A C T A   U N I V E R S I T A T I S   L O D Z I E N S I S
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QUASI-ELLIPTICAL SYMMETRY AND DECOMPOSABILITY BY THE PAIR OF PROBABILITY MEASURES[1]

The problem of elliptical symmetry of an operator stable measure on finite dimensional vector space was studied by J.P. Holmes, W.N. Hudson and J.D. Mason [1]. Characterization of an elliptically symmetric full operator semi-stable measure was given by A. Luczak [5]. The paper deals with some analogon of the elliptical symmetry for full measure, which is decomposable by the pair (r, T_a), where r is real and positive and T_a is the multiplication operator.

1. Introduction

Let $\mathbf{V}$ denote a finite dimensional vector space over reals with an inner product $(,)$ and μ be a probability measure on $\mathbf{V}$. For an arbitrary linear operator A acting in $\mathbf{V}$ and Borel subset B of $\mathbf{V}$ a measure $A\mu$ is defined by

$$A\mu(B) = \mu(A^{-1}(B)),$$

where $A^{-1}(B)$ is an inverse image of B. From elementary calculations we get equalities for measures

$$A(B\mu) = (AB)\mu, \qquad A(\mu * \nu) = A\mu * A\nu,$$

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where A, B - linear operators, μ, ν - probability measures, asterisk denotes convolution, and for characteristic function

$$(A\hat{\mu})(y) = \hat{\mu}(A^*y),$$

where A^* is adjoint of A. Symbol δ_x will stand for the probability measure concentrated at point x. As an infinitely divisible measure μ has the unique representation $[x, D, M]$, where $x \in \mathbf{V}$, D is non-negative linear operator on $\mathbf{V}$, M is the Levy spectral measure of μ, so, it is easy to verify that the representation of $A\mu$ has the form

$$[x', ADA^*, AM] \tag{1}$$

for some $x \in \mathbf{V}$.

We recall now some basic definitions. The measure is *full* on $\mathbf{V}$, if it is not concentrated on any proper hyperplane of $\mathbf{V}$. The probability measure μ on $\mathbf{V}$ is operator semi-stable if

$$\mu = \lim_{n\to\infty} A_n v^{k_n} * \delta_{h_n},$$

where v stands for some probability measure on $\mathbf{V}$, $\{A_n\}$ is a sequence of linear operators on $\mathbf{V}$, k_n - positive integers fullfiling condition $k_{n+1}/k_n \to r$, $1 \leq r < \infty$, k_n-th power - in sense of convolution.

An infinitely divisible measure μ on $\mathbf{V}$ is decomposable by the pair (r, A), $r > 0$, $r \neq 1$, $A \in$ End $\mathbf{V}$ - set of all linear operators on $\mathbf{V}$, if

$$\mu^r = A\mu * \delta_h, \tag{2}$$

for some $h \in \mathbf{V}$.

The useful tool in describing properties of measures is so-called *the symmetry group of the measure* μ - the set of linear authomorphisms defined as follows

$$S(\mu) = \{a \in \mathrm{Aut}\mathbf{V};\quad \exists h \in \mathbf{V},\quad \mu = A\mu * \delta_h\}. \tag{3}$$

The measure μ is said to be *elliptically symmetric* if

$$S(\mu) = w^{-1}Ow$$

for some positive linear operator w on $\mathbf{V}$, O stands for the group of orthogonal operators. A. Luczak gave full characterization of operator semi-stable measures in [4] and of full elliptically symmetric operator semi-stable measures in [5]. He proved that these last measures are simply semi-stable in classical sense. Semi-stable probability measures were fully characterized by Jajte [2]

The paper deals with a special case of vector space $\mathbf{V}$, when dimension of $\mathbf{V}$ is n. It can be then regarded as the space of all linear operators on n-dimensional vector space (or equivalently - with all $n \times n$ real matrices). We denote it by $\mathbf{L}$. In this case some natural group of operators appears for modyfying measures, namely, the group of multiplication operators.

2. Properties of multiplication operators

For $a, x \in \mathbf{L}$, by T_a we mean left-side multiplication by a, $T_a(x) = a \circ x$ and by $_aT$ - the right-side mutiplication by a. We will omitt the sign " $\circ$ " in further text for simplicity. Algebraically operators

$$\{T_a; a \in \mathbf{L}\}$$

form a subalgebra $T_{\mathbf{L}}$. It has some specific properties :

(i) T_a is nonsingular iff a is nonsingular and $T_a^{-1} = T_{a^{-1}}$,

(ii) $T_a^* = T_{a^*}$, the asterisk means adjoint,

(iii) $\mathrm{sp}T_a = \mathrm{sp}a$, sp denotes the spectrum of an operator,

(iv) the subagebra $T_{\mathbf{L}}$ is closed,

(v) T_a is orthogonal if a is orthogonal.

Moreover if the matrix

$$A = [a_{i,j}], \quad i, j = 1, ..., n$$

corresponds to an operator a, then the matrix (of dimension $n^2 \times n^2$) corresponding to the operator T_a (by some standard basis) is of the form

$$(4) \qquad \begin{bmatrix} a_{11}I & \dots & a_{1n}I \\ \vdots & \ddots & \vdots \\ a_{n1}I & \dots & a_{nn}I \end{bmatrix}$$

where I - unite $n \times n$ matrix , and the matrix of an operator $_aT$ has the form

$$(5) \qquad \begin{bmatrix} A^* & & & \\ & A^* & & \\ & & \ddots & \\ & & & A^* \end{bmatrix}.$$

It can be shown, that operators T_a and $_{a^*}T$ are similar.

3. Quasi - elliptical symmetry of the measure and decomposability by the pair (γ, T_a).

For the probability measure μ we define the set $S(\mu)$

$$S(\mu) = \{a \in \text{Aut}\mathbf{V};\ \exists x \in \mathbf{L}\ \ \mu = T_a\mu * \delta_x\}.$$

It is obvious that $a \in S(\mu)$ iff $T_a \in S(\mu)$.

Definition. The probability measure μ is *quasi elliptically symmetric* if

$$S(\mu) = w^{-1}Ow$$

for some positive $w \in \mathbf{L}$, where O is orthogonal group contained in $\mathbf{L}$.

Directly from definition we see that the symmetry group of such measure has the form $T_{w^{-1}Ow}$. Moreover,

$$T_{w^{-1}\Theta w} = T_{w^{-1}}T_\Theta T_w$$

and T_w is positive, T_Θ is orthogonal ($\Theta \in O$) but it doesn't mean that quasi elliptical symmetry implies elliptical symmetry of the measure or vice versa.

Lemma 1. *Let μ be infinitely divisible measure such that $S(\mu) = O$. Then there exist $b \in \mathbf{L}$ and a probability measure v on $\mathbf{L}$ for which the equalities*

$$\mu = v * \delta_b,$$

and

$$T_u v = v,$$

hold for some $u \in O$.

Proof. Since $(-e) \in O$, (e - identity operator) so there exists some $x \in \mathbf{L}$, such that $\mu = T_{-e}\mu * \delta_x$. In terms of characteristic function we have

$$\hat{\mu}(y) = \overline{\hat{\mu}}(y)e^{i(x,y)}$$

and also

$$\hat{\mu}^2(y) = |\hat{\mu}(y)|^2 e^{i(x,y)}.$$

As $|\hat{\mu}|$ is the Fourier transform of the symmetrization ${}^{\circ}\mu^{1/2}$ of the measure $\mu^{1/2}$, the last equality can be rewritten in form

$$\mu * \mu = ({}^{\circ}\mu^{1/2} * \delta_{x/2}) * ({}^{\circ}\mu^{1/2} * \delta_{x/2}).$$

The infinite divisibility of ${}^{\circ}\mu^{1/2}$ implies $\mu = {}^{\circ}\mu^{1/2} * \delta_{x/2}$. Putting $b = x/2$ and $v = {}^{\circ}\mu^{1/2}$ we obtain

$$T_u v = T_u {}^{\circ}\mu^{1/2}(T_u {}^{\circ}\mu)^{1/2} = {}^{\circ}\mu^{1/2} = v,$$

for each $u \in O$, which ends the proof of the lemma.

Lemma 2. *Let μ be quasi elliptical symmetric probability measure on $\mathbf{L}$ decomposable by the pair (γ, T_a). Then two following conditions are satisfied*

(i) *$a = \alpha b$ for some $\alpha > 0$ and $b \in S(\mu)$,*

(ii) *the measure μ is decomposable by the pair $(\gamma, T_{\alpha e})$.*

Proof. We start with the case when $S(\mu) = O$. For each $u \in O$, from the decomposability of μ we get

$$T_{aua^{-1}}\mu = T_{au}\mu^{1/\gamma} * \delta_{x_1} = T_a(T_u\mu)^{1/\gamma} * \delta_{x_2} = T_a\mu^{1/\gamma} * \delta_{w_3} = \mu * \delta_{x_4}.$$

so $aua^{-1} \in S(\mu)$. Since the adjoint of an orthogonal operator is the converse we have

$$(a^{-1})^* u^* a^* = au^* a^{-1}.$$

Putting $|a|^2 = a^* a$, we obtain

$$u^* |a|^2 = a^2 u^*,$$

for each $u \in O$. Thus $|a|^2 = \alpha^2 e$, because operator $|a|^2$ commutes with the whole group O. We have then

$$|a| = \alpha e, \quad \alpha > 0.$$

From the polar decomposition

$$a = u_\circ |a|, \quad u_\circ \in O$$

and so

$$a = \alpha u_\circ, \quad \alpha > 0, \quad u_\circ \in S(\mu).$$

Now, we assume that

$$S(\mu) = w^{-1} O w.$$

It is easy to see that $S(T_w \mu) = O$ and $T_w \mu$ is decomposable by the pair $(\gamma, T_{waw^{-1}})$. From the consideration above we have

$$waw^{-1} = \alpha u'_\circ$$

for some $u'_\circ \in O$.

Putting $b = w^{-1} u'_\circ w$ we see that

$$b \in S(\mu)$$

and finally

$$a = \alpha b,$$

which ends the proof of (i).

Decomposability of μ by the pair (γ, T_a), condition (i) and the fact that $b \in S(\mu)$ imply

$$\mu^\gamma = T_a \mu * \delta_x = T_{\alpha b} \mu * \delta_x = T_{\alpha e} \mu * \delta_x,$$

which ends the proof of the lemma.

Theorem. *Let μ be the full, infinitely divisible probability measure on $\mathbf{L}$, decomposable by the pair (γ, T_a), $0 < \gamma < 1$, $\mathrm{sp}a \subset \{z : |z|^2 < \gamma\}$. If μ is quasi elliptically symmetric, then μ is semi-stable in classical sense. Moreover, if μ is pure Gaussian then its covariance operator is some composition of multiplication operators.*

Proof. From the assumption we have $S(\mu) = w^{-1}Ow$ for some positive linear operator w, and μ is decomposable by the pair (γ, T_a). Lemma 2 implies the decomposability of μ by the pair $(\gamma, T_{\alpha e})$ for some $\alpha > 0$. From equalities

$$\mu^{\gamma} = T_a\mu * \delta_x = T_{\alpha e}\mu * \delta_x$$

it yields that $\alpha^{-1}a \in S(\mu)$. It can be shown, that if λ is eigenvalue of an operator from $S(\mu)$, then $|\lambda| = 1$. Thus, if λ is an eigenvalue of a, then $|\lambda| = \alpha$. From the assumption we have $\alpha^2 < \gamma < 1$.

Putting $s = 1/\gamma$ and iterating n-times the equality

$$\mu = T_{\alpha e}\mu^{s} * \delta_x,$$

we get

$$\mu = T^{n}_{\alpha e}\mu^{s^n} * \delta_{x_n}.$$

Let $k_n = [s^n]$ - the entier of s^n and $v_n = T^n_{\alpha e}\mu^{k_n} * \delta_{x_n}$. As $k_{n+1}/k_n \to s$ and $T^n_{\alpha e} = T_{\alpha^n e} \to 0$ - zero operator, so we have

$$\frac{\hat{\mu}(y)}{\hat{v}(y)} = \{\mu\,(T^*_{\alpha^n e}(y))\}^{(s^n - [s^n])} \to 1,$$

as $n \to \infty$.

It means that $v_n \Rightarrow \mu$ and μ is semi-stable measure. According to Kruglow [3], μ is either Gaussian or purely Poissonian. As μ is decomposable by the pair (γ, T_a), so $T_w\mu$ is decomposable by $(\gamma, T_{waw^{-1}})$ and also by the pair $(\gamma, T_{\alpha e})$ - on account of Lemma 2. Since $S(T_w\mu) = O$, according to Lemma 1, there exists $x' \in \mathbf{L}$ and a probability measure ν, such that

$$\nu = T_w\mu * \delta_{(-x')} \quad \text{and} \quad T_u\nu = \nu \;\; \text{for all} \;\; u \in O. \tag{6}$$

Consequently $T_{(-e)}\nu = \nu$, and $\hat{\nu}(y) = \overline{\hat{\nu}}(y)$, so the characteristic function of the measure ν is real. If μ is purely Gaussian, so is the measure ν (on account of first equality in (6)), and its characteristic fnction has the form

$$\hat{\nu}(y) = \exp\{-1/2(Dy, y)\}$$

where covariance operator D of ν satisfies - according to second equality in (6) and equality (1) - condition $D = T_u D T_u^*$ for each $u \in O$. Thus we have

$$T_u^* D = D T_u^*. \tag{7}$$

As it was mentioned in introduction, by some standard basis in $\mathbf{L}$, matrix of an operator T_u is of the form (4). From similarity of the operators T_{u^*} and $_uT$, there exists another basis, by which, the matrix of the operator T_u^* $(= T_{u^*})$ has the diagonal form (5), where U - the $n \times n$ matrix of the operator u appears n-times on the diagonal. Dividing the matrix of D into n^2 minors of dimension $n \times n$, multiplication of matrices corresponding to $T_u^* D$ has the form

$$\begin{bmatrix} U & & & \\ & U & & \\ & & \ddots & \\ & & & U \end{bmatrix} \begin{bmatrix} D_{11} & \dots & D_{1n} \\ \vdots & & \vdots \\ D_{n1} & \dots & D_{nn} \end{bmatrix} = \begin{bmatrix} UD_{11} & \dots & UD_{1n} \\ \vdots & & \vdots \\ UD_{n1} & \dots & UD_{nn} \end{bmatrix}.$$

As the matrix of DT_u^* consists of minors $D_{i,j}U$, $i, j = 1, \dots, n$, so, from the equality

$$UD_{i,j} = D_{i,j}U \quad i, j = 1, \dots, n,$$

for any U - matrix of an orthogonal operator from O, which is the consequence of (7), we conclude that $D_{i,j} = \alpha_{i,j} I$, $\alpha_{i,j}$ - reals, and I - the $n \times n$ unit matrix. Thus,the matrix of D has the form (4), but turning back to the first basis, it is of the form

$$\begin{bmatrix} A & & & \\ & A & & \\ & & \ddots & \\ & & & A \end{bmatrix} \tag{8}$$

where

$$A = \{\alpha_{i,j};\ \ i,j = 1,\ldots,n\}.$$

Thus, according to (5), it is the matrix corresponding to right hand side multiplication operator $_{a^*}T$. From the symmetry of covariance operator, we conclude that $_{a^*}T = {}_aT$. On account of (1) and (6), the covariance operator of the measure μ is $T_w^{-1}{}_aT(T_w^{-1})^*$. Reflecting the symmetry of w, after simple calculations, it can be written in the form $T_{(w^{-1})^2a}T$, which ends the proof.

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QUASI-ELIPTYCZNA SYMETRIA I ROZKŁADALNOŚĆ PRZEZ PARĘ MIAR PRAWDOPODOBIEŃSTWA

Zagadnienie eliptycznej symetrii miary operatorowo-stabilnej w skończenie wymiarowych przestrzeniach wektorowych było badane przez J.P. Holmesa, W.N. Hudsona i J.D. Masona. Charakteryzacje pełnej, eliptycznie symetrycznej, operatorowo półstabilnej miary podał A. Łuczak. Niniejsza praca zajmuje się pewnym analogonem eliptycznej symetrii dla pełnej miary, ktora jest jednocześnie rozkładalna przez parę (r, T_a), gdzie r jest pewną liczbą rzeczywistą dodatnią, zaś T_a jest operatorem mnożenia.

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