A C T A   U N I V E R S I T A T I S   L O D Z I E N S I S
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POROSITY PRESERVING HOMEOMORPHISMS

In the paper the notion of a homeomorphism preserving porosity points is introduced. Several properties of such functions are proved and, among them, the following one: if a homeomorphism f and the inverse homeomorphism f^{-1} satisfy the Lipschitz condition, then f preserves points of porosity.

The notion of a point of porosity was defined by Z a j i č e k [2]. We shall give his definitions in the form which is suitable for our purposes.

Let $M \subset R$, $x \in R$. We say that x is a point of porosity of M if and only if

$$\limsup_{r \to 0^+} \frac{\gamma(x, r, M)}{r} > 0,$$

where $\gamma(x, r, M)$ is the supremum of the set $\{a > 0$, for some $z \in R$, $K(z, a) \subset K(x, r)$ and $K(z, a) \cap M = \emptyset\}$. $K(x, r)$ denotes the open sphere with the centre $x \in R$ and the radius $r > 0$.

In this work we shall define homeomorphisms preserving porosity points and we shall study some of their properties.

Definition 1. We shall say that a homeomorphism $f : R \xrightarrow{\text{onto}} R$ preserves points of porosity if and only if for every set $M \subset R$, for every $x_0 \in R$, which is a point of porosity of M, a point $f(x_0)$ is a point of porosity of $f(M)$.

Definition 2. We shall say that a homeomorphism $f : R \xrightarrow[\text{onto}]{} R$ preserves the porosity at a point $x_o \in R$ if and only if for every set $M \subset R$ having x_o as a point of porosity $f(x_o)$ is a point of porosity of $f(M)$.

Remark 1. Obviously a homeomorphism f preserves points of porosity if and only if it preserves porosity at every point $x_o \in R$.

Theorem 1. A point x_o is a point of porosity of a set M if and only if there exists a sequence $\{(a_k, b_k)\}_{k \in N}$ of open intervals which are mutually disjoint and disjoint with M such that $b_k \searrow x_o$ and

$$\lim_{k \to \infty} \frac{b_k - a_k}{2(b_k - x_o)} = \alpha > 0$$

or $a_k \nearrow x_o$ and

$$\lim_{k \to \infty} \frac{b_k - a_k}{2(a_k - x_o)} = \alpha > 0.$$

P r o o f. Necessity. Suppose that

$$\lim_{R \to 0^+} \sup \frac{\gamma\ (x_o, R, M)}{R} = \alpha > 0.$$

From the assumption it follows that there exists a sequence $\{R_n\}_{n \in N}$ such that $R_n \to 0^+$ and

$$\lim_{n \to \infty} \frac{\gamma\ (x_o, R_n, M)}{R_n} = \alpha.$$

Let $\gamma_n = \gamma\ (x_o, R_n, M)$. Suppose that x_o is a point of accumulation of M from both sides (in a contrary case the theorem is obvious). Using the mathematical induction we choose a sequence $\{(a_k, b_k)\}_{k \in N}$ of intervals such that.

1) $x_o < x_o + R_{n_k+1} < a_k < b_k < x_o + R_{n_k}$,

2) $(a_k, b_k) \cap M = \emptyset$,

3) $\gamma_{n_k} - \frac{R_{n_k}}{k} \leqslant \frac{b_k - a_k}{2}$.

We can (and shall) suppose that these intervals lie to the right from x_o (if it is impossible to construct such a sequence, then it is easy to see that we are able to construct a sequence with required properties lying to the left from x_o).

We obtain the inequality.

$$\frac{\gamma_{n_k} - \frac{1}{k} R_{n_k}}{R_{n_k}} \leq \frac{b_k - a_k}{2(b_k - x_o)},$$

which ends the proof.

Sufficiency. Put $R_k = b_k - x_o$. Then $R_k \to 0^+$

and

$$\gamma_k = \gamma\,(x_o, b_k, M) \geq \frac{b_k - a_k}{2}.$$

From the assumption we obtain

$$\limsup_{k \to \infty} \frac{\gamma_k}{R_k} \geq \alpha,$$

so x_o is a point of porosity of M.

From now we shall suppose that all homeomorphisms under considerations are increasing.

Theorem 2. A homeomorphism f preserves points of porosity if and only if preserves points of porosity of monotonely convergent sequences.

P r o o f. Necessity is obvious.

Sufficiency. Let x_o be a point of porosity of M. In virtue of *Theorem 1* there exists two sequences $\{a_n\}$ and $\{b_n\}$ such that for every $n \in N$ $b_{n+1} < a_n < b_n$, $(a_n, b_n) \cap M = \emptyset$, $a_n \searrow x_o$ and

$$\lim_{n\to\infty} \frac{b_n - a_n}{2(b_n - x_o)} = \alpha > 0.$$

From the assumption it follows that

$$\limsup_{n\to\infty} \frac{f(b_n) - f(a_n)}{2(f(b_n) - f(x_o))} = \beta > 0.$$

Indeed, suppose that

$$\limsup_{n\to\infty} \frac{f(b_n) - f(a_n)}{2(f(b_n) - f(x_o))} = 0.$$

Let us divide every interval $[f(b_n), f(a_{n-1})]$ onto equal subintervals having lengths smaller that

$$f(b_n) - f(a_n).$$

If $[y_1, y_2]$ is one of these subintervals, then obviously

$$\frac{y_2 - y_1}{2(y_2 - f(x_o))} \leq \frac{f(b_n) - f(a_n)}{2(f(b_n) - f(x_o))}.$$

Consider the set consisting of all points $f(a_n)$, $f(b_n)$ and of all points of subdivision. The elements of this set form o monotone sequence tending to $f(x_o)$. Obviously $f(x_o)$ is not a point of porosity of this sequence, but x_o is still a point of porosity of its inverse image - a contradiction.

But $(f(a_n), f(b_n)) \cap f(M) = \emptyset$, so $f(x_o)$ is a point of porosity of $f(M)$. If $a_n \nearrow x_o$, the proof is analogous.

Theorem 3. If f is a hemeomorphism such that $f(0) = 0$ and $0 < f'(0) < +\infty$, then f preserves porosity at zero.

P r o o f. Suppose that f does not preserve the porosity of some decreasing sequence at zero (for increasing sequences the proof if analogous). Then

$$\lim_{n\to\infty} \sup \frac{a_n - a_{n+1}}{2a_n} = \alpha > 0$$

and

$$\lim_{n\to\infty} \sup \frac{f(a_n) - f(a_{n+1})}{2f(a_n)} = 0.$$

Obviously there exists a subsequence $\{a_{n_k}\}$ of $\{a_n\}$ such that

$$(1) \qquad \lim_{k\to\infty} \frac{a_{n_k} - a_{n_k+1}}{2a_{n_k}} = \alpha, \quad \text{where} \quad 0 < \alpha \le \frac{1}{2}$$

and

$$(2) \qquad \lim_{k\to\infty} \frac{f(a_{n_k}) - f(a_{n_k+1})}{2f(a_{n_k})} = 0.$$

If $0 < \alpha < \frac{1}{2}$, then from (1) it follows that

$$\lim_{k\to\infty} \frac{a_{n_k} - a_{n_k+1}}{a_{n_k+1}} = \frac{2\alpha}{1 - 2\alpha} = \beta,$$

so β is a positive number. If $\alpha = \frac{1}{2}$, then

$$(3) \qquad \lim_{k\to\infty} \frac{a_{n_k} - a_{n_k+1}}{a_{n_k+1}} = +\infty.$$

Let

$$\varepsilon \in \left(0, \ \frac{2 + \beta - \sqrt{4 + \beta^2}}{2}\right).$$

Then $(1 - \varepsilon)(1 + \beta - \varepsilon) > 1$.

Let K_1 be a natural number such that

$$(4) \qquad (\beta - \varepsilon)\, a_{n_k+1} < a_{n_k} - a_{n_k+1} \quad \text{for} \quad k \geq K_1.$$

Let K_2 be a natural number such that

$$f(a_{n_k}) - f(a_{n_k+1}) < \frac{\varepsilon}{2}\, 2f(a_{n_k}) \quad \text{for} \quad k \geq K_2.$$

Hence

$$f(a_{n_k}) < \frac{1}{1-\varepsilon}\, f(a_{n_k+1}) \quad \text{for} \quad k \geq K_2,$$

$$(5) \qquad f(a_{n_k}) - f(a_{n_k+1}) < \frac{\varepsilon}{1-\varepsilon}\, f(a_{n_k+1}) \quad \text{for} \quad k \geq K_2.$$

Let now $K = \max(K_1, K_2)$. From (4) and (5) we obtain

$$\frac{f(a_{n_k})}{a_{n_k}} = \frac{f(a_{n_k+1}) + f(a_{n_k}) - f(a_{n_k+1})}{a_{n_k+1} + a_{n_k} + a_{n_k+1}} < \frac{\left(1 + \frac{\varepsilon}{1-\varepsilon}\right) f(a_{n_k+1})}{(1 + \beta - \varepsilon)\, a_{n_k+1}} =$$

$$= \frac{1}{(1 - \varepsilon)(1 + \beta - \varepsilon)} \cdot \frac{f(a_{n_k+1})}{a_{n_k+1}} \quad \text{for} \quad k \geq K.$$

Let $f'(0) = a$. Hence

$$a \leq \frac{1}{(1-\varepsilon)(1+\beta-\varepsilon)}\, a,$$

so $(1-\varepsilon)(1+\beta-\varepsilon) \leq 1$, a contradiction. In the case $\alpha = \frac{1}{2}$ we can find K_1 such that for every $k \geq K_1$, $a_{n_k+1} < a_{n_k} - a_{n_k+1}$.

Let $\varepsilon < \frac{1}{2}$. Then

$$\frac{f(a_{n_k})}{a_{n_k}} \leq \frac{1}{2(1-\varepsilon)} \cdot \frac{f(a_{n_k+1})}{a_{n_k+1}} \quad \text{for} \quad k \geq \max(K_1, K_2),$$

so $a \leq \frac{1}{2(1-\varepsilon)}\, a$. Hence $\varepsilon \geqslant \frac{1}{2}$ - a contradiction.

Remark 2. If f is a homeomorphism preserving the porosity at x_0 and g is a homeomorphism preserving the porosity at $y_0 = f(x_0)$, then the homeomorphism $h = g \circ f$ preserves the porosity at x_0.

Remark 3. If f is a homeomorphism preserving the porosity at x_0, then for every $c \in R$ the homeomorphism g of the form $g(x) = f(x) + c$ preserves the porosity at x_0.

Remark 4. If f is a homeomorphism for which $0 < f'(x_0) < +\infty$, then f preserves the porosity at x_0.

P r o o f. To prove this fact it suffices to consider the homeomorphism $g(x) = f(x + x_0) - f(x_0)$ and to apply *Theorem 3*.

Theorem 4. There exists a homeomorphism f such that $f(0) = 0$, $f'(0) = 0$ and f does not preserve the porosity at zero.

P r o o f. Put

$$f(x) = \begin{cases} \dfrac{1}{4^n} & \text{for} \quad x = \dfrac{1}{2^n} \quad n \in N \\[2ex] \dfrac{4^{n+1}+1}{16^{n+1}} & \text{for} \quad x = \dfrac{3}{2^{n+2}} \quad n \in N \\[2ex] \text{linear in the intervals of the form} \\ \left[\dfrac{1}{2^n}, \dfrac{3}{2^{n+2}}\right] \text{ and } \left[\dfrac{3}{2^{n+2}}, \dfrac{2}{2^{n+1}}\right] \\[2ex] 0 & \text{for} \quad x = 0 \end{cases}$$

If we denote $a_n = \dfrac{1}{2^{n+1}}$, $b_n = \dfrac{3}{2^{n+2}}$,

then

$$\lim_{n\to\infty} \frac{b_n - a_n}{a_n} = \frac{1}{3},$$

so zero is a point of porosity of the set $\bigcup\limits_{n=1}^{\infty} [b_{n+1}, a_n]$. Simultaneously

$$\lim_{n\to\infty} \frac{f(b_n) - f(a_n)}{f(b_n)} = \lim_{n\to\infty} \frac{(4^{n+1} + 1 - 4^{n+1}) \cdot 16^{n+1}}{16^{n+1}(1 + 4^{n+1})} = 0,$$

so $f(0) = 0$ is not a point of porosity of the set

$$\bigcup_{n=1}^{\infty} [f(b_{n+1}), f(a_n)].$$

To prove that $f'(0) = 0$ it suffices only to observe that for $x \in \left[\dfrac{1}{2^{n+1}}, \dfrac{1}{2^n}\right]$ we have $\dfrac{f(x)}{x} < \dfrac{1}{2^{n-1}}$.

Theorem. 5. If f is a homeomorphism such that $f(0) = 0$. A homeomorphism f preserves the porosity at zero if and only if for every sequence $\{a_n\}$, $a_n \searrow 0$ there exist a pair of numbers

$\alpha, \beta \in [0,1)$ such that the following implication holds:
if

$$\lim_{n\to\infty} \inf \frac{a_{n+1}}{a_n} = \alpha,$$

then

$$\lim_{n\to\infty} \inf \frac{f(a_{n+1})}{f(a_n)} = \beta.$$

P r o o f. In virtue of *Theorem 2* f preserves the porosity at zero if and only if f preserves the porosity at zero of every monotone sequence $\{a_n\}$ tending to zero. The last fact is equivalent to the following implication:
if

$$\lim_{n\to\infty} \sup \frac{a_n - a_{n+1}}{a_n} = \hat{\alpha},$$

then

$$\lim_{n\to\infty} \sup \frac{f(a_n) - f(a_{n+1})}{f(a_n)} = \hat{\beta},$$

where $\hat{\alpha}, \hat{\beta}$ are some numbers in $(0;1]$. If we put $\alpha = 1 - \hat{\alpha}$, $\beta = 1 - \hat{\beta}$, we obtain the thesis.

Example 1. Let $f(x) = x^p$, where $p > 0$. Then $f'(0) = +\infty$ for $0 < p < 1$ and $f'(0) = 0$ for $p > 1$.

So from *Theorem 3* we are not able to conclude if f preserves the porosity at zero.

Let $\{x_n\}_{n\in N}$ be an arbitrary sequence such that

$$\lim_{n\to\infty} \inf \frac{x_n}{x_{n+1}} = \alpha, \quad 0 \le \alpha < 1.$$

Hence there exists a subsequence $\{x_{n_k}\}$ such that

$$\lim_{k\to\infty} \frac{x_{n_k}}{x_{n_k+1}} = \alpha, \quad \text{then} \quad \lim_{k\to\infty} \left(\frac{x_{n_k}}{x_{n_k+1}}\right)^p = \alpha^p,$$

so

$$\liminf_{n\to\infty} \frac{f(x_n)}{f(x_{n+1})} \le \alpha^p < 1.$$

and in virtue of *Theorem 5* f preserves the porosity at zero.

Example 2. Let

$$f(x) = \begin{cases} \log(x^{-1})^{-1} & \text{for} \quad x > 0 \\ 0 & \text{for} \quad x = 0. \end{cases}$$

We have $f'(0) = +\infty$. Let $\{x_n\}$ be a sequence such that $x_n \searrow 0$ and $\{x_{n_k}\}$ such subsequence that

$$\lim_{k\to\infty} \frac{x_{n_k}}{x_{n_k+1}} = \alpha, \quad 0 \le \alpha < 1.$$

Then

$$\lim_{k\to\infty} \left(\frac{-\log x_{n_k}}{-\log x_{n_k+1}}\right)^{-1} = \lim_{k\to\infty} \frac{x_{n_k}}{x_{n_k+1}} = \alpha,$$

so

$$\liminf_{n\to\infty} \frac{f(x_n)}{f(x_{n+1})} \le \alpha < 1$$

and f preserves the porosity.

Remark 5. From the above examples it follows that the Lipschitz condition is not necessary for preserving the porosity.

Theorem 6. If a homeomorphism f and the inverse homeomorphism f^{-1} fulfills the Lipschitz condition, then f preserves points of porosity.

P r o o f. Let $M \subset R$ and $x_o \in R$ be a point of porosity of the set M. In virtue of *Theorem 1* there exists a sequence of disjoint intervals $\{(a_n, b_n)\}$ such that $b_n \searrow x_o$ or $b_n \nearrow x_o$,

$$\lim_{n\to\infty} \frac{b_n - a_n}{2(b_n - x_o)} = \alpha > 0$$

and $(a_n, b_n) \cap M = \emptyset$ for every $n \in N$.

We shall consider the first case, the proof in the second is quite similar. From the assumption there exist two positive constants such that for every $x, y \in R$

$$C_1 \, |f(y) - f(x)| \le |y - x| \le C_2 \, |f(y) - f(x)|.$$

Hence for every $n \in N$

$$|b_n - a_n| \le C_2 \, |f(b_n) - f(a_n)|$$

$$|b_n - x_o| \geqslant C_1 \, |f(b_n) - f(x_o)|$$

So for every $n \in N$

$$\frac{f(b_n) - f(a_n)}{2(f(b_n) - f(x_o))} \ge \frac{C_1(b_n - a_n)}{2C_2(b_n - x_o)}.$$

Obviously $(f(a_n), f(b_n)) \cap f(M) = \emptyset$ for every $n \in N$, so

$$\lim_{n\to\infty} \sup \frac{f(b_n) - f(a_n)}{2(f(b_n) - f(x_o))} \ge \alpha \frac{C_1}{C_2} > 0.$$

From the last inequality it follows that f preserves the porosity at x_o, so from the arbitrariness of x_o f preserves points of porosity.

Example 3. We shall construct a homeomorphism preserving points of porosity such that the inverse homeomorphism does not preserve points f porosity.

Let

$$x_n = \frac{1}{2^{n-1}}, \quad y_n = \sum_{i=n}^{\infty} \frac{1}{i^2}, \quad f(x_n) = y_n \quad \text{for every} \quad n \in N.$$

Let f be a linear function on every interval of the form $[x_{n+1}, x_n]$ and let $f(0) = 0$ (this homeomorphism was constructed in [1] as an example of the homeomorphism which preserves points of density, for which the inverse homeomorphism does not preserve points of density).

It is easy to see that f does not preserve the porosity at zero. Indeed, we have

$$\lim_{n\to\infty} \frac{x_n - x_{n+1}}{x_n} = \frac{1}{2} \quad \text{and} \quad \lim_{n\to\infty} \frac{f(x_n) - f(x_{n+1})}{f(x_n)} = 0.$$

Now we shall prove that f^{-1} preserves the porosity at zero.

Let $M \subset R$ be a set having zero as a point of porosity. From the *Theorem 1* there exists a sequence of disjoint intervals $\{(f(a_n), f(b_n))\}$ such that $((f(a_n), f(b_n)) \cap M = 0$ for every $n \in N$ and

$$\lim_{n\to\infty} \frac{f(b_n) - f(a_n)}{f(b_n)} = \alpha > 0.$$

Suppose, that there exists a subsequence $\{(f(a_{n_k}), f(b_{n_k})\}$ the elements of which are included in some interval $[y_{n+2}, y_n]$. Then.

$$\lim_{n\to\infty} \frac{f(b_{n_k}) - f(a_{n_k})}{f(b_{n_k})} = 0,$$

which is impossible.

After a while of thinking one can see that almost every interval $[f(a_n), f(b_n)]$ must contain some interval $[y_{k+1}, y_k]$, so almost every interval $[a_n, b_n]$ must contain an interval $[x_{k+1}, x_k]$.

Hence

$$\lim_{n\to\infty} \sup \frac{b_n - a_n}{b_n} > 0.$$

Remark 6. As in the case of homeomorphisms preserving density points (compare [1]) one can observe that the sum and the product of two homeomorphisms defined on $[0, +\infty)$ preserving points of porosity, and vanishing at zero also preserves points of porosity, but the limit of uniformly convergent sequence of homeomorphism preserving points of porosity needs not preserve points of porosity.

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HOMEOMORFIZMY ZACHOWUJĄCE POROWATOŚĆ

W pracy tej wprowadzone jest pojęcie homeomorfizmu zachowującego punkty porowatości.

Dowodzi się kilku własności takich funkcji, między innymi, że jeżeli homeomorfizm f i homeomorfizm odwrotny f^{-1} spełniają warunek Lipschitza, to f zachowuje punkty porowatości.