A C T A U N I V E R S I T A T I S L O D Z I E N S I S
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GRAVITY UNFOLDING ANALYSIS FOR ASYMMETRIC SIMILARITIES MATRIX

Abstract. The multidimensional unfolding analysis which uses in the structure the gravity function allows consideration and analysis of additional factors (such as market share or brand loyalty), which affect the assessments of respondents preference scores. In classical unfolding models these factors are ignored or assumed to be the same for all objects.

The gravity unfolding analysis can also be used for graphical presentation of the data contained in the asymmetric similarity matrix expressing information about changes in respondents' preferences in a definite period. As a result, instead of the configuration of points representing objects and respondents, the configuration of points representing objects at different periods is obtained.

Key words: unfolding analysis, gravity model, preference maps.

I. GRAVITY MODELS IN SOCIAL SCIENCES

Gravity models are an adaptation of Newton's law of universal gravity to the economic phenomena. In social sciences, gravity models are widely used to describe the spatial interactions, which encompass any movement over space that results from a human process. For example, it can be applied to estimate international trade flows. This model assumes that the value of trade between the countries is directly proportional to the gross domestic products of the two countries, and inversely proportional to the distance between them. Other applications can be found for example in migration patterns, telecommunications and in transportation (see: Haynes and Fotheringham (1984)). In marketing, gravity models have been used to forecast the demand for leisure travel and shopping centres, to select the optimal size and location for shopping centres and to estimate market area boundaries (see: Haynes and Fotheringham (1984), Mayo *at al*. (1988)).

The idea of the gravity model can also be used in preference studies, including those in which the results are presented in a graphical form. The unfolding analysis, which is a special case of multidimensional scaling, represents such a method. It assumes that different individuals perceive various objects of choice

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in the same space, but differ with respect to what they consider an ideal combination of the objects' attributes. In unfolding, the data are usually preference scores of different individuals for a set of choice objects. These data can be conceived as proximities between the elements of two sets, individuals and choice objects.

II. UNFOLDING ANALYSIS BASIS

Unfolding attempts to produce a configuration **Y** of points in the r-dimensional space with each point $\mathbf{y}_j$ ($j = 1, \ldots, m$) representing one of m judges, together with another configuration **X** of points $\mathbf{x}_i$ ($i = 1, \ldots, n$) in the same space, these points representing choice objects. Individuals are represented as „ideal" points in the multidimensional space, so that the distances from each ideal point to the object points correspond to the preference scores.

For preference judgements p_{ij} unfolding attempts to find configurations **X** and **Y** that minimize STRESS function:

$$S = \sqrt{\sum_{i<j} \left(d_{ij} - \hat{d}_{ij}\right)^2 \Big/ \sum_{i<j} d_{ij}^2}\ , \tag{1}$$

where:

$d_{ij} = \sqrt{\sum_{a=1}^{r} \left(x_{ia} - y_{ja}\right)^2}$ – distance between $\mathbf{x}_i$ and $\mathbf{y}_j$,

$\hat{d}_{ij} = f(d_{ij})$ – monotonic regression of d_{ij} on p_{ij}.

For non-metric preference judgements disparities $\hat{d}_{ij}$ must satisfy the monotonic restriction:

$$p_{ij} \prec p_{i'j'} \Rightarrow \hat{d}_{ij} \le \hat{d}_{i'j'}$$

Unfolding solution can be computed by the majorization algorithm, where STRESS is reduced by iteratively taking Guttman transform. After K step of iteration the updates of **X** and **Y** becomes (see: Borg and Groenen (2005)):

$$\begin{aligned} \mathbf{X}^K &= \mathbf{V}_{11}^{+}\left[\mathbf{B}_{11}(\mathbf{X}^*, \mathbf{Y}^*)\mathbf{X}^* + \mathbf{B}_{12}(\mathbf{X}^*, \mathbf{Y}^*)\mathbf{Y}^*\right] \\ \mathbf{Y}^K &= \mathbf{V}_{22}^{+}\left[\mathbf{B}_{12}(\mathbf{X}^*, \mathbf{Y}^*)^T\mathbf{X}^* + \mathbf{B}_{22}(\mathbf{X}^*, \mathbf{Y}^*)\mathbf{Y}^*\right] \end{aligned}, \tag{2}$$

where:

$[\mathbf{V}_{11}^{+}]_{n\times n} = m^{-1}(\mathbf{I}-(n+m)^{-1}\mathbf{1}\mathbf{1}^{T})$,

$[\mathbf{V}_{22}^{+}]_{m\times m} = n^{-1}(\mathbf{I}-(n+m)^{-1}\mathbf{1}\mathbf{1}^{T})$,

1 – column vector of ones,

$\mathbf{B}_{12}(\mathbf{X}^*,\mathbf{Y}^*)$ – matrix with elements $b_{ij} = \begin{cases} \dfrac{-f_{ij}}{d_{ij}(\mathbf{X}^*,\mathbf{Y}^*)} & \text{for } d_{ij}(\mathbf{X}^*,\mathbf{Y}^*)\neq 0 \\ 0 & \text{for } d_{ij}(\mathbf{X}^*,\mathbf{Y}^*)=0 \end{cases}$,

$\mathbf{B}_{11}(\mathbf{X}^*,\mathbf{Y}^*)$ – diagonal matrix with elements $b_{ii} = -\sum_k b_{ik}$,

$\mathbf{B}_{22}(\mathbf{X}^*,\mathbf{Y}^*)$ – diagonal matrix with elements $b_{kk} = -\sum_i b_{ik}$,

$\mathbf{X}^*, \mathbf{Y}^*$ – configurations **X** and **Y** after $K-1$ step of iteration.

III. GRAVITY UNFOLDING

In the classical analysis of unfolding, it is assumed that the preferences of all respondents are determined by the same dimensions. However, it does not take into account certain factors, which influence the preferences. One of these factors may be, for example market share of different brands. Consumers prefer more popular brands because they are perceived as representing beliefs, thoughts and expectations of others brands. On the other hand, if preferences are measured by the size of shopping, the evaluation of brand preferences can be affected by the size of household income. The preferences can also affect the number of children in the family, etc. These additional factors in the model will be referred as gravitational masses. Depending on whether one of the factors is associated with ownership of the respondents or with brands we call it the consumer-mass or the brand-mass.

Similarly, as in any gravity model, here also the assumption is made, that assessment preferences are directly proportional to the product of the respondent's and object's masses and inversely proportional to the distance between the respondent's point and the object's point on perceptual map.

In unfolding gravity model preferences of individuals are presented in the following form (see: DeSarbo *at al.* (2002)):

$$p_{ij} = \varphi_i\left(\frac{M_j^{\beta} S_i^{\chi}}{d_{ij}^2}\right), \tag{3}$$

where:

p_{ij} – preference score of individual *i* for object *j*,

φ_i – monotonous non-decreasing function for individual *i*,

$d_{ij} = \sqrt{\sum_{a=1}^{r}(x_{ja} - y_{ia})^2}$ – distance between $\mathbf{x}_i$ and $\mathbf{y}_j$,

M_j – brand mass of object *j*,

S_i – consumer mass of individual *i*,

β and χ – mass parameters.

The unfolding analysis algorithm using a gravity model is an iterative process and is similar to other methods of multidimensional scaling. The iterative non-metric procedure of gravity unfolding consists of the following steps (for details see: Imaizumi (2005), Zaborski (2013)):

1. To obtain initial joint configuration **X**, **Y** (if the volume of the masses are not known, then it is assumed that their initial values are all equal to 1).

2. To normalize current joint configuration, compute value of loss function, and check whether this iterative process is converged or not.

3. To update joint configuration if it is not converged, and if the volume of the masses are not known, to update $\{M_j\}$ and $\{S_i\}$.

4. To repeat steps 2 and 3.

IV. THE APPLICATION OF GRAVITY UNFOLDING FOR ASYMMETRIC PREFFERENCES MATRIX

The idea of gravity unfolding can also be used for the asymmetric matrix containing information about the changes of preferences in time.

The example shows changes in voter preferences for the political parties that have been selected to the lower chamber of the Polish Parliament in 2007 and in 2011. The research conducted by TNS OBOP on the movement of the electorate shows that 57.4% voters who voted for the Left and Democrats (LiD) in parliamentary elections in 2007, in 2011 cast their votes for Democratic Left Alliance (SLD), 17.9% decided to cast their vote for Civic Platform (PO), 15.9% for the Palikot’s Movement (RP), 4.4% voted for Polish People's Party (PSL), and just over 2% for Law and Justice (PiS). 74.4% voters who voted for Civic Platform in 2007 voted for this party again, 10.9% voted for Palikot’s Movement, 4.6% for Left and Democrats, 3.9% for Polish People's Party, and 3.9% for Law and Justice. Law and Justice received again 84.4% votes from its electorate in 2007, 3.9% voted for Civic Platform, 3.6% for the Polish People's Party, 2.7% for Poland Comes First (PJN), and 2.5% for the Palikot's Movement. 67.7% Polish People's Party voters in 2007 backed up the party in 2011, 8.5% supported Left and Democrats, 8.4% voted for Civic Platform, 7.3% for Law and Justice and 5.5% for the Palikot's Movement.

Table 1 presents electorate switching data, where element in *i*-th row and in *j*-th column shows the approximate number of voters (in thousands) who voted in 2007 for *i*-th party but in 2011 voted for *j*-th party. We equate m_i and m_j with the row and column sums of the electorate switching data, and they are treated as the masses of the parties rightly in 2007 and in 2011.

Table 1. Electorate switching data among parties which have been selected to the lower chamber of the Polish Parliament in 2007 and in 2011

From	To					
	SLD	PO	RP	PSL	PiS	$m_i = \sum_j p_{ij}$
LID	1.219	380	338	93	42	2.072
PO	308	4.985	730	261	261	6.545
PSL	122	121	79	937	105	1.364
PiS	52	202	129	187	4.515	5.085
$m_j = \sum_i p_{ij}$	1.701	5.688	1.276	1.478	4.923	

Source: author's estimation.

The data in Table 1 can be interpreted as similarities, because large values indicate that voters easily switch between the two parties, and hence, consider them similar. For unfolding, we need to transform these similarities into dissimilarities. Transformation in the sense of the gravity model:

$$\delta_{ij} = \sqrt{\frac{m_i \cdot m_j}{p_{ij}}}$$

gives an asymmetric dissimilarity matrix (Table 2) on which unfolding can be performed.

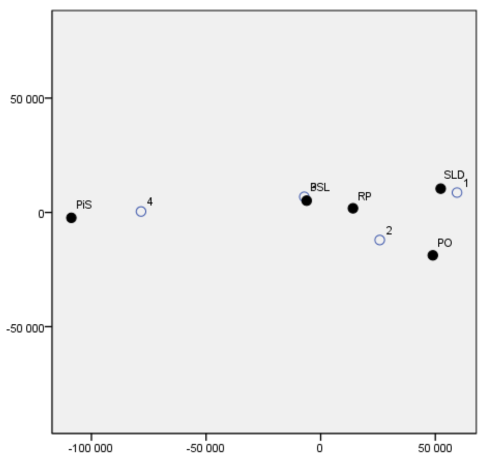
Table 2. Electorate switching data converted to the dissimilarities
in the sense of the gravity model

	SLD	PO	RP	PSL	PiS
LID	2.891	31.015	7.822	32.929	242.868
PO	36.146	7.468	11.440	37.063	123.452
PSL	19.018	64.119	22.031	2.151	63.952
PiS	166.338	143.185	50.298	40.190	5.544

Source: author's estimation.

The joint representation obtained by unfolding analysis is given in Figure 1, where rows (the parties in 2007) are plotted as open circles and columns (the parties in 2011) as solid points.

Fig. 1. Unfolding on the parties switching data after being converted by the gravity model



1 – Left and Democrats (LiD)
2 – Civic Platform (PO)
3 – Polish People's Party (PSL)
4 – Law and Justice (PiS)

(○ – parties in 2007; ● – parties in 2011)
Source: author's elaboration.

The layout of points on the perceptual map shows that voter preferences for Polish People's Party likewise for Left and Democrats and for Democratic Left Alliance have hardly altered in 2007 and in 2012, but we can notice a bigger diversification in preferences for Civic Platform and for Law and Justice. If we take into consideration voters preferences expressed by the electorate switching, in 2011 a coalition between Civic Platform and Democratic Left Alliance should be formed.

V. CONCLUSIONS

The use of gravity unfolding analysis allows the consideration of additional factors affecting the preferences of the respondents. An additional advantage of the model is that it can be used for analysing data contained in the asymmetric similarities matrix. This data informs us about preference changes in time. As a result of unfolding, a joint configuration of objects at different periods is obtained in this model.

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GRAWITACYJNA ANALIZA UNFOLDING DLA NIESYMETRYCZNEJ MACIERZY PODOBIEŃSTW

Wielowymiarowa analiza unfolding wykorzystująca w swojej konstrukcji funkcję grawitacji pozwala na uwzględnienie w analizie dodatkowych czynników (takich jak udział w rynku lub lojalność względem marki), które mają wpływ na oceny preferencji respondentów. W klasycznej analizie unfolding udział ten jest ignorowany lub przyjmuje się, że jest on taki sam dla wszystkich obiektów.

Grawitacyjna analiza unfolding może być również wykorzystana do graficznej prezentacji danych zawartych w niesymetrycznej macierzy podobieństw. Macierz taka zawiera informacje na temat zmian preferencji respondentów w pewnym okresie. W rezultacie nie otrzymujemy na mapie percepcyjnej, tak jak w przypadku analizy unfolding, konfiguracji punktów i respondentów, ale konfigurację punktów reprezentujących obiekty w różnych okresach.