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# ON SOME ISOPERIMETRIC PROBLEM IN THE SPACE OF GENERALIZED FINITE MEASURES

In the paper there has been considered a minimal value of the functional

$$F_o(\varphi) = F_o(\int_I f_o(t)\varphi(t)dt), \ldots, \int_I f_n(t)\varphi(t)dt)$$

in the family $L_{c,k}$ of functions defined at least on the interval $I = [a,b]$, measurable, Lebesgue intergrable and satisfying the conditions

$$\int_I \varphi(t)dt = c, \qquad \int_I |\varphi(t)|dt \leqslant k$$

for $k \geqslant |c|$, $k > 0$, under the constraints

$$F_j(\varphi) = F_j(\int_I f_o(t)\varphi(t)dt, \ldots, \int_I f_n(t)\varphi(t)dt) = l_j,$$

$$j = 1, \ldots, m.$$

The functions $f_o, \ldots, f_n$ are defined and continous on the interval I, whereas the functions $F_o, \ldots, F_m$, defined in the space $R^{n+1}$, are of class $C^1$.

## 1. INTRODUCTION

Denote by $f_o, \ldots, f_n$ a finite system of real-valued functions defined and continuous on the interval $I = [a, b]$, whereas by $F_o, \ldots, F_m$ - a system of functions of the class $C^1$, defined in the space $R^{n+1}$. Let further $L_{c,k}$ denote the family of functions $\varphi$ defined at least on I, measurable, Lebesgue integrable, and satisfying the conditions:

$$\int_I \varphi(t)\,dt = c, \qquad \int_I |\varphi(t)|\,dt \leqslant k, \tag{1.1}$$

where c and k are any fixed real numbers such that $k \geqslant |c|$, $k > 0$.

Let the topology in $L_{c,k}$ be a weak topology induced from the space $L^1$ of all summable functions with the first power on the interval I.

Consider the following variational problem:

*Problem 1.* Determine the minimal value of the functional

$$F_0(\varphi) = F_0\Big(\int_I f_0(t)\,\varphi(t)dt,\ \int_I f_1(t)\,\varphi(t)dt,\ \ldots,\ \int_I f_n(t)\,\varphi(t)dt\Big) \tag{1.2}$$

in the family $L_{c,k}$, under the conditions

$$F_j(\varphi) = F_j\Big(\int_I f_0(t)\,\varphi(t)dt,\ \int_I f_1(t)\,\varphi(t)dt,\ \ldots,\ \int_I f_n(t)\,\varphi(t)dt\Big) = l_j, \qquad j = 1, 2, \ldots, m, \tag{1.3}$$

where $l_j$ are any preassigned finite real numbers.

Simple examples show that the variational problem thus formulated does not always possess a solution. This follows from the fact that the family $L_{c,k}$, with arbitrary values of c,k, $k \geqslant |c|$, $k > 0$, is not weakly compact in itself.

Consequently, we shall extend *Problem 1* to a variational problem in the space of generalized finite measures, in the sense of I o f f e and T i k h o m i r o v' s definition [1]. For it turns out that the problem thus extended possesses a solution and preserves the lower bound of the functional.

The method of extending variational problems was initiated by H i l b e r t and applied, among others, by Y o u n g [2] and K r o t o v [3].

The present paper includes a method of solving the extended problem.

## 2. NOTATION AND AUXILIARY THEOREMS

Denote by $\Omega$ the family of all distribution functions $\Phi$ satisfying the following conditions:

$1^{\circ}$ $\Phi$ is a non-decreasing function,

$2^{\circ}$ $\Phi$ is a function continuous on the right,

$3^{\circ}$ $\Phi(t) = 0$ for $t \in (-\infty, a)$,

$4^{\circ}$ $\Phi(t) = \Phi(b)$ for $t \in [b, \infty)$.

For further considerations, we shall introduce the following notation:

$\mathfrak{M}$ - the family of generalized finite measures defined on the $\sigma$-field of Borel sets of the space R, cumulated on the interval I,

$\mathfrak{M}_o$ - the family of finite (in the usual sense) measures defined on the $\sigma$-field of Borel sets of the space R, cumulated on the interval I,

$\tilde{\mathfrak{M}}_o$ - the subset of the family $\mathfrak{M}_o$, composed of measures absolutely continuous with respect to the Lebesgue measure,

$\mathfrak{M}(M)$ - the subset of the family $\mathfrak{M}$, composed of generalized measures absolutely continuous with respect to the Lebesgue measure, with the derivative $\varphi(t) \in [-M,M]$ for almost all $t \in R$,

$\mathfrak{M}_o(M)$ - the subset of the family $\mathfrak{M}_o$, composed of measures absolutely continuous with respect to the Lebesgue measure, with the derivative $\varphi(t) \in [0,M]$ for almost all $t \in R$,

$\mathfrak{M}_{c,k}$ - the subset of the family $\mathfrak{M}$, composed of those generalized measures which satisfy the condition $\mu(I) = c$ and can be represented in the form

$$\mu = \mu_1 - \mu_2,$$

where $\mu_1, \mu_2$ are measures in the usual sense, such that

$$\mu_1(I) + \mu_2(I) = k$$

for $k \geq |c|$, $k > 0$.

It is not hard to show that the family $\mathfrak{M}_{c,k}$ is identical with the family of those generalized measures $\mu \subset \mathfrak{M}$ which satisfy the conditions

$$\mu(I) = c, \qquad |\mu|(I) \leqslant k$$

for $k \geqslant |c|$, $k > 0$.

From the form of a measure absolutely continuous with respect to the Lebesgue measure it follows that all the families of measures we have introduced are non-empty.

Let further $C([a,b])$ denote the space of functions continuous on the closed interval $[a,b]$, whereas $C^*([a,b])$ - the space conjugate to it (cf. [4], p. 28). Let the topology in the family $\mathfrak{M}$ be a weak-*-topology induced from the space $C^*([a,b])$.

In order to formulate a variational problem extended with respect to *Problem 1*, we shall prove

*Lemma 2.1.* Let $\varepsilon$ be any positive number, and let $g_j$, $j = 1, \ldots, m$, be fixed continuous functions on R.

For any measure $\mu \in \mathfrak{M}_0$, there exists a measure $\mu_0 \in \mathfrak{M}_0(M_0)$, with some $M_0$, such that $\mu(I) = \mu_0(I)$ and

$$(2.1) \qquad \left|\int_I g_j(t)d\mu - \int_I g_j(t)d\mu_0\right| < \varepsilon, \quad j = 1, \ldots, m.$$

P r o o f. For a given measure $\mu \in \mathfrak{M}_0$, let us put

$$\Phi(t) = \mu((-\infty, t]).$$

The function $\Phi$ thus defined is non-decreasing, continuous on the right and satisfying the conditions

$$\Phi(t) = 0 \quad \text{for} \quad t \in (-\infty, a),$$
$$\Phi(t) = \Phi(b) \quad \text{for} \quad t \in [b, \infty),$$

and consequently, it belongs to $\Omega$. Conversely, let us take any distribution function $\Phi \in \Omega$. It can be shown (cf. [5], p. 176) that the distribution function $\Phi$ induces some measure $\mu \in \mathfrak{M}_0$ called the Lebesgue-Stieltejes measure.

Consequently, between elements of the families $\Omega$ and $\mathfrak{M}_0$ there, exists a one-to-one correspondence and, moreover, (2.2) $$\int_I g_j(t)d\Phi = \int_I g_j(t)d\mu, \quad j = 1, \ldots, m,$$

where $\mu$ is the measure induced by the distribution function $\Phi$.

So, let $\mu$ be any measure belonging to the family $\mathfrak{M}_o$, $\Phi$ - the distribution function induced by it. For every non-decreasing function $\Phi$, there exists a piecewise constant function $\bar{\Phi} \in \Omega$, such that

(2.3) $$\left|\int_I g_j(t)d\Phi - \int_I g_j(t)d\bar{\Phi}\right| < \frac{\varepsilon}{2}, \quad j = 1, \ldots, m.$$

Denote, successively, by $\tau_1, \ldots, \tau_n$ jump points of the function $\bar{\Phi}$. The function $\bar{\Phi}$ can be so chosen that $a < \tau_1 < \tau_2 < \ldots < \tau_n < b$. Through the point $(\tau_i, \bar{\Phi}(\tau_i))$ let us draw a straight line with the equation $y = M_o^i(t - \tau_i) + \bar{\Phi}(\tau_i)$. With $M_o^i$ sufficiently large, the line intersects the graph of the function $\bar{\Phi}$ at one point. Denote it by $(\tau_i', \bar{\Phi}(\tau_i'))$. Since the number of jumps is finite, one may choose $M_o$ universal for every $i$. Let us now define the function

$$\tilde{\Phi}_{M_o}(t) = \begin{cases} M_o(t - \tau_i) + \bar{\Phi}(\tau_i) & \text{for} \quad \tau_i' \leqslant t \leqslant \tau_i, \\ & \qquad i = 1, \ldots, n \\ \bar{\Phi}(t) & \text{for the remaining } t\text{'s}. \end{cases}$$

The function $\tilde{\Phi}_{M_o}(t)$ thus defined is continuous, with the derivative equal to zero or $M_o$. With $M_o \to \infty$, $\tilde{\Phi}_{M_o}(t) \to \bar{\Phi}(t)$ for every $t$. By Helly's theorem,

(2.4) $$\int_I g_j(t)\, d\tilde{\Phi}_{M_o}(t) \longrightarrow \int_I g_j(t)d\bar{\Phi}(t), \quad j = 1, \ldots, m,$$

consequently, there exists some $M_o$ such that the function $\tilde{\Phi} = \tilde{\Phi}_{M_o}$ satisfies the following inequalities

(2.5) $$\left|\int_I g_j(t)d\bar{\Phi}(t) - \int_I g_j(t)d\tilde{\Phi}(t)\right| < \frac{\varepsilon}{2}, \quad j = 1, \ldots, m.$$

Let $\mu_o$ be the measure induced by the distribution function $\tilde{\Phi}$. Since $\tilde{\Phi}$ is an absolutely continuous function, the measure $\mu_o$, corresponding to it, is absolutely continuous with respect to the Lebesgue measure. Moreover, the derivative $\varphi(t)$ of the measure $\mu_o$ with respect to the Lebesgue measure is almost everywhere equal to the derivative of the function $\tilde{\Phi}$. Consequently, the function $\varphi$ satisfies the inequality $0 \leqslant \varphi(t) \leqslant M_o$ for almost all $t \in R$. Hence it appears that $\mu_o \in \mathfrak{M}_o(M_o)$.

In virtue of inequalities (2.3), (2.5) and equality (2.2), we obtain condition (2.1) of the proposition.

Moreover, $\mu(I) = \Phi(b) - \Phi(a^-) = \tilde{\Phi}(b) - \tilde{\Phi}(a^-) = \mu_o(I)$, which concludes the proof.

By the last lemma, the definitions of the families $\mathfrak{M}_o(M)$, $\tilde{\mathfrak{M}}_o$ and of the weak $*$-neighbourhood, we obtain

*Lemma 2.2.* The family $\tilde{\mathfrak{M}}_o$ is dense in $\mathfrak{M}_o$, that is, $\bar{\tilde{\mathfrak{M}}}_o = \mathfrak{M}_o$, where $\bar{\tilde{\mathfrak{M}}}_o$ denotes the closure in the weak-$*$-topology.

### 3. FORMULATION OF THE EXTENDED PROBLEM

Let a mapping assigning to elements of the set $L_{c,k}$ generalized measures absolutely continuous with respect to the Lebesgue measure, belonging to the family $\mathfrak{M}_{c,k}$, be defined by the equality

$$\mu(A) = \int_A \varphi(t)\, dt. \tag{3.1}$$

So, we shall formulate a problem extended with respect to *Problem 1* from the space $L_{c,k}$ to the space $\mathfrak{M}_{c,k}$.

*Problem 2.* Determine the minimal value of the functional

$$F_o(\mu) = F_o\Big(\int_I f_o(t)\,d\mu, \ldots, \int_I f_n(t)\,d\mu\Big) \tag{3.2}$$

in the family $\mathfrak{M}_{c,k}$ under the conditions

$$F_j(\mu) = F_j\left(\int_I f_o(t)d\mu, \ldots, \int_I f_n(t)d\mu\right) = l_j, \tag{3.3}$$

$$j = 1, 2, \ldots, m.$$

From (3.1) and *Lemma 2.2* it follows that *Problem 2* satisfies all the definition conditions of the extended problem.

Note that, if there exists at least one generalized measure $\mu \in \mathfrak{M}_{c,k}$ satisfying conditions (3.3), then variational *Problem 2* possesses a solution. This follows from the fact that, by the Alaoglu theorem [6], the family $\mathfrak{M}_{c,k}$ is weakly-*-compact in itself, and the functionals $F_i$, $i = 0, 1, \ldots, m$, are continuous in the weak-*-topology of the space $\mathfrak{M}_{c,k}$.

In the sequel, we shall give a method of solving *Problem 2*. With that end in view, we shall consider some other auxiliary sets.

## 4. GENERAL CHARACTERIZATION OF THE FAMILIES $L_o(M)$, $L(M)$, $\mathfrak{M}_o(M)$

Denote by $L(M)$ the set of Lebesgue measurable functions $\varphi$ satisfyinf almost everywhere the conditions:

$$(4.1)\qquad \begin{aligned} &\varphi(t) \in [-M, M] \quad \text{for} \quad t \in I, \\ &\varphi(t) = 0 \quad \text{for} \quad t \in R \setminus I, \end{aligned}$$

and by $L_o(M)$ - the set of Lebesgue measurable functions $\varphi$ satisfying almost everywhere the conditions:

$$(4.2)\qquad \begin{aligned} &\varphi(t) \in [0, M] \quad \text{for} \quad t \in I, \\ &\varphi(t) = 0 \quad \text{for} \quad t \in R \setminus I. \end{aligned}$$

From the definitions of the families $L(M)$ and $L_o(M)$ there follows at once

*Lemma 4.1.* The set $L(M)$ can be represented in the form of the algebraic difference $L_o(M) - L_o(M)$.

Moreover, there takes place

*Lemma 4.2.* Between elements of the sets $L_o(M)$ and $\mathfrak{M}_o(M)$, as well as $L(M)$ and $\mathfrak{M}(M)$, there holds a one-to-one correspondence.

The proof follows from the definitions of the families $L(M)$ and $L_o(M)$ we have adopted and the fact that the assignment in question is given by formula (3.1).

Let $\mu = \mu_1 - \mu_2$ be a Jordan decomposition of a generalized measure $\mu \in \mathfrak{M}$.

*Lemma 4.3.* The set $\mathfrak{M}$ can be represented in the form of the algebraic difference $\mathfrak{M}_o - \mathfrak{M}_o$, whereas $\mathfrak{M}(M)$ - in the form of the algebraic difference $\mathfrak{M}_o(M) - \mathfrak{M}_o(M)$.

The proof follows from the theorem on the Jordan decomposition of a generalized measure $\mu$ and from the Radon-Nikodym theorem [7].

## 5. PROPERTIES OF THE SET $V_o(M)$

Let us denote

$$V_o = \left\{ x \in R^{n+2};\ \ x = x(\mu) = \left(\int_I f_o(t)\,d\mu,\ \ldots,\ \int_I f_n(t)\,d\mu,\ \int_I d\mu\right),\ \mu \in \mathfrak{M}_o \right\}$$

and

$$V_o(M) = \left\{ x \in R^{n+2};\ \ x = x(\mu) = \left(\int_I f_o(t)\,d\mu,\ \ldots,\ \int_I f_n(t)\,d\mu,\ \int_I d\mu\right),\ \mu \in \mathfrak{M}_o(M) \right\}.$$

The sets $V_o$ and $V_o(M)$ may be treated as sets of $(n+2)$-dimensional vectors of the space $R^{n+2}$ with coordinates

$$\int_I f_o(t)\,d\mu,\ \ldots,\ \int_I f_n(t)\,d\mu,\ \int_I d\mu.$$

*Lemma 5.1.* The set $V_o(M)$ is sequentially compact in itself and convex.

P r o o f. Note that the set $V_o(M)$ can be represented in the following equivalent form

$$(5.1)\qquad V_o(M) = \Big\{ x \in R^{n+2};\ x = \Big(\int_I f_o(t)\varphi(t)dt, \ldots, \int_I f_n(t)\varphi(t)dt, \int_I \varphi(t)dt\Big), \varphi \in L_o(M)\Big\}.$$

The proposition follows from the convexity and weak - * -compactness of the family $L_o(M)$.

Let now $v = (v^o, \ldots, v^{n+1})$ be any vector of the space $R^{n+2}$. Consider a function $(v, f)$ where $f = (f_o, \ldots, f_{n+1})$, $f_{n+1} = 1$.

In the sequel, we shall assume that, with each $v \in R^{n+2}$,

$$(5.2)\qquad (v,f) = \sum_{i=0}^{n+1} v^i f_i(t) \neq 0$$

on any set of the positive Lebesgue measure.

Under this assumption we shall prove the following

*Lemma 5.2.* Int $V_o(M) \neq \emptyset$.

P r o o f. Suppose, centrariwise, that Int $V_o(M) = \emptyset$. Denote by $H$ a carrying hyperplane of the set $V_o(M)$. It is well known (cf. [8], p. 199) that, if Int $V_o(M) = \emptyset$, then dim $H < n + 2$. Consequently, there exists a vector $v_o = (v_o^o, \ldots, v_o^{n+1}) \neq 0$ orthogonal to $H$, i.e. such that $(v_o, x) = 0$ for any $x \in V_o(M)$. So,

$$(v_o, (\int_I f_o(t)\varphi(t)dt, \ldots, \int_I f_n(t)\varphi(t)dt, \int_I \varphi(t)dt)) =$$

$$= \int_I \Big(\sum_{i=0}^{n+1} v_o^i f_i(t)\Big)\varphi(t)dt = 0$$

for any function $\varphi \in L_o(M)$. Thus, $(v_o, f(t)) = 0$ almost everywhere on I. This fact, contradicting assumption (5.2) proves the proposition.

It follows from *Lemmas* 5.1. and 5.2 that, at each boundary point $x^*$ of the set $V_o(M)$, there exists a vector $v_M$ supporting the set.

*Definition* 5.1. The measure $\mu_o^* \in \mathcal{M}_o(M)$, satisfying the condition

$$x^* = x(\mu_o^*),$$

is called a boundary measure with respect to the set $V_o(M)$.

Let $\varphi_o^*$ be the derivative of the boundary measure $\mu_o^*$. We shall prove

*Theorem* 5.1. The derivative of the boundary measure with respect to the set $V_o(M)$ is double-valued, that is,

$$\varphi_o^*(t) = \begin{cases} M & \text{for} \quad t \in Z(v_M) \quad \text{a.e.,} \\ 0 & \text{for} \quad t \in R \setminus Z(v_M) \quad \text{a.e.,} \end{cases} \tag{5.3}$$

where

$$Z(v) = \left\{ t \in I;\ (v, f(t)) = \sum_{i=0}^{n+1} v^i f_i(t) \geq 0, f_{n+1} = 1; \quad v \in R^{n+2} \right\}, \tag{5.4}$$

$v_M = (v_M^o, \ldots, v_M^{n+1})$ is a vector supporting the set $V_o(M)$ at the point $x^*$.

P r o o f. From the definition of a supporting vector it follows that $(v_M, x^* - x) \geq 0$ for any $x \in V_o(M)$. Then

$$(v_M, x^*) - (v_M, x) = \int_I (v_M, f(t)) \left[ \varphi_o^*(t) - \varphi(t) \right] dt \geq 0$$

for any function $\varphi \in L_o(M)$. Hence we get

$$\varphi_o^*(t) = \begin{cases} M & \text{when} \quad (v_M, f(t)) \geqslant 0 \quad \text{a.e.,} \\ 0 & \text{when} \quad (v_M, f(t)) < 0 \quad \text{a.e.,} \end{cases}$$

which ends the proof.

## 6. PROPERTIES OF THE SET $V_o$

Let us recall that

$$V_o = \{x \in R^{n+2};$$

$$x = x(\mu) = (\int_I f_o(t)d\mu, \ \dots, \ \int_I f_n(t)d\mu, \ \int_I d\mu), \ \mu \in \mathfrak{M}_o\}.$$

We shall prove

*Lemma 6.1.* The set $V_o$ is a convex cone with apex at zero. Moreover, dim $V_o$ = n+2, and Int $V_o \neq \emptyset$.

P r o o f. Since, with any $M > 0$, the set $V_o(M) \subset V_o$, therefore, by *Lemma 5.2,* Int $V_o \neq \emptyset$. The remaining properties of the set $V_o$ follow immediately from the definitions of a cone and of a convex set.

In the sequel, we shall prove that the set $V_o$ is a closed set, and characterize its boundary points. For the purpose, consider a hyperplane $H_d$ in the space $R^{n+2}$, with the equation $x_{n+1} = d$, $d > 0$. Let $V_o \cap H_d$ be a cross section of the cone $V_o$ by this hyperplane. The set $V_o \cap H_d \neq \emptyset$ since the set of measures satisfying the condition $x_{n+1} = d$, $d > 0$, is not empty.

*Lemma 6.2.* The set $V_o \cap H_d$ is sequentially compact in itself.

P r o o f. The set $V_o \cap H_d$ is the image of the set of those measures belonging to $\mathfrak{M}_o$ which satisfy the condition $\int_I d\mu = d$. The set of those measures is weakly-*-compact in itself, therefore, from any sequence $\{x_m\}$ of elements of the set $V_o \cap H_d$ of the form

$$x_m = x(\mu_m) = (\int_I f_o(t)d\mu_m, \ldots, \int_I f_n(t)d\mu_m, \int_I d\mu_m)$$

one may choose a subsequence convergent to the element

$$x_o = x(\mu_o) = (\int_I f_o(t)d\mu_o, \ldots, \int_I f_n(t)d\mu_o, \int_I d\mu_o)$$

belonging to $V_o \cap H_d$, where $\int_I d\mu_o = d$, which completes the proof.

From the fact that the cross section $V_o \cap H_d$ is sequentially compact in itself with any $d > 0$ there follows

*Lemma 6.3.* $V_o$ is a closed set.

*Remark 6.1.* It follows from *Lemma 2.1.* that each point $x \in V_o \cap H_d$ is the limit of a sequence of point $\{x_n\}$ where $x_n \in V_o(M_n) \cap H_d$ with some $M_n$. If $x$ is a boundary point of the set $V_o$, then the sequence $\{x_n\}$ can be so chosen that $x_n$ should be a boundary point of the set $V_o(M_n)$ with every $n$.

*Remark 6.2.* The sets $V_o(M)$ form a family of ascending sets.

## 7. CHARACTERIZATION OF BOUNDARY POINTS OF THE SET $V_o$

Let now $\{\Phi_n\}$ be a sequence of absolutely continuous distribution functions belonging to $\Omega$ and such that $\Phi_n(b) = d$, $d > 0$. Then

$$\Phi_n(t) = \int_a^t \varphi_n(t)dt.$$

Denote by $\{M_n\}$ a sequence of positive numbers, tending to $+\infty$. Assume that $\varphi_n(t)$, $n = 1, 2, \ldots$, is a piecewise constant function taking the values $M_n$ on $k_n$ disjoint intervals $[t_1^n, t_2^n], \ldots, [t_{2k_n-1}^n, t_{2k_n}^n]$ contained in I, and zero at the remaining points of the interval I. The number of intervals $k_n$ may alter as $n$ does.

*Theorem 7.1.* Let N be a fixed positive integer. If, with every $n$, the number of intervals $k_n$, in which $\varphi_n(t) = M_n$, does not exceed N, then from the sequence $\{\Phi_n\}$ one may choose a subsequence convergent to some non-decreasing function $\Phi_o$ which is piecewise constant, possesses at most N jump points, and $\Phi_o(b) = d$.

P r o o f. The number of maximal intervals in which the functions $\Phi_n$ are different from a constant is bounded by N, therefore, from the sequence $\{\Phi_n\}$ let us choose a subsequence composed of those functions which are different from a constant on exactly $s$, $s \leqslant N$, disjoint maximal intervals. Of course, on these intervals $\varphi_n = M_n$. Denote them by $(t_1^n, t_2^n)$, ..., $(t_{2s-1}^n, t_{2s}^n)$. Consider now the sequences $\{t_i^n\}$ with any fixed $i = 1, 2, \dots, 2s$. From every sequence $\{t_i^n\}$ let us choose a subsequence monotonically convergent to $t_i^o \in I$. In the sequel, subsequences will be denoted by the same letters as sequences are. From the conditions $\int_I \varphi_n(t)dt = d$ and $M_n \to \infty$ it follows that the length of the interval $t_{2k}^n - t_{2k-1}^n \to 0$, $k = 1, 2, \dots, s$. So, we have $t_{2k-1}^n \to \tau_k$ and, at the same time, $t_{2k}^n \to \tau_k$, $k = 1, \dots, s$. Consequently, $t_1^o = t_2^o = \tau_1, \dots, t_{2s-1}^o = t_{2s}^o = \tau_s$. Then, from the sequence $\{\Phi_n\}$ one may choose a subsequence convergent at each point to some non-decreasing function $\Phi_o$. We shall show that it is piecewise constant.

Let $t'$, $t''$ be any points of the interval $(\tau_i, \tau_{i+1})$, $\varepsilon$ - any positive number. We then have

$$|\Phi_o(t') - \Phi_n(t')| < \frac{\varepsilon}{2} \tag{7.1}$$

and

$$|\Phi_n(t'') - \Phi_n(t'')| < \frac{\varepsilon}{2} \tag{7.2}$$

for $n$ sufficiently large. Moreover,

$$\Phi_n(t') = \Phi_n(t'') \tag{7.3}$$

for $n$ sufficiently large. From conditions (7.1), (7.2) and (7.3) it follows that

$$|\varphi_o(t') - \Phi_o(t'')| < \varepsilon.$$

In view of the arbitrariness of $\varepsilon$, we get

$$\Phi_o(t') = \Phi_o(t'').$$

Since $\Phi_n(b) = d$, $n = 1, 2, \ldots,$ therefore $\Phi_o(b) = d$. Consequently, $\Phi_n$ is a piecewise constant non-decreasing function whose number of jump points does not exceed $N$. Finally, in order that $\Phi_o$ be a function continuous on the right, let us make a change of its values at a finite number of points. So, $\Phi_o$ thus obtained is the distribution function of the measure cumulated at the points $\tau_1, \ldots, \tau_s \in I$, $s \leqslant N$, $\Phi_o \in \Omega$, which ends the proof.

*Remark 7.1.* A subset of the interval $(a,b)$ has a positive measure when it contains at least one jump point of the function $\Phi_o$ since

$$\mu(\tau_i) = \Phi_o(\tau_i) - \Phi_o(\tau_i^-).$$

The measure of any interval contained in the interval $(\tau_k, \tau_{k+1})$ is equal to zero.

Let now $x^*$ be a boundary point of the set $V_o$. Since $V_o$ is a closed set, there exists a measure $\mu^* \in \mathfrak{M}_o$ such that $x^* = x(\mu^*)$. The measure $\mu^*$ will be called a boundary measure with respect to the set $V_o$.

*Theorem 7.2.* If, with any non-zero $v \in R^{n+2}$, the condition:

(7.4) the set $Z(v)$ is the union of at most $N$ disjoint intervals

is satisfied, then

$1^o$ for each boundary point $x^*$ of the set $V_o$, there exists a boundary measure $\mu^*$ cumulated at s, $s \leqslant N$, points of the interval I;

$2^o$ for any $\varepsilon > 0$, there exists a measure $\mu \in \mathfrak{M}_o(M)$ with some M, cumulated on s, $s \leqslant N$, disjoint subintervals of I, such that $|x^* - x(\mu)| < \varepsilon$.

P r o o f. Let $x^* = x(\mu^*) = (x_o^*, \ldots, x_n^*, d)$ be any boundary point of the set $V_o$. By *Remark 6.1*, for the boundary measure $\mu^*$ with respect to the set $V_o$, there exists a sequence of boundary measures $\mu^*_n \in \mathfrak{M}_o(M_n)$ with respect to the sets $V_o(M_n)$, such that the sequence of images of these measures $x(\mu_n^*)$ is convergent to the point $x(\mu^*)$, and $\mu_n^*(I) = d$.

Denote by $\Phi_n^*$ the distribution function of the measure $\mu_n^*$, whereas its derivative - by $\varphi_n^*$. It follows from *Theorem 5.1* that $\varphi_n^*$ is a piecewise constant function taking the values O or $M_n$ only. On the ground of assumption (7.4) and *Theorem 7.1*, from the sequence of distribution functions $\{\Phi_n\}$ induced by the measures $\mu_n^*$ one may choose a subsequence convergent to the piecewise constant distribution function $\Phi_o$ possessing at most N jump points. Denote by $\mu_o$ the measure induced by the distribution function $\Phi_o$. By Helly's theorem, the sequence of integrals

$$\int_I f_i(t)\, d\mu_n^* = \int_I f_i(t) d\, \Phi_n^*(t),$$

with every $i = 0, 1, \ldots, n$, converges to the integral

$$\int_I f_i(t) d\, \Phi_o(t) = \int_I f_i(t) d\, \mu_o.$$

Consequently, the sequence $\{x(\mu_n^*)\}$ of images of the measures $\mu_n^*$ tends to the image of the measure $\mu_o$, and on the other hand, it tends to $x(\mu^*) = x^*$.
Hence

$$x(\mu^*) = x(\mu_o),$$

which means that any boundary point $x^*$ of the set $V_o$ is the image of the measure cumulated at s, $s \leqslant N$, points of the interval I.

The other part of the proposition follows directly from *Lemma 2.1*, which completes the proof.

Let $\mu$ be any generalized measure belonging to the family $\mathfrak{M}_{c,k}$, and $\mu = \mu_1 - \mu_2$ - some decomposition of this measure. We then have

$$\int_I d\mu = \int_I d\mu_1 - \int_I d\mu_2 = c$$

and

$$\int_I d\mu_1 + \int_I d\mu_2 = k.$$

Hence

$$\int_I d\mu_1 = \frac{k+c}{2}, \quad \int_I d\mu_2 = \frac{k-c}{2}.$$

Furthermore, denote by G the algebraic difference of the sets $V_o \cap H_{\frac{k+c}{2}}$, $V_o \cap H_{\frac{k-c}{2}}$.

*Lemma 7.1.* The set G is sequentially compact in itself and convex.

The proof follows directly from the sequential compactness in itself as well as from the convexity of the set $V_o \cap H_d$ with any $d > 0$.

Also, it is not hard to prove

*Lemma 7.2.* Each boundary point of the set G is brought into the set G by boundary points of the sets $V_o \cap H_{\frac{k+c}{2}}$, $V_o \cap H_{\frac{k-c}{2}}$.

From this the following corollary results:

*Corollary.* Under assumption (7.4) each boundary point of the set G is the image of a generalized measure cumulated at at most 2N points of the interval I.

*Theorem 7.3.* If there exists at least one generalized measure $\mu \in \mathfrak{M}_{c,k}$ satisfying conditions (3.3), then *Problem 2* possesses a solution.

If, moreover, assumption (7.4) is satisfied, and

(7.5) $$\text{grad } F_o, \ldots, \text{grad } F_m$$

are linearly independent vectors, then there exists a generalized measure $\mu_o^*$ cumulated at $k$, $k \leq 2N$, points of the interval $I$, which is the solution of *Problem 2*.

P r o o f. Note that the solution of *Problem 2* is reduced to the determination of the minimum of the function $F_o(x) = F_o(x_o, \ldots, x_n)$ on the set $G$, under the conditions $F_j(x) = F_j(x_o, \ldots, x_n) = l_j$, $j = 1, \ldots, m$. Since all functionals $F_o(\mu), \ldots, F_m(\mu)$ do not depend on the coordinate $x_{n+1}$, the search of the minimum of the function $F_o(x)$ on the set $G$ under the conditions $F_j(x) = l_j$, $j = 1, \ldots, m$, is equivalent to the search of the conditional minimum of the function $F_o(x)$ on the projection $\tilde{G}$ of the set $G$ onto the space $R^{n+1}$. The set $G$ has the non-empty interior relative with respect to the hyperplane $H_c$ being the algebraic difference of the hyperplanes $H_{\frac{k+c}{2}}$, $H_{\frac{k-c}{2}}$ and, by *Lemma 7.1*, it is sequentially compact in itself and convex.

So, the set $\tilde{G}$ is sequentially compact in itself and convex and possesses the non-empty interior relative with respect to $R^{n+1}$. By assumption (7.5), the conditional minimum must be attained on the boundary of the set $\tilde{G}$. For, if the conditional minimum were attained at an internal point $x_o$ of the set $\tilde{G}$, then, by the Lagrange-Lyusternik theorem (cf. [4], p. 75), there would exist constants $\lambda_o, \ldots, \lambda_n$ not vanishing simultaneously, such that $x_o$ would be a stationary point of the function $\lambda_o F_o + \ldots + \lambda_m F_m$. Hence it would follow that grad $F_o, \ldots,$ grad $F_m$ are linearly dependent vectors, which is contradictory. Consequently, in virtue of the corollary to *Lemma 7.2*, we obtain the proposition.

Assumptions (7.4) and (7.5) play an essential role in the proof of *Theorem 7.3*. The number $N$ occurring in assumption (7.4) can be effectively determined in many cases. This happens when the functions $f_o(t), \ldots, f_n(t)$ are polynomials or linearly independent rational functions. In particular, $N$ can be determined when the functionals $F_o, \ldots, F_m$, described in *Problem 2*, are defined in some families of complex functions having familiar structural representations.

*Theorem 7.3* also holds true without assumption (7.5), but then, the extremum of the function $F_o$ can be attained at an internal point of the set $G$ which, in the general case, is the image of the measure cumulated at at most $4N$ points of the interval $I$.

So, variational *Problem 2* under assumption (7.4) is reduced to a mathematical programming problem.

## 8. APPLICATION OF *THEOREM 7.3* TO SOME EXTREMAL PROBLEMS IN THE CLASS OF FUNCTIONS WITH BOUNDED BOUNDARY ROTATION

In 1931 P a a t e r o [9] introduced and investigated the class $V_k$ of analytic functions which he called functions with bounded rotation of the image boundary.

A function $f$ belongs to the class $V_k$ if it satisfies the following conditions:

$1^o$ $f(z)$ is holomorphic, and $f'(z) \neq 0$ for $z \in K = \{z: |z| < 1\}$,

$2^o$ $f(z)$ is normalized by the conditions $f(0) = 0$, $f'(0) = 1$,

$3^o$ $f(z)$ maps the disc $K$ onto the domain $G$ with bounded boundary rotation (cf. [9]).

P a a t e r o [9] proved that $f \in V_k$ if and only if

$$f'(z) = \exp \int_{-\pi}^{\pi} - \log (1 - ze^{-it})\, d\mu, \tag{8.1}$$

where $\mu$ is the generalized finite measure cumulated on the interval $[-\pi, \pi]$, normalized by the conditions

$$\int_{-\pi}^{\pi} d\mu = 2, \qquad \int_{-\pi}^{\pi} |d\mu| \leqslant k \quad \text{for} \quad k \geqslant 2.$$

Moreover, $\log (1 - ze^{-it})$, denotes that branch of the logarithm which, for $z = 0$, takes the value zero.

*Theorem 7.3* allows one to characterize the generalized extremal measure $\mu^*$ in all the extremal problems in the class $V_k$ under consideration which can be written down in the formalism of *Problem 2*.

Consider, for example, a problem of $\max_{f \in V_k} |f'(z_o)|$ where $z_o$ is a fixed point of the disc $|z| < 1$.

Denote by $B_k$ the class of functions $h$ having a structural representation of the form

$$h(z) = \int_{-\pi}^{\pi} \log (1 - ze^{-it})\, d\mu, \tag{8.2}$$

where

$$\int_{-\pi}^{\pi} d\mu = 2, \qquad \int_{-\pi}^{\pi} |d\mu| \leqslant k.$$

On account of (8.1), the problem considered is reduced to the finding of the minimum of the functional

$$F_o(h) = \mathrm{re}\, h(z_o) = \mathrm{re} \int_{-\pi}^{\pi} \log (1 - z_o e^{-it})\, d\mu$$

in the class $B_k$.

It is not difficult to verify that in this case the number of intervals in which the function $(v, f(t))$ takes its values greater than or equal to zero in the interval $[-\pi, \pi]$, with any $v \in R^2$, is at most equal to two; consequently, by *Theorem 7.3*, the extremal measure is cumulated at at most four points of the interval $[-\pi, \pi]$.

Let us now consider the special case of *Problem 2* for $c = k = 2$, $a = -\pi$, $b = \pi$. The family $\mathcal{M}_{2,2}$ is the set of finite measures cumulated on the interval $[-\pi, \pi]$, normed by the condition $\int_{-\pi}^{\pi} d\mu = 2$. In this case, with assumptions (7.4) and (7.5) satisfied, the extremal measure is cumulated at at most $N$ points of the interval $[-\pi, \pi]$.

This result is consonant to theorem 6 of paper [10], p. 28. In particular, the application of the theorem obtained to isoperimetric problems considered in some families of complex functions, such as starlike, convex, with positive real part, and other ones, whose integral representations depend on the measure in the usual sense, allows one to obtain a characterization of extremal measures identical with that in paper [10].

So, *Theorem 7.3* and its special case for $c = k = 2$ give a general characterization of boundary functions with respect to a wide class of functionals considered in many well-known classes of analytic functions.

G o o d m a n [11] was one of the first scientists to pay attention to the possibility of expressing extremal problems for continuous functionals, defined in some classes of univalent functions, as problems of optimal control. This idea was made use of later in papers [12], [13], [10] and [14].

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# O PEWNYM ZADANIU IZOPERYMETRYCZNYM W PRZESTRZENI UOGÓLNIONYCH MIAR SKOŃCZONYCH

W pracy rozważa się następujące zadanie wariancyjne. Wyznaczyć minimum funkcjonału

$$F_o(\varphi) = F_o\left(\int_I f_o(t)\varphi(t)dt, \ldots, \int_I f_n(t)\varphi(t)dt\right)$$

w rodzinie $L_{c,k}$ funkcji $\varphi$, określonych przynajmniej na przedziale $I = [a, b]$, mierzalnych, całkowalnych w sensie Lebesgue'a i spełniających warunki

$$\int_I \varphi(t)dt = c, \quad \int_I |\varphi(t)|dt \leqslant k$$

dla $k \geqslant |c|$, $k > 0$ przy ograniczeniach

$$F_j(\varphi) = F_j\left(\int_I f_o(t)\varphi(t)dt, \ldots, \int_I f_n(t)\varphi(t)dt\right) = l_j,$$

$$j = 1, \ldots, m.$$

Funkcje $f_o, \ldots, f_n$ są określone i ciągłe na przedziale I, zaś funkcje $F_o, \ldots, F_m$, określone w przestrzeni $R^{n+1}$ są klasy $C^1$.

Ponieważ rodzina $L_{c,k}$ nie jest słabo zwarta w sobie, sformułowane zadanie nie zawsze posiada rozwiązanie. Dlatego zadanie to rozszerzono w sensie definicji A. Ioffego i B. Tichomirowa do zadania wariacyjnego w przestrzeni $\mathfrak{M}_{c,k}$ uogólnionych miar skończonych, spełniających warunek $\mu(I) = c$ i $|\mu|(I) \leq k$. Takie rozszerzenie zachowuje kres dolny funkcjonału. Ponieważ rodzina $\mathfrak{M}_{c,k}$ jest słabo * zwarta w sobie, więc zadanie rozszerzone zawsze posiada rozwiązanie.

Najważniejszym wynikiem pracy jest *Twierdzenie 7.3* dające charakteryzację miar ekstremalnych.