ACTA UNIVERSITATIS LODZIENSIS
FOLIA OECONOMICA 225, 2009

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ON TESTING LINEARITY OF TREND FUNCTION

Abstract

Testing the goodness of fit between a hypothetical trend function and its non-parametric variant will be considered. This problem was analysed e.g. by D o m a ń s k i (1979, 1990), W y w i a ł (1990, 1995). Our result can be treated as a modification of the test proposed by A z z a l i n i and B o w m a n (1993). The hypothetical trend function will be denoted by $f(t, \theta)$. It is estimated by an unbiased method. A trend function can be estimated by means of an non-parametric method. Azzalini and Bowman suggested testing the hypothesis on the linearity of the trend on the basis of the ratio of two residual variances. One of them is the residual variance of the trend estimated by means of the least square method and the other one by means of a non-parametric method. The well known Pearson curves are used for an approximation of the distribution function of the ratio. We use a different method in order to approximate the distribution of the test statistic. The table with quantiles of the test statistic are evaluated.

Key words: kernel estimator, trend, testing, approximation of distribution function.

1. Introduction

Let us rewrite the time series $\{Y_t,\ t = 1,\ ...,\ n\}$ by means of the $\mathbf{Y}^T = [Y_1, Y_2, ..., Y_n]$ vector of independent and normally distributed random variables. We assume that $\mathbf{Y} \sim N(\mu, \mathbf{I}_n\sigma^2)$, where $\mathbf{I}_n$ is a unit matrix of n degree. Let $\mu^T=[\mu_1\ \mu_2\ ...\ \mu_n]$, $\mu_t = f(t)$, where f(t) is the trend function. So, $E(\mu_t)=f(t)$. Let us assume that $Y_t=f(t)+\varepsilon_t$, where $\boldsymbol{\varepsilon}^T=[\varepsilon_1\ \varepsilon_2\ ...\ \varepsilon_n]$ and $\varepsilon \sim N(\mathbf{0}, \mathbf{I}_n\sigma^2)$. Let $\mathbf{e}=[e_1...e_n]$ be the following well known residual vector of estimator of the linear trend, obtained by means of least square method:

$$\mathbf{e} = \mathbf{Y}\mathbf{P}^T \tag{1}$$

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where

$$\mathbf{P} = \mathbf{I}_n - \mathbf{X}\left(\mathbf{X}^T\mathbf{X}\right)^{-1}\mathbf{X}^T, \tag{2}$$

$$\mathbf{X}^T = \begin{bmatrix} N \\ J \end{bmatrix}, \quad \mathbf{N}=[1\ 2\ \dots\ n], \quad \mathbf{J}=[1\ 1\ \dots\ 1].$$

The parameters of the **e** vector are as follows:

$$\delta_e = \mu\mathbf{P}^T, \quad \Sigma_{ee} = \sigma^2\mathbf{P}\mathbf{P}^T. \tag{3}$$

Hence, $\mathbf{e} \sim N(\delta_e, \Sigma_{ee})$ and $R(\mathbf{P}) = n-2$.

The well known (see e.g. Härdle (1991)) non-parametric estimator of the trend function $f(t)$ is as follows:

$$\hat{\mu}_t = \sum_{j=1}^{n} Y_j a_{tj} \tag{4}$$

Where

$$a_{tj} = \frac{g\{(t-j)/h\}}{\sum_{j=1}^{m} g\{(t-j)/h\}}, \tag{5}$$

the window parameter is denoted by $h > 0$ while $g\{.\}$ is the density function of standard normal distribution. Let $\hat{\mu} = [\hat{\mu}_1 \dots \hat{\mu}_n]$ and $\mathbf{A} = [a_{tj}]$. So:

$$\hat{\mu} = \mathbf{Y}\mathbf{A}^T \tag{6}$$

The residuals of the non-parametric estimator of the trend are defined by

$$\hat{\mathbf{e}} = \mathbf{Y} - \hat{\mu} \tag{7}$$

or

$$\hat{\mathbf{e}} = \mathbf{Y}\mathbf{B}^T \tag{8}$$

where:

$$\mathbf{B} = \mathbf{I}_n - \mathbf{A}. \tag{9}$$

Moreover, $\hat{\mathbf{e}} \sim N(\Delta, \Sigma_e)$, where:

$$\Delta = \mu \mathbf{B}^T, \tag{10}$$

$$\Sigma_{\hat{e}} = \sigma^2 \mathbf{B}\mathbf{B}^T, \tag{11}$$

$R(\Sigma_{\hat{e}}) = R(\mathbf{B}) = k < n$. We assume that $\mu \mathbf{A}^T = \mu$, $\Delta = \mathbf{O}$ and $\hat{\mathbf{e}} \sim N(\mathbf{O}; \Sigma_{\hat{e}})$.

2. Tests of trend linearity

When the postulated trend agrees with the true one, $\boldsymbol{\delta}_e = \mathbf{0}$. Hence, our aim is to test the hypothesis $H_0 : \boldsymbol{\delta}_e = \mathbf{0}$ against the alternative one $H_1 : \boldsymbol{\delta}_e \neq \mathbf{0}$.

Azzalini and Bowman (1993) proposed the following test statistic:

$$R_n = \frac{\mathbf{e}\mathbf{e}^T - \mathbf{e}\mathbf{B}^T\mathbf{B}\mathbf{e}^T}{\mathbf{e}\mathbf{B}^T\mathbf{B}\mathbf{e}^T} \tag{12}$$

Sufficiently large value of the R_n statistic leads to a rejection of the hypothesis H_0. The p-value of the test can be evaluated on the basis of the following expression:

$$p = P(R_n \geq r) = P\left(e\left(I - (1+r)B^T B\right)e^T > 0\right) \tag{13}$$

The distribution function of the quadratic form $e\left(I - (1+r)B^T B\right)e^T$ is approximated by means of Pearson curves.

Let us consider the following test statistic:

$$G_n = \frac{Q(\mathbf{e})}{Q_1(\hat{\mathbf{e}})} \tag{14}$$

where

$$Q(\mathbf{e}) = \frac{\mathbf{e}\mathbf{e}^T}{k}, \quad Q_1(\hat{\mathbf{e}}) = \frac{\hat{\mathbf{e}}\Sigma_{\hat{e}}^{-}\hat{\mathbf{e}}^T}{k} \tag{15}$$

$\Sigma_{\hat{e}}^{-}$ is the pseudo-inverse of the $\Sigma_{\hat{e}}$ matrix. The basic definitions and theorems of mathematical statistics lead to the conclusion that $(n-2)Q(\mathbf{e})$ and $kQ_1(\hat{\mathbf{e}})$ have chi-square distributions with $(n-2)$ and k degrees of freedoms, respectively. Moreover,

$$E\left(Q(\mathbf{e}) \mid H_0\right) = E\left(Q_1(\hat{\mathbf{e}}) \mid H_0\right) \tag{16}$$

$$E\left(Q(\mathbf{e}) \mid H_1\right) \geq E\left(Q_1(\hat{\mathbf{e}}) \mid H_1\right) \tag{17}$$

Hence, significantly large value of the G_n test statistic leads to the rejection of the H_0 hypothesis. The p-value of the test can be evaluated on the basis of the following expressions:

$$p = P\left(G_n \geq g \mid H_0\right) \tag{18}$$

$$p = P\left(Q(\mathbf{e}) - gQ_1(\hat{\mathbf{e}}) \geq 0 \mid H_0\right) \tag{19}$$

$$p = P\left(U(g) \geq 0 \mid H_0\right) \tag{20}$$

where

$$U(g) = \mathbf{YM}(g)\mathbf{Y}^T \tag{21}$$

$$\mathbf{M}(g) = \frac{1}{n-2}\mathbf{P} - \frac{g}{k}\mathbf{B}^T\left(\mathbf{BB}^T\right)^{-1}\mathbf{B} \tag{22}$$

Similarly as it is in the case of the Azzalini and Bowman's test, the distribution function of the G_n statistic can be approximated by means of the Pearson curves. Below, we are going to consider one of the other methods of approximating a probability distribution function.

3. Approximation of the distribution function

On the basis of the general method pointed out by Provost and Rudiuk (1991), see Mathai, Provost (1992, pp. 152–153), too, we have the following algorithm. Let **L** be such an orthogonal matrix that $\mathbf{LL}^T = \mathbf{I}_n$ and

$$\mathbf{Y} = \mathbf{ZL} \tag{23}$$

$$\mathbf{L}^T\mathbf{M}(g)\mathbf{L} = \mathbf{D}(g) \tag{24}$$

where $\mathbf{D}(g)$ is the diagonal matrix of eigenvalues of the $\mathbf{M}(g)$ matrix. Let $\mathbf{D}(g)$ equal diag$(d_1, \ldots, d_e, d_{e+1}, \ldots, -d_r)$ where $d_i > 0$ for $i = 1, \ldots, r$. Hence, the expression (20) may be rewritten in the following way:

$$p=P\{\mathbf{Z}^{\mathrm{T}}\mathbf{D}(g)\mathbf{Z}<0\} \tag{25}$$

where $\mathbf{Z}\sim N_r(\mathbf{0}, \boldsymbol{I}_n)$.

The general results of Imhof (1961) lead to the following result (see Mathai and Provost (1992, pp. 141, too):

$$p=P\{\mathbf{Y}^T\mathbf{M}(g)\mathbf{Y}<0\}=P\{\mathbf{Z}^T\mathbf{D}(g)\mathbf{Z}<0\}=\frac{1}{2}-\frac{1}{\pi}\int_0^{\infty}\frac{\sin(\varepsilon(\nu))}{\nu\gamma(\nu)}d\nu \tag{26}$$

where:

$$\varepsilon(\nu)=\frac{1}{2}\sum_{i=1}^{m}tg^{-1}(d_i\nu) \tag{27}$$

$$\gamma(\nu)=\prod_{i=1}^{m}\sqrt[4]{1+d_i^2\nu^2} \tag{28}$$

The value x is substituted for ∞ in such a way that the following inequality should be fulfilled (see Koerts and Abrahams, 1969):

$$\left|\frac{1}{\pi}\int_0^{x}\frac{\sin(\varepsilon(\nu))}{\nu\gamma(\nu)}d\nu\right|<\frac{2}{m}\frac{1}{\pi}\left(\prod_{i=1}^{m}\sqrt{|d_i|}\right)^{-1}x^{-\frac{m}{2}}=e \tag{29}$$

Next, the $\frac{1}{\pi}\int_0^{x}\frac{\sin(\varepsilon(\nu))}{\nu\gamma(\nu)}d\nu$ integral is evaluated approximately by means of an appriopriate numeric method.

4. Approximate evaluation of quantiles of the test statistic

The approximate values of the quantiles of order 0.9, 0.95 and 0.99 were evaluated on the basis of the above method. The optimal value of the window parameter is fixed on the $h_g=\frac{1,06}{n^{0,2}}$ level as it was suggested by Gajek and Kałuszka (2000). The well known method of trapezium was used for an approximation of the integral. The quantiles are presented in the Table 1. The Table 1 shows the approximate dependence between the size of the sample and the values of the quantiles.

Table 1

Quantiles of the G_n test statistic

N	q_α quantile			n	q_α quantile		
	0.9	0.95	0.99		0.9	0.95	0.99
5	5.41	6.33	7.48	18	10.24	11.90	16.21
6	6.13	7.69	11.26	19	10.57	12.23	16.49
7	6.70	8.33	13.24	20	10.91	12.57	16.79
8	7.12	8.87	13.91	21	11.25	12.92	17.11
9	7.47	9.22	14.36	22	11.61	13.27	17.45
10	7.80	9.55	14.59	23	11.97	13.64	17.81
11	8.10	9.84	14.76	24	12.34	14.02	18.18
12	8.40	10.12	14.92	25	12.72	14.41	18.56
13	8.69	10.40	15.09	26	13.11	14.81	18.97
14	8.99	10.68	15.27	27	13.50	15.22	19.38
15	9.29	10.97	15.47	28	13.91	15.64	19.81
16	9.60	11.27	15.70	29	14.33	16.07	20.23
17	9.92	11.58	15.94	30	14.75	16.50	20.71

S o u r c e: authors' own calculations.

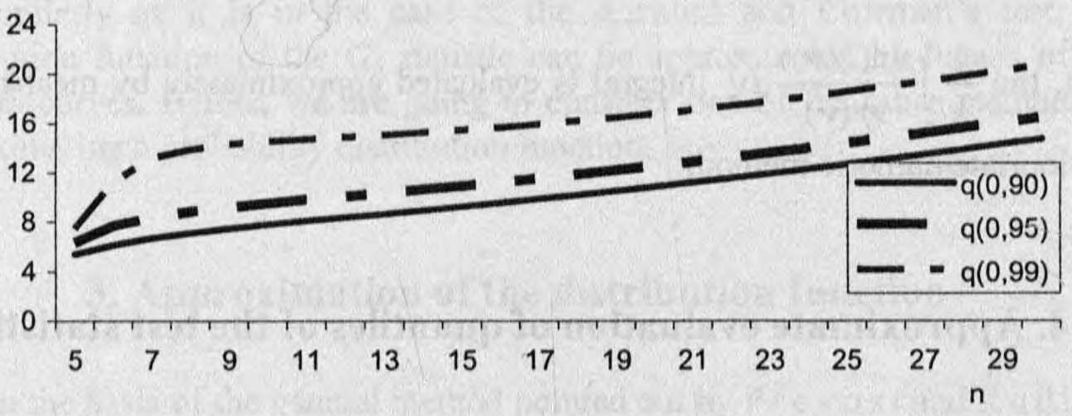

Fig. 1. Quantiles of the test statistic

S o u r c e: own study.

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O testowaniu hipotezy o liniowości trendu

W artykule rozważano pewną modyfikację testu dla hipotezy głoszącej, że trend szeregu czasowego ma postać liniową, który zaproponowali Azzalini i Bowman (1993). Statystyka testowa jest ilorazem wariancji resztowej estymatora wartości funkcji liniowej trendu uzyskanej metodą najmniejszych kwadratów i pewnej formy kwadratowej reszt oceny trendu otrzymanego metodą estymacji jądrowej. Wysokie wartości tego ilorazu świadczą przeciwko hipotezie o liniowości funkcji trendu. Ze względu na złożoną postać proponowanej statystyki Azzalini i Bowman wykorzystali do aproksymacji jej rozkładu prawdopodobieństwa tzw. krzywe Pearsona. W niniejszym artykule stosuje się do przybliżenia rozkładu sprawdzianu testu inną metodę wykorzystującą technikę całkowania numerycznego. Przy założeniu liniowej postaci funkcji trendu pozwoliło to wyznaczyć numerycznie kwantyle rzędu 0,9, 0,95 i 0,99 rozważanej statystyki testowej. Przedstawione wartości kwantyli mogą stanowić podstawę do podjęcia decyzji o ewentualnym odrzuceniu hipotezy o postaci trendu liniowego.