Research Article

Variational Methods for a Fractional Dirichlet Problem Involving Jumarie's Derivative

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We investigate a fractional Dirichlet problem involving Jumarie's derivative. Using some variational methods a theorem on the existence and uniqueness of a solution to such problem is proved. In the proof of the main result we use a fractional counterpart of the du Bois-Reymond fundamental lemma.

1. Introduction

In the last time, fractional calculus plays an essential role in the fields of mathematics, physics, electronics, mechanics, engineering, and so forth (cf. [1–5]). Many processes in physics and engineering can be described accurately by using systems of differential equations containing different type of fractional derivatives. Among definitions of derivatives of fractional order we can pick the Riemann-Liouville and the Caputo derivatives out. Unfortunately, each of them has different unusual properties. For instance, the Riemann-Liouville derivative of a constant is not zero and the Caputo derivative is defined only for differentiable functions (alternatively, for such functions that have no first order derivative but then they might have fractional derivatives of all orders less than one, see [6]).

Recently, Jumarie proposed a new definition of the fractional derivative being a little modification of the Riemann-Liouville derivative (cf. [7–10]). His definition eliminates disadvantages of mentioned earlier derivatives, because the Jumarie derivative of a constant is equal to zero and it is defined for any continuous (nondifferentiable) functions.

In the paper we consider the following fractional boundary problem:

$$\left(x^{(\alpha)}\right)^{(\alpha)}(t) = F_x\left(t, x\left(t\right)\right), \quad \text{for a.e. } t \in [a, b], \quad (1)$$

$$x\left(a\right) = x\left(b\right) = 0, \quad (2)$$

where $\alpha \in (1/2, 1)$, $F : [a, b] \times \mathbb{R}^n \to \mathbb{R}$, and $x^{(\alpha)}$ denotes Jumarie's derivative of a function x. The above problem is a generalization of the classical Dirichlet problem of the form

$$x''\left(t\right) = F_x\left(t, x\left(t\right)\right),$$
$$x\left(a\right) = x\left(b\right) = 0. \quad (3)$$

We discuss the problem of the existence of solutions to above problem. In our investigations we use some variational method given in [11]. First, we consider some integral functional depending on the Jumarie derivative, for which (1) is the Euler-Lagrange equation. Next, we prove existence of a critical point of mentioned functional in an appropriate space of functions and under suitable assumptions of regularity, coercivity, and convexity. In order to do it, we use the following.

Proposition 1 (see [11]). *If X is a reflexive Banach space and the functional $\mathscr{L} : X \to \mathbb{R}$ is coercive and sequentially weakly lower semicontinuos, then it possesses at least one minimum at $x_0 \in X$.*

Let us remind that a functional $\mathscr{L}$ defined on a Banach space X is coercive if $\mathscr{L}(x) \to \infty$ whenever $\|x\| \to \infty$, and $\mathscr{L}$ is sequentially weakly lower semicontinuous at $x_0 \in X$ if $\liminf_{n \to \infty} \mathscr{L}(x_n) \geq \mathscr{L}(x_0)$ for any sequence $\{x_n\} \subset X$ such that $x_n \to x_0$ weakly in X.

Of course such critical point is a minimizer of this functional and it generates the solution to problem (1)-(2). In order to prove that a minimum point gives a solution it is sufficient to apply a fractional version of the du Bois-Reymond lemma obtained in Section 3. Results of a such type for the Dirichlet problem involving the Riemann-Liouville derivative have been obtained in [12].

The paper is organized as follows. In Section 2, we review some basic definitions and facts concerning fractional calculus that we need in the sequel. Moreover, we introduce the space of solutions to considered problem and give some useful properties of this space. In Section 3, we formulate and prove some fractional version of the du Bois-Reymond Lemma, which we use in the proof of the main result. Mentioned main results of the work, namely, a theorem on the existence and uniqueness of a solution to problem (1)-(2), are formulated and proved in Section 4.

2. Preliminaries

In the first part of this paper we recall some basic facts concerning fractional calculus (cf. [3, 7–10, 13]). Next, we introduce some function spaces, which will be used later.

2.1. Fractional Calculus. We will assume that $[a, b] \subset \mathbb{R}$ is a bounded interval.

Let $\alpha > 0$ and $f \in L^1([a, b], \mathbb{R}^n)$. The left-sided Riemann-Liouville integral of the function f of order α is defined by

$$\left(I_{a+}^{\alpha} f\right)(t) := \frac{1}{\Gamma(\alpha)} \int_a^t \frac{f(\tau)}{(t-\tau)^{1-\alpha}} d\tau, \quad t \in [a, b] \text{ a.e.} \quad (4)$$

In the rest of this paper we will assume that $\alpha \in (0, 1)$.

The left-sided Riemann-Liouville derivative $D_{a+}^{\alpha} f$ of the function f of order α is defined in the following way:

$$\left(D_{a+}^{\alpha} f\right)(t) := \frac{d}{dt} \left(I_{a+}^{1-\alpha} f\right)(t), \quad t \in [a, b] \text{ a.e.,} \quad (5)$$

provided that $I_{a+}^{1-\alpha} f$ has an absolutely continuous representant on $[a, b]$ (i.e., there exists an absolutely continuous function on $[a, b]$ which is equal a.e. on $[a, b]$ to $I_{a+}^{1-\alpha} f$).

Now, let us assume that $f \in C([a, b], \mathbb{R}^n)$.

Jumarie's modified Riemann-Liouville derivative of the function f of order α is defined by

$$f^{(\alpha)}(t) := \left(D_{a+}^{\alpha} \left(f(\cdot) - f(a)\right)\right)(t)$$

$$= \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_a^t \frac{f(\tau) - f(a)}{(t-\tau)^{\alpha}} d\tau, \quad (6)$$

$$t \in [a, b] \text{ a.e.,}$$

provided that $I_{a+}^{1-\alpha} (f(\cdot) - f(a))$ has an absolutely continuous representant on $[a, b]$.

Remark 2. It is easy to see that if $f(a) = 0$, then defined above derivatives coincide. Moreover, Jumarie's modified Riemann-Liouville derivative of a constant equals zero.

Remark 3. The definition of fractional derivative given by (6) is a consequence of the following fractional derivative via difference reads defined by Jumarie:

$$f^{(\alpha)}(t) := \lim_{h \to 0} h^{-\alpha} \sum_{k=1}^{\infty} (-1)^k \binom{\alpha}{k} f[x + (\alpha - k) h]. \quad (7)$$

The $(dt)^{\alpha}$ integral of f is given by

$$\int_a^x f(t) (dt)^{\alpha} := \alpha \int_a^x f(t) (x - t)^{\alpha-1} dt, \quad x \in [a, b]. \quad (8)$$

We have the following theorem on the integration by parts.

Theorem 4. *Let $f, g \in C([a, b], \mathbb{R}^n)$ and there exist derivatives $f^{(\alpha)}$ and $g^{(\alpha)}$. Then*

$$\int_a^b f^{(\alpha)}(t) g(t) (dt)^{\alpha}$$

$$= \Gamma(\alpha + 1) \left(f(b) g(b) - f(a) g(a)\right) \quad (9)$$

$$- \int_a^b f(t) g^{(\alpha)}(t) (dt)^{\alpha}.$$

2.2. Space $L^{p,\alpha}([a, b], \mathbb{R}^n)$. Let $1 \leq p < \infty$.

Let us define the set

$$L^{p,\alpha}([a, b], \mathbb{R}^n) := \Big\{ f : [a, b]$$

$$\longrightarrow \mathbb{R}^n; \ f\text{-measurable}, \ (b - \cdot)^{(\alpha-1)/p} f \quad (10)$$

$$\in L^p([a, b], \mathbb{R}^n) \Big\}$$

with the norm

$$\|f\|_{L^{p,\alpha}} := \left\| (b - \cdot)^{(\alpha-1)/p} f \right\|_{L^p}$$

$$= \left(\int_a^b |f(t)|^p (b - t)^{\alpha-1} dt \right)^{1/p}. \quad (11)$$

We will identify two functions belonging to $L^{p,\alpha}([a, b], \mathbb{R}^n)$ that coincide a.e. on $[a, b]$.

It is easy to show that $L^{p,\alpha}([a, b], \mathbb{R}^n)$ is a Banach space. In particular, the space $L^{2,\alpha}([a, b], \mathbb{R}^n)$, equipped with the inner product

$$\langle f, g \rangle_{L^{2,\alpha}} := \int_a^b f(t) g(t) (b - t)^{\alpha-1} dt, \quad (12)$$

is a Hilbert space.

Now, we give some properties of the space $L^{p,\alpha}([a, b], \mathbb{R}^n)$.

Proposition 5. *The injection*

$$L^{p,\alpha}([a, b], \mathbb{R}^n) \subset L^p([a, b], \mathbb{R}^n) \quad (13)$$

is continuous.

Proof. Let $f \in L^{p,\alpha}([a,b], \mathbb{R}^n)$. Then

$$\int_a^b |f(t)|^p \, dt$$

$$= \int_a^b (b-t)^{1-\alpha} (b-t)^{\alpha-1} |f(t)|^p \, dt \tag{14}$$

$$\leqslant (b-a)^{1-\alpha} \int_a^b \left((b-t)^{(\alpha-1)/p} |f(t)| \right)^p \, dt.$$

Consequently,

$$\|f\|_{L^p} \leqslant (b-a)^{(1-\alpha)/p} \|f\|_{L^{p,\alpha}}. \tag{15}$$

The proof is completed. $\qquad\square$

Proposition 6 (Hölder inequality). *Let $f \in L^{p,\alpha}([a,b], \mathbb{R}^n)$, $g \in L^{q,\alpha}([a,b], \mathbb{R}^n)$, and $1/p + 1/q = 1$. Then $fg \in L^{1,\alpha}([a,b], \mathbb{R}^n)$ and*

$$\int_a^b |f(t) g(t)| (b-t)^{\alpha-1} \, dt$$

$$\leqslant \left(\int_a^b |f(t)|^p (b-t)^{\alpha-1} \, dt \right)^{1/p} \tag{16}$$

$$\cdot \left(\int_a^b |g(t)|^q (b-t)^{\alpha-1} \, dt \right)^{1/q}.$$

Proof. From the Hölder inequality for the space $L^p([a,b], \mathbb{R}^n)$ we obtain

$$\int_a^b |f(t) g(t)| (b-t)^{\alpha-1} \, dt$$

$$= \int_a^b \left((b-t)^{(\alpha-1)/p} |f(t)| \right) \left((b-t)^{(\alpha-1)/q} |g(t)| \right) dt$$

$$\leqslant \left(\int_a^b |f(t)|^p (b-t)^{\alpha-1} \, dt \right)^{1/p} \tag{17}$$

$$\cdot \left(\int_a^b |g(t)|^q (b-t)^{\alpha-1} \, dt \right)^{1/q}.$$

$\qquad\square$

Lemma 7. *The operator $I_{a+}^\alpha : L^{p,\alpha}([a,b], \mathbb{R}^n) \to L^{p,\alpha}([a,b], \mathbb{R}^n)$ is bounded, and it means there exists a constant $C_p > 0$ such that*

$$\|I_{a+}^\alpha f\|_{L^{p,\alpha}} \leqslant C_p \|f\|_{L^{p,\alpha}}, \quad f \in L^{p,\alpha}([a,b], \mathbb{R}^n). \tag{18}$$

Proof. Using Fubini's Theorem and [14, Lemma 1], we obtain

$$\|I_{a+}^\alpha f\|_{L^{p,\alpha}}^p = \int_a^b (b-t)^{\alpha-1} |(I_{a+}^\alpha f)(t)|^p \, dt \leqslant c_0 \int_a^b (b$$

$$-t)^{\alpha-1} I_{a+}^\alpha |f(t)|^p \, dt = \frac{c_0}{\Gamma(\alpha)} \int_a^b \left(\int_a^t (b-t)^{\alpha-1} \right.$$

$$\cdot (t-\tau)^{\alpha-1} |f(\tau)|^p \, d\tau \Big) dt = \frac{c_0}{\Gamma(\alpha)} \int_a^b |f(\tau)|^p$$

$$\cdot \left(\int_\tau^b (b-t)^{\alpha-1} (t-\tau)^{\alpha-1} \, dt \right) d\tau$$

$$= \left\{ \begin{matrix} t = b - s(b-\tau) \\ dt = -ds(b-\tau) \end{matrix} \right\} = \frac{c_0}{\Gamma(\alpha)} \int_a^b |f(\tau)|^p$$

$$\cdot \left(\int_0^1 s^{\alpha-1} (b-\tau)^{\alpha-1} (1-s)^{\alpha-1} (b-\tau)^{\alpha-1} \right. \tag{19}$$

$$\cdot (b-\tau) \, ds \Big) d\tau = \frac{c_0}{\Gamma(\alpha)} \int_a^b (b-\tau)^{2\alpha-1} |f(\tau)|^p$$

$$\cdot \left(\int_0^1 s^{\alpha-1} (1-s)^{\alpha-1} \, ds \right) d\tau = \frac{\beta(\alpha,\alpha) c_0}{\Gamma(\alpha)} \int_a^b (b$$

$$-\tau)^{\alpha-1} (b-\tau)^\alpha |f(\tau)|^p \, d\tau \leqslant \frac{\beta(\alpha,\alpha) c_0 (b-a)^\alpha}{\Gamma(\alpha)}$$

$$\cdot \int_a^b (b-\tau)^{\alpha-1} |f(\tau)|^p \, d\tau$$

$$= \frac{\beta(\alpha,\alpha) c_0 (b-a)^\alpha}{\Gamma(\alpha)} \|f\|_{L^{p,\alpha}}^p,$$

where $c_0 = ((b-a)^\alpha/\Gamma(\alpha+1))^{p-1}$, $\beta(z,u) := \int_0^1 r^{z-1}(1-r)^{u-1} dr$, $z, u > 0$.

Since $\beta(z,u) = \Gamma(z)\Gamma(u)/\Gamma(z+u)$, it suffices to put $C_p = (b-a)^\alpha (\Gamma(\alpha)/\Gamma(2\alpha)\Gamma(\alpha+1)^{p-1})^{1/p}$.

The proof is completed. $\qquad\square$

2.3. Space $I_{a+}^\alpha (L^{p,\alpha})$. Let us define the set $I_{a+}^\alpha (L^{p,\alpha}([a,b], \mathbb{R}^n))$ (shortly $I_{a+}^\alpha (L^{p,\alpha})$) as follows:

$$I_{a+}^\alpha (L^{p,\alpha}) := \{ f : [a,b] \longrightarrow \mathbb{R}^n; \ f$$

$$= I_{a+}^\alpha \varphi, \text{ a.e. on } [a,b], \ \varphi \in L^{p,\alpha}([a,b], \mathbb{R}^n) \}. \tag{20}$$

Functions belonging to $I_{a+}^\alpha (L^{p,\alpha})$ and equal a.e. on $[a,b]$ are identified.

From Proposition 5, we immediately obtain the following.

Proposition 8. *Consider*

$$I_{a+}^\alpha (L^{p,\alpha}) \subset I_{a+}^\alpha (L^p), \tag{21}$$

where $I_{a+}^\alpha (L^p) := \{ f : [a,b] \to \mathbb{R}^n; f = I_{a+}^\alpha h, \text{ a.e. on } [a,b], h \in L^p([a,b], \mathbb{R}^n) \}$.

From the above fact and [14, Proposition 2] it follows that if $f \in I_{a+}^{\alpha}(L^{p,\alpha})$ then there exists the Riemann-Liouville derivative $D_{a+}^{\alpha} f$ almost everywhere on $[a, b]$.

Moreover, one can show that $I_{a+}^{\alpha}(L^{p,\alpha})$ with the norm $\| \cdot \|_{I_{a+}^{\alpha}(L^{p,\alpha})} : I_{a+}^{\alpha}(L^{p,\alpha}) \to [0, \infty)$ given by

$$\|f\|_{I_{a+}^{\alpha}(L^{p,\alpha})} := \|D_{a+}^{\alpha} f\|_{L^{p,\alpha}} \tag{22}$$

is complete and, consequently, is a Banach space. In particular, the space $I_{a+}^{\alpha}(L^{2,\alpha})$, equipped with the inner product

$$\langle f, g \rangle_{I_{a+}^{\alpha}(L^{2,\alpha})}$$
$$:= \int_a^b \left(D_{a+}^{\alpha} f \right)(t) \left(D_{a+}^{\alpha} g \right)(t) (b - t)^{\alpha-1} dt, \tag{23}$$

is a Hilbert space.

Remark 9. Let us note that in the case $p > 1/\alpha$, from Proposition 8 and [15, Property 4], it follows that

$$I_{a+}^{\alpha}\left(L^{p,\alpha} \right) \subset I_{a+}^{\alpha}\left(L^p \right) \subset C_0\left([a, b], \mathbb{R}^n \right), \tag{24}$$

where $C_0([a, b], \mathbb{R}^n) := \{u \in C([a, b], \mathbb{R}^n) : u(a) = 0\}$. Consequently, if $f \in I_{a+}^{\alpha}(L^{p,\alpha})$ then f possesses Jumarie's modified Riemann-Liouville derivative $f^{(\alpha)}$ and $f^{(\alpha)} = D_{a+}^{\alpha} f$. Of course, then

$$\|f\|_{I_{a+}^{\alpha}(L^{p,\alpha})} := \|f^{(\alpha)}\|_{L^{p,\alpha}}. \tag{25}$$

Remark 10. From Proposition 8, monography [3, Lemma 2.5(a)] and Remarks 2 and 9 it follows that if $f \in I_{a+}^{\alpha}(L^{p,\alpha})$ with $p > 1/\alpha$ then

$$f(t) = I_{a+}^{\alpha} f^{(\alpha)}(t), \quad t \in [a, b] \text{ a.e.} \tag{26}$$

From the above remark and Lemma 7, we immediately obtain the following.

Lemma 11 (fractional Poincaré Inequality). *Let $p > 1/\alpha$. Then*

$$\left(\int_a^b |f(t)|^p (b - t)^{\alpha-1} dt \right)^{1/p}$$
$$\leq C_p \left(\int_a^b \left| f^{(\alpha)}(t) \right|^p (b - t)^{\alpha-1} dt \right)^{1/p} \tag{27}$$

for $f \in I_{a+}^{\alpha}(L^{p,\alpha})$, where $C_p > 0$ is the constant from Lemma 7.

3. Du Bois-Reymond Lemma

In this section, we will prove the du Bois-Reymond lemma for nondifferentiable functions.

We have the following.

Lemma 12 (du Bois-Reymond lemma). *Let $\alpha \in (1/2, 1)$, $f \in L^{2,\alpha}([a, b], \mathbb{R}^n)$, and*

$$\int_a^b f(t) h^{(\alpha)}(t) (b - t)^{\alpha-1} dt = 0 \tag{28}$$

for any function $h \in I_{a+}^{\alpha}(L^{2,\alpha})$ such that $h(b) = 0$. Then there exists a constant $c \in \mathbb{R}^n$ such that $f(t) = c$ for a.e. $t \in [a, b]$ and, consequently, $f^{(\alpha)}(t) = 0$ for a.e. $t \in [a, b]$.

Proof. First, let us note that from the Hölder inequality (cf. Proposition 6) it follows that the integral (28) is well-defined. Let $w(t) = c$ for $t \in [a, b]$, where $c = (\alpha/(b - a)^{\alpha}) \int_a^b f(t)(b - t)^{\alpha-1} dt$. Then, for any function $h \in I_{a+}^{\alpha}(L^{2,\alpha})$ such that $h(b) = 0$ (in view of Remark 9 the condition $h(a) = 0$ is satisfied also), from assumption (28) and Theorem 4, we obtain

$$0 = \int_a^b f(t) h^{(\alpha)}(t) (b - t)^{\alpha-1} dt - 0$$
$$= \int_a^b f(t) h^{(\alpha)}(t) (b - t)^{\alpha-1} dt$$
$$\quad - \frac{1}{\alpha} \int_a^b w^{(\alpha)}(t) h(t) (dt)^{\alpha}$$
$$= \int_a^b f(t) h^{(\alpha)}(t) (b - t)^{\alpha-1} dt$$
$$\quad - \frac{1}{\alpha} \int_a^b w(t) h^{(\alpha)}(t) (dt)^{\alpha}$$
$$\quad + \frac{\Gamma(\alpha + 1)}{\alpha} (w(b) h(b) - w(a) h(a))$$
$$= \int_a^b \left(f(t) - w(t) \right) h^{(\alpha)}(t) (b - t)^{\alpha-1} dt. \tag{29}$$

Thus,

$$\int_a^b \left(f(t) - w(t) \right) h^{(\alpha)}(t) (b - t)^{\alpha-1} dt = 0,$$
$$h \in I_{a+}^{\alpha}\left(L^{2,\alpha} \right), \quad h(b) = 0. \tag{30}$$

Let us consider the function $\widehat{h}(t) = (I_{a+}^{\alpha}(f - w))(t)$ for a.e. $t \in [a, b]$. It is easy to see that $\widehat{h} \in I_{a+}^{\alpha}(L^{2,\alpha})$ and, in view of Remark 9, $\widehat{h}(a) = 0$. We will show that $\widehat{h}(b) = 0$. Indeed, we have

$$\widehat{h}(b) = \frac{1}{\Gamma(\alpha)} \int_a^b \left(f(t) - w(t) \right) (b - t)^{\alpha-1} dt$$
$$= \frac{1}{\Gamma(\alpha)} \int_a^b f(t) (b - t)^{\alpha-1} dt$$
$$\quad - \frac{c}{\Gamma(\alpha)} \int_a^b (b - t)^{\alpha-1} dt$$
$$= \frac{1}{\Gamma(\alpha)} \int_a^b f(t) (b - t)^{\alpha-1} dt - \frac{c}{\Gamma(\alpha)} \frac{(b - a)^{\alpha}}{\alpha} = 0. \tag{31}$$

Consequently, the function $\widehat{h}$ satisfies equality (30). So,

$$\int_a^b \left(f(t) - w(t) \right)^2 (b - t)^{\alpha-1} dt = 0. \tag{32}$$

It means that

$$f(t) = w(t) = c, \quad t \in [a,b] \text{ a.e.} \tag{33}$$

The proof is completed. □

Remark 13. In [16] result of such a type, but for Caputo derivative (for differentiable functions h), had been proved.

Using Lemma 12, we will prove the next lemma, which will play a key role in the next section. We have the following.

Lemma 14. *Let* $\alpha \in (1/2, 1)$, $f_1 \in L^{2,\alpha}([a,b], \mathbb{R}^n)$, $f_2 \in L^2([a,b], \mathbb{R}^n)$, *and*

$$\int_a^b \left(f_1(t) h^{(\alpha)}(t) + f_2(t) h(t) \right) (b-t)^{\alpha-1} dt = 0 \tag{34}$$

for any function $h \in I_{a+}^\alpha(L^{2,\alpha})$ *such that* $h(b) = 0$. *Then,*

$$f_1(t) = \left(I_{a+}^\alpha f_2 \right)(t) + c, \quad t \in [a,b] \text{ a.e.,} \tag{35}$$

where $c \in \mathbb{R}^n$ *is the constant from Lemma 12, and consequently*

$$f_1^{(\alpha)}(t) = f_2(t), \quad t \in [a,b] \text{ a.e.} \tag{36}$$

Proof. Using the Hölder inequality for spaces L^p and $L^{p,\alpha}$, we check that integral (34) exists. Let us put $v = I_{a+}^\alpha f_2$. Then $v \in I_{a+}^\alpha(L^2)$ and $v^{(\alpha)} = f_2$. From Theorem 4 and assumption (34), we obtain

$$0 = \alpha \int_a^b \left(f_1(t) h^{(\alpha)}(t) + f_2(t) h(t) \right) (b-t)^{\alpha-1} dt$$

$$= \alpha \int_a^b f_1(t) h^{(\alpha)}(t) (b-t)^{\alpha-1} dt$$

$$\quad + \int_a^b v^{(\alpha)}(t) h(t) (dt)^\alpha$$

$$= \alpha \int_a^b f_1(t) h^{(\alpha)}(t) (b-t)^{\alpha-1} dt \tag{37}$$

$$\quad - \int_a^b v(t) h^{(\alpha)}(t) (dt)^\alpha$$

$$\quad + \Gamma(\alpha+1) \left(v(b) h(b) - v(a) h(a) \right)$$

$$= \alpha \int_a^b \left(f_1(t) - v(t) \right) h^{(\alpha)}(t) (b-t)^{\alpha-1} dt,$$

$$h \in I_{a+}^\alpha \left(L^{2,\alpha} \right), \; h(b) = 0.$$

Thus

$$\int_a^b \left(f_1(t) - v(t) \right) h^{(\alpha)}(t) (b-t)^{\alpha-1} dt = 0,$$

$$h \in I_{a+}^\alpha \left(L^{2,\alpha} \right), \; h(b) = 0. \tag{38}$$

From Lemma 12 it follows that there exists a constant $c \in \mathbb{R}^n$ such that

$$f_1(t) - v(t) = c, \quad t \in [a,b] \text{ a.e.} \tag{39}$$

Consequently,

$$f_1(t) = v(t) + c = \left(I_{a+}^\alpha f_2 \right)(t) + c, \quad t \in [a,b] \text{ a.e.} \tag{40}$$

It is well known that the function $I_{a+}^\alpha f_2$ possesses the left-sided Riemann-Liouville derivative and

$$\left(D_{a+}^\alpha I_{a+}^\alpha f_2 \right)(t) = f_2(t), \quad t \in [a,b] \text{ a.e.} \tag{41}$$

Since $\alpha \in (1/2, 1)$, from [15, Property 4] it follows that the function $I_{a+}^\alpha f_2$ is continuous and $(I_{a+}^\alpha f_2)(a) = 0$. Consequently, it possesses also the Jumarie modified Riemann-Liouville derivative, wchich equals f_2. It means that the function f_1 has the Jumarie modified Riemann-Liouville derivative and (using the second part of Remark 2)

$$f_1^{(\alpha)}(t) = f_2(t), \quad t \in [a,b] \text{ a.e.} \tag{42}$$

The proof is completed. □

4. Main Result

Let us consider Dirichlet problem (1)-(2). By a solution to such problem we will mean a function $x \in I_{a+}^\alpha(L^{2,\alpha})$, satysfying condition $x(b) = 0$, such that $x^{(\alpha)} \in (I_{a+}^\alpha(L^2) + \{c; c \in \mathbb{R}\})$.

Let us notice that since $\alpha \in (1/2, 1)$, from Remark 9 it follows that x and $x^{(\alpha)}$ are continuous and x satisfies the initial condition $x(a) = 0$.

In order to prove the existence of solutions to problem (1)-(2), we use variational methods.

Let us consider a functional $\mathscr{L}$ of the form

$$\mathscr{L}(x) := \int_a^b \left(\frac{1}{2} \left| x^{(\alpha)}(t) \right|^2 + F(t, x(t)) \right) (b-t)^{\alpha-1} dt \tag{43}$$

defined on the following space

$$K_0^{2,\alpha}\left([a,b], \mathbb{R}^n\right) = K_0^{2,\alpha}$$
$$= \left\{ x \in I_{a+}^\alpha \left(L^{2,\alpha} \right) : x(b) = 0 \right\}. \tag{44}$$

We impose the following assumption on the function F:

(A1) The function

$$[a,b] \ni t \longmapsto F(t,x) \in \mathbb{R} \tag{45}$$

is measurable on $[a,b]$ for any $x \in \mathbb{R}^n$ and the function

$$\mathbb{R}^n \ni x \longmapsto F(t,x) \in \mathbb{R} \tag{46}$$

is of class C^1 on $\mathbb{R}$ for a.e. $t \in [a,b]$.

(A2) There exist constants $c_1, c_2 \geq 0$ and functions $\gamma_1 \in L^{1,\alpha}([a,b], \mathbb{R}^+)$ and $\gamma_2 \in L^2([a,b], \mathbb{R}^+)$ such that

$$|F(t,x)| \leq c_1 |x| + \gamma_1(t),$$
$$|F_x(t,x)| \leq c_2 |x| + \gamma_2(t) \tag{47}$$

for almost every $t \in [a,b]$ and all $x \in \mathbb{R}^n$.

We say that $\mathscr{L}$ possesses the first variation $\delta\mathscr{L}(x,h)$ at the point $x \in K_0^{2,\alpha}$ in the direction $h \in K_0^{2,\alpha}$ (cf. [17]) if there exists a finite limit

$$\delta\mathscr{L}(x,h) = \lim_{\lambda \to 0} \frac{\mathscr{L}(x + \lambda h) - \mathscr{L}(x)}{\lambda}. \qquad (48)$$

We will prove that, under assumptions (A1) and (A2), $\mathscr{L}$ possesses its minimum at a point x_0 which is a solution to (1).

To begin with, we will prove the following.

Theorem 15. *Let us assume that conditions (A1)-(A2) are satisfied. Then the functional $\mathscr{L}$ is well-defined on $K_0^{2,\alpha}$ and possesses the first variation $\delta\mathscr{L}(x,h)$ at any point $x \in K_0^{2,\alpha}$ and in any direction $h \in K_0^{2,\alpha}$ given by*

$$\delta\mathscr{L}(x,h)$$
$$= \int_a^b \left(\left(x^{(\alpha)} \right)(t) \left(h^{(\alpha)} \right)(t) + F_x(t, x(t)) h(t) \right) \qquad (49)$$
$$\cdot (b-t)^{\alpha-1} dt.$$

Proof. The fact that $\mathscr{L}$ and $\delta\mathscr{L}$ are well-defined follows directly from (A1)-(A2) and the Hölder inequality (cf. Proposition 6). Let us fix $x \in K_0^{2,\alpha}$ and $h \in K_0^{2,\alpha}$ and write the functional $\mathscr{L}$ as

$$\mathscr{L} = \mathscr{L}_2 \circ \mathscr{L}_1, \qquad (50)$$

where

$$\mathscr{L}_1 : K_0^{2,\alpha} \ni x(\cdot)$$
$$\longrightarrow \frac{1}{2} \left| x^{(\alpha)}(\cdot) \right|^2 + F(\cdot, x(\cdot))$$
$$\in L^{1,\alpha}([a,b], \mathbb{R}), \qquad (51)$$
$$\mathscr{L}_2 : L^{1,\alpha}([a,b], \mathbb{R}) \ni y(\cdot) \longrightarrow \int_a^b y(t)(b-t)^{\alpha-1} dt$$
$$\in \mathbb{R}.$$

It is clear that $\mathscr{L}_1$ and $\mathscr{L}_2$ are well-defined and $\mathscr{L}_2$ is linear. Moreover,

$$\left| \mathscr{L}_2(y) \right| \leq \int_a^b |y(t)|(b-t)^{\alpha-1} dt = \|y\|_{L^{1,\alpha}}, \qquad (52)$$
$$y \in L^{1,\alpha}([a,b], \mathbb{R}),$$

so $\mathscr{L}_2$ is continuous. Consequently, it is differentiable in the sense of Frechet on $L^{1,\alpha}([a,b], \mathbb{R})$ and the differential at any point $y \in L^{1,\alpha}([a,b], \mathbb{R})$ is equal to $\mathscr{L}_2$. Using the Lebesque dominated convergence theorem and the mean value theorem, we assert that the mapping $\mathscr{L}_1$ has the first variation $\delta\mathscr{L}_1(x,h)$ at any point $x \in K_0^{2,\alpha}$ and in any direction $h \in K_0^{2,\alpha}$ given by

$$\delta\mathscr{L}_1(x(\cdot), h(\cdot)) = \left(x^{(\alpha)} \right)(\cdot) \left(h^{(\alpha)} \right)(\cdot)$$
$$+ F_x(\cdot, x(\cdot)) h(\cdot). \qquad (53)$$

This means (cf. [17, Section 2.2.2]) that there exists the first variation of the mapping $\mathscr{L}$ given by equality (49).

The proof is completed. $\square$

Using the same arguments as in [15, Proposition 1 (3.3)], we can obtain the following.

Theorem 16. *Let $\alpha \in (1/2, 1)$ and $(f_k)_{k \in \mathbb{N}} \subset K_0^{2,\alpha}$. If $f_k \rightharpoonup f$ weakly in $K_0^{2,\alpha}$ (with topology induced from $I_{a+}^{\alpha}(L^{2,\alpha})$), then $f_k \rightrightarrows f$ uniformly on $[a,b]$.*

Now, we will prove the main result of this paper, namely, a theorem on the existence of a unique solution to problem (1)-(2). We have the following.

Theorem 17. *Let $\alpha \in (1/2, 1)$ and assume that assumptions (A1)-(A2) are satisfied. If there are constants $a_1 < \alpha\Gamma(2\alpha)/(b-a)^{2\alpha}$, $b_1, b_2 \geq 0$, such that*

$$F(t,x) \geq -\frac{a_1}{2}|x|^2 - b_1|x| - b_2, \qquad (54)$$
$$t \in [a,b] \quad a.e., \quad x \in \mathbb{R}^n,$$

then problem (1)-(2) possesses at least one solution which minimizes functional $\mathscr{L}$. Moreover, if the function

$$\mathbb{R}^n \ni x \longmapsto F(t,x) + \frac{a_1|x|^2}{2} \in \mathbb{R} \qquad (55)$$

is convex for a.e. $t \in [a,b]$, then the solution is unique.

Proof. Let $x \in K_0^{2,\alpha}$. Then from condition (54), Proposition 6, and Lemma 11 it follows that

$$\mathscr{L}(x) = \frac{1}{2}\|x\|^2_{I_{a+}^{\alpha}(L^{2,\alpha})} + \int_a^b F(t,x(t))(b-t)^{\alpha-1} dt$$
$$\geq \frac{1}{2}\|x\|^2_{I_{a+}^{\alpha}(L^{2,\alpha})} - \frac{a_1}{2}\|x\|^2_{L^{2,\alpha}}$$
$$- b_1\sqrt{\frac{(b-a)^\alpha}{\alpha}}\|x\|_{L^{2,\alpha}} - b_2\frac{(b-a)^\alpha}{\alpha} \qquad (56)$$
$$\geq \frac{1 - a_1 C_2^2}{2}\|x\|^2_{I_{a+}^{\alpha}(L^{2,\alpha})}$$
$$- b_1 C_2\sqrt{\frac{(b-a)^\alpha}{\alpha}}\|x\|_{I_{a+}^{\alpha}(L^{2,\alpha})} - b_2\frac{(b-a)^\alpha}{\alpha},$$

where $C_2 = (b-a)^\alpha / \sqrt{\alpha\Gamma(2\alpha)}$ is the constant from Lemma 7.

Consequently, since $a_1 < \alpha\Gamma(2\alpha)/(b-a)^{2\alpha}$, $1 - a_1 C_2^2 > 0$; so $\mathscr{L}$ is coercive.

Now, Let $x_k \rightharpoonup x_0$ weakly in $K_0^{2,\alpha}$. From Theorem 16 it follows that $x_k \rightrightarrows x_0$, on $[a,b]$. Thus and from assumption (A2), using the dominated convergence theorem, we get that

$$\lim_{k \to \infty} \int_a^b F(t, x_k(t)) dt = \int_a^b F(t, x_0(t)) dt. \qquad (57)$$

Moreover, $K_0^{2,\alpha}$ is a Hilbert space as the closed subspace of $I_{a+}^\alpha (L^{2,\alpha})$. Consequently, the mapping $x \mapsto \|x\|_{I_{a+}^\alpha (L^{2,\alpha})}$ is weakly lower semicontinuous, so

$$\liminf_{k \to \infty} \mathscr{L}(x_k) = \liminf_{k \to \infty} \frac{1}{2} \|x_k\|^2_{I_{a+}^\alpha (L^{2,\alpha})}$$

$$+ \lim_{k \to \infty} \int_a^b F(t, x_k(t))\, dt \qquad (58)$$

$$\geq \mathscr{L}(x_0).$$

This means that the functional $\mathscr{L}$ is sequentially weakly lower semicontinuous and, by the virtue of Proposition 1, we conclude that it possesses minimum at the point $x_0 \in K_0^{2,\alpha}$.

From Theorem 15 and Fermat lemma it follows that

$$\int_a^b \left(x_0^{(\alpha)}(t)\, h^{(\alpha)}(t) + F_x(t, x_0(t))\, h(t) \right)(b-t)^{\alpha-1}\, dt \qquad (59)$$

$$= 0$$

for any $h \in K_0^{2,\alpha}$. Applying Lemma 14, we get

$$\left(x^{(\alpha)} \right)^{(\alpha)}(t) = F_x(t, x(t)) \quad \text{for a.e. } t \in [a, b]. \qquad (60)$$

Since $x_0 \in K_0^{2,\alpha}$, boundary conditions $x(a) = x(b) = 0$ are satisfied.

The proof of the existence part is completed.

Now, we will show that, under assumption (55), the solution to problem (1)-(2) is unique. First, let us note that for $x, y \in K_0^{2,\alpha}$, $x \neq y$, and $\lambda \in (0, 1)$ we have

$$(1-\lambda) \int_a^b \left(\left| x^{(\alpha)}(t) \right|^2 - a_1 |x(t)|^2 \right)(b-t)^{\alpha-1}\, dt$$

$$+ \lambda \int_a^b \left(\left| y^{(\alpha)}(t) \right|^2 - a_1 |y(t)|^2 \right)(b-t)^{\alpha-1}\, dt$$

$$- \int_a^b \left(\left| (1-\lambda) x^{(\alpha)}(t) + \lambda y^{(\alpha)}(t) \right|^2 \right.$$

$$\left. - a_1 \left| (1-\lambda) x(t) + \lambda y(t) \right|^2 \right)(b-t)^{\alpha-1}\, dt = (1$$

$$-\lambda) \int_a^b \left(\left| x^{(\alpha)}(t) \right|^2 - a_1 |x(t)|^2 \right)(b-t)^{\alpha-1}\, dt$$

$$+ \lambda \int_a^b \left(\left| y^{(\alpha)}(t) \right|^2 - a_1 |y(t)|^2 \right)(b-t)^{\alpha-1}\, dt$$

$$- \int_a^b \left((1-\lambda)^2 \left| x^{(\alpha)}(t) \right|^2 \right.$$

$$+ 2(1-\lambda)\lambda \left\langle x^{(\alpha)}(t), y^{(\alpha)}(t) \right\rangle + \lambda^2 \left| y^{(\alpha)}(t) \right|^2 \right)(b$$

$$-t)^{\alpha-1}\, dt + \int_a^b \left(a_1 (1-\lambda)^2 |x(t)|^2 \right.$$

$$+ 2a_1 (1-\lambda)\lambda \left\langle x(t), y(t) \right\rangle + a_1 \lambda^2 |y(t)|^2 \right)(b$$

$$-t)^{\alpha-1}\, dt = \int_a^b \left((1-\lambda) \left| x^{(\alpha)}(t) \right|^2 \right.$$

$$- (1-\lambda)^2 \left| x^{(\alpha)}(t) \right|^2$$

$$\left. - 2(1-\lambda)\lambda \left\langle x^{(\alpha)}(t), y^{(\alpha)}(t) \right\rangle \right)(b-t)^{\alpha-1}\, dt$$

$$+ \int_a^b \left(\lambda \left| y^{(\alpha)}(t) \right|^2 - \lambda^2 \left| y^{(\alpha)}(t) \right|^2 \right)(b-t)^{\alpha-1}\, dt$$

$$+ \int_a^b \left(-a_1 (1-\lambda) |x(t)|^2 + a_1 (1-\lambda)^2 |x(t)|^2 \right)(b$$

$$-t)^{\alpha-1}\, dt + \int_a^b \left(2a_1 (1-\lambda)\lambda \left\langle x(t), y(t) \right\rangle \right.$$

$$\left. - a_1 \lambda |y(t)|^2 + a_1 \lambda^2 |y(t)|^2 \right)(b-t)^{\alpha-1}\, dt$$

$$= \int_a^b \left((1-\lambda)\lambda \left| x^{(\alpha)}(t) \right|^2 \right.$$

$$- 2(1-\lambda)\lambda \left\langle x^{(\alpha)}(t), y^{(\alpha)}(t) \right\rangle$$

$$\left. + (1-\lambda)\lambda \left| y^{(\alpha)}(t) \right|^2 \right)(b-t)^{\alpha-1}\, dt$$

$$- a_1 \int_a^b \left((1-\lambda)\lambda |x(t)|^2 \right.$$

$$\left. - 2(1-\lambda)\lambda \left\langle x(t), y(t) \right\rangle + (1-\lambda)\lambda |y(t)|^2 \right)(b$$

$$-t)^{\alpha-1}\, dt = (1-\lambda)$$

$$\cdot \lambda \left(\int_a^b \left| x^{(\alpha)}(t) - y^{(\alpha)}(t) \right|^2 (b-t)^{\alpha-1}\, dt \right.$$

$$\left. - a_1 \int_a^b |x(t) - y(t)|^2 (b-t)^{\alpha-1}\, dt \right). \qquad (61)$$

If $a_1 \leq 0$ then

$$(1-\lambda)\lambda \left(\int_a^b \left| x^{(\alpha)}(t) - y^{(\alpha)}(t) \right|^2 (b-t)^{\alpha-1}\, dt \right.$$

$$\left. - a_1 \int_a^b |x(t) - y(t)|^2 (b-t)^{\alpha-1}\, dt \right) \geq (1-\lambda) \qquad (62)$$

$$\cdot \lambda \int_a^b \left| x^{(\alpha)}(t) - y^{(\alpha)}(t) \right|^2 (b-t)^{\alpha-1}\, dt > 0,$$

otherwise, from Lemma 11 and assumption $a_1 < \alpha\Gamma(2\alpha)/(b - a)^{2\alpha}$, it follows that

$$
\begin{aligned}
(1 - \lambda)\,\lambda & \left(\int_a^b \left| x^{(\alpha)}(t) - y^{(\alpha)}(t) \right|^2 (b-t)^{\alpha-1}\,dt \right. \\
& \left. - a_1 \int_a^b \left| x(t) - y(t) \right|^2 (b-t)^{\alpha-1}\,dt \right) \geq (1 - \lambda) \\
& \cdot \lambda \left(\int_a^b \left| x^{(\alpha)}(t) - y^{(\alpha)}(t) \right|^2 (b-t)^{\alpha-1}\,dt \right. \\
& \left. - a_1 \frac{(b-a)^{2\alpha}}{\alpha\Gamma(2\alpha)} \int_a^b \left| x^{(\alpha)}(t) - y^{(\alpha)}(t) \right|^2 (b-t)^{\alpha-1}\,dt \right) \\
& = (1 - \lambda)\,\lambda \left(1 - a_1 \frac{(b-a)^{2\alpha}}{\alpha\Gamma(2\alpha)} \right) \\
& \cdot \int_a^b \left| x^{(\alpha)}(t) - y^{(\alpha)}(t) \right|^2 (b-t)^{\alpha-1}\,dt > 0.
\end{aligned}
\tag{63}
$$

It means that the functional

$$
\begin{aligned}
K_0^{2,\alpha} & \ni x \\
& \longmapsto \frac{1}{2} \int_a^b \left(\left| x^{(\alpha)}(t) \right|^2 - a_1 |x(t)|^2 \right) (b-t)^{\alpha-1}\,dt \\
& \in \mathbb{R}
\end{aligned}
\tag{64}
$$

is strictly convex. Consequently, using assumption (55), we assert that the functional

$$
\begin{aligned}
\mathscr{L}(x) = & \int_a^b \left(F(t, x(t)) + \frac{a_1}{2} |x(t)|^2 \right) (b-t)^{\alpha-1}\,dt \\
& + \frac{1}{2} \int_a^b \left(\left| x^{(\alpha)}(t) \right|^2 - a_1 |x(t)|^2 \right) (b-t)^{\alpha-1}\,dt
\end{aligned}
\tag{65}
$$

is strictly convex, so its minimum point $x_0 \in K_0^{2,\alpha}$ is unique. On the other hand if $x \in K_0^{2,\alpha}$ is a solution to (1)-(2) then

$$
\left(x_0^{(\alpha)} \right)^{(\alpha)}(t) = F_x(t, x_0(t)), \quad t \in [a, b] \text{ a.e.}
\tag{66}
$$

Thus,

$$
\begin{aligned}
& \int_a^b \left(x_0^{(\alpha)} \right)^{(\alpha)}(t)\,h(t)\,(b-t)^{\alpha-1}\,dt \\
& \quad - \int_a^b F_x(t, x_0(t))\,h(t)\,(b-t)^{\alpha-1}\,dt = 0
\end{aligned}
\tag{67}
$$

for any $h \in K_0^{2,\alpha}$ and from Theorem 4 we get

$$
\begin{aligned}
& \int_a^b x_0^{(\alpha)}(t)\,h^{(\alpha)}(t)\,(b-t)^{\alpha-1}\,dt \\
& \quad + \int_a^b F_x(t, x_0(t))\,h(t)\,(b-t)^{\alpha-1}\,dt = \delta\mathscr{L}(x_0, h) \\
& = 0.
\end{aligned}
\tag{68}
$$

This means that the solution x_0 to problem (1)-(2) is a minimum point of $\mathscr{L}$, so it is unique. The proof is completed. ☐

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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