co}\,K}$ the following inequality holds

$$p_k(f_k - f_l) > 0 = p_k(f' - f_l),$$

where $f' \in E_{\overline{co}\,K} \setminus \{f_k, f_l\}$. It shows that in this example considered assumption is not fulfilled.

2. *Theorem 1* can be used for finding diameters of some compacta or sets having compact closure in locally convex and separable spaces - in particular in the space H of holomorfic functions on the unit disc Δ with the topology of uniform convergence on compacta of Δ. This space will be considered now.

Let $\{r_n\}$ be an arbitrary sequence of real numbers from the interval $(0,1)$, increasing to 1 and let $\overline{\Delta}_n = \{z : |z| \leq r_n\}$. In the space H we define a countable family of seminorms $\{p_n\}$:

$$p_n(f) = \max_{z \in \overline{\Delta}_n} |f(z)|, \quad f \in H \tag{3}$$

and the metric given by the formula (1).

Let us consider the Caratheodory class $\mathfrak{P}$ of holomorfic functions on Δ having a positive real part and normalized by the condition $f(0) = 1$. It is known ([2]) that his class forms a compact and convex subset of H and that $E_{\mathfrak{P}} = \{f(z) = \frac{\varepsilon + z}{\varepsilon + z} \in H : |\varepsilon| = 1\}$.

It follows from the principle of subordination for holomorfic functions that in the case of functions of the form $f(z) = \frac{1+z}{1-z}$, $g(z) = \frac{1-z}{1+z}$ the function $|f(z) - g(z)|$ assumes its maximum equal to $\frac{4r_n}{1 - r_n^2}$ on the disc $|z| \leq r_n$ in the point $z = r_n$.

By theorem 1 we obtain

$$\varrho(\mathfrak{P}) = d(f,g) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{4r_n}{1 + 4r_n - r_n^2}.$$

3. In [1] antipodal points were defined for arbitrary subset of a linear space.

Definition. Let X be a linear space (real or complex) and let K be a subset of X having at least two points. We shall say that two different points $p_1, p_2 \in K$ are antipodal in K if and only if for every $x_1, x_2 \in K$ and for every real number t the equality $t(p_1 - p_2) = x_1 - x_2$ implies $|t| \leqslant 1$.

It is easy to show that the following theorem is true:

Theorem 2. Let X be a linear topological space with the metric given by the formula (1) and K be an arbitrary compact subset of X. If $\tilde{x}, \tilde{y}$ are such points of K that $d(\tilde{x}, \tilde{y}) = \max_{x,y \in K} d(x,y)$, then $\tilde{x}, \tilde{y}$ are antipodal in K.

Let $\tilde{f}, \tilde{g}$ be any extremal points of the Caratheodory class $\mathfrak{P}$. Then

$$\tilde{f}(z) = \frac{e^{i\alpha} + z}{e^{i\alpha} - z}, \quad \tilde{g}(z) = \frac{e^{i\beta} + z}{e^{i\beta} - z}, \quad z \in \Delta,$$

where $\alpha, \beta \in \langle 0, 2\pi \rangle$. Let $z = re^{i\theta}$ be a fixed point of Δ. Thus

$$|\tilde{f}(re^{i\theta}) - \tilde{g}(re^{i\theta})| \leqslant \frac{4r}{1 - r^2}, \quad \theta \in \langle 0, 2\pi \rangle$$

and from the inequality

$$(4) \qquad |\cos \frac{\alpha - \beta}{2}| \leqslant \frac{2r}{1 + r^2}$$

it follows that then exists $\theta_0 \in \langle 0, 2\pi \rangle$ such that

$$(5) \qquad |\tilde{f}(re^{i\theta_0}) - \tilde{g}(re^{i\theta_0})| = \frac{4r}{1 - r^2}.$$

Assume that $\alpha, \beta \in \langle 0, 2\pi)$ and $\alpha \neq \beta$. There exists a real number $r \in (0,1)$ for which

$$|\cos \frac{\alpha - \beta}{2}| \leqslant \frac{2r}{1 + r^2}.$$

Let $\{r_n\} \subset \langle r,1)$ be an arbitrary sequence increasing to 1. From the conditions (4) and (5) it follows that $\tilde{f},\tilde{g}$ are such elements of $E_{\mathfrak{p}}$ for which

$$p_n(f - g) \leqslant p_n(\tilde{f} - \tilde{g})$$

for every seminorm (3) and every points $f,g \in E_{\overline{co}\,\mathfrak{p}}$. So we see that for each pair $\tilde{f},\tilde{g}$ of different points of $\mathfrak{p}$ there exists a sequence $\{r_n\}$ and a metric d generated by $\{r_n\}$ such that $d(\tilde{f},\tilde{g}) = \rho(\mathfrak{p})$. Because of the *Theorem 2* $\tilde{f},\tilde{g}$ are antipodal in $\mathfrak{p}$.

One can ask if the pairs of different extremal points of are the only pairs of antipodal points in $\mathfrak{p}$. A negative answer and a full characterization of antipodal points in $\mathfrak{p}$ are given in [1].

REFERENCES

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O ŚREDNICACH ZBIORÓW W LINIOWYCH PRZESTRZENIACH TOPOLOGICZNYCH

W podanej pracy bada się zależność średnicy zbioru zwartego w przestrzeni liniowej X z metryką

$$d(x,y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{p_n(x-y)}{1+p_n(x-y)} \quad \text{dla } x,y \in X,$$

gdzie $\{p_n\}_{n \in N}$ oznacza rodzinę półnorm określonych na X i rozdzielającą punkty tej przestrzeni.

W *Twierdzeniu 1* podaje się warunek dostateczny na to, aby średnica zbioru zwartego w tej przestrzeni była równa średnicy zbioru punktów ekstremalnych domknięcia otoczki wypukłej tego zbioru. Podaje się również przykład wskazujący na istotę założeń w *Twierdzeniu 1* oraz oblicza się przykładowo średnicę klasy $\mathfrak{P}$ Caratheodory'ego funkcji f holomorficznych w kole jednostkowym $\Delta = \{z \in C : |z| < 1\}$ o dodatniej części rzeczywistej i unormowanych warunkiem $f(0) = 1$.

W dalszej części pracy podaje się definicję punktów antypodycznych oraz warunek dostateczny na to, aby punkty ekstremalne były punktami antypodycznymi (*Twierdzenie 2*). Na podstawie twierdzenia 2 pokazano, że każde dwa różne punkty ekstremalne klasy Caratheodory'ego $\mathfrak{P}_1$ są punktami antypodycznymi.