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BAYESIAN INTERVAL ESTIMATION OF SHARPE STYLE WEIGHTS IN THE MODEL OF STYLE ANALYSIS OF THE MANAGEMENT OF OPEN PENSION FUNDS [1]

Abstract. According to the Law on the Organization and Functioning of the Retirement Pay and Pension Funds of 28 August 1997, the Polish Open Pension Funds (OPFs) invest in assets carrying different degrees of risk, from different sectors. Since the investment effectiveness varies between sectors, it is difficult to distinguish the profits which result from the choice of particular sectors from the profits earned from the choice of the assets within a given sector. Style analysis, developed by William Sharpe, is a method of identifying results of the assets portfolio management which allows to form conclusions on the influences exerted on the investment portfolios of the funds and their investment management styles.

The restrictions imposed on the parameters of Sharpe style analysis result in the fact that the distribution of the estimator of the Ordinary Least Squares Method (OLS) is unknown. The aim of this paper is to implement the Bayesian method in the interval estimation of Sharpe style weights based on the model of the style analysis of the management of a selected OPF.

Key words: Sharpe Style Analysis, Bayesian confidence interval, highest posteriori density interval, Open Pension Fund.

I. SHARPE STYLE ANALYSIS MULTI-FACTOR MODEL

The concept of *style*, developed by William Sharpe, applies to such investment policy of the fund which allows to achieve its target results.

The relationship between the fund's rate of return and a set of style indices' rates of return representing the returns on particular asset classes that the fund invested in time t, $t = 1,2,\ldots,T$ has the form:

$$R_t = \beta_1 F_{t1} + \beta_2 F_{t2} + \ldots + \beta_k F_{tk} + \varepsilon_t, \; t = 1,2,\ldots,T \qquad (1)$$

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where R_t means the fund's rate of return in time t; F_{ti}, $i = 1,2,\dots,k$ is the return rate of the i-th style index in time t; β_i, $i = 1,2,\dots,k$ is the i-th parameter of the model and it is the i-th sensitivity factor of the rate R_t to the rate F_{ti}, whereas ε_t, $t = 1,2,\dots,T$ are independent random variables such that $\varepsilon_t \sim N(0,\sigma^2)$ and such that $F_{t1},\dots,F_{tk}$ and ε_t are independent.

The vector $\boldsymbol{\beta} = (\beta_1, \beta_2, \dots, \beta_k)'$ of the unknown parameters of the model (1) is the vector of coefficients determining the structure of the portfolio of k asset classes represented by corresponding indices. If $\mathbf{F}_t = (F_{t1}, F_{t2}, \dots, F_{tk})'$ denotes the vector of the rates of return of the style indices in time t, then the product $\mathbf{F}_t' \boldsymbol{\beta}$ is referred to as the return on a *"style portfolio"*. The value of the $\hat{\boldsymbol{\beta}}_{\text{MNK}}$ estimator of the vector of parameters $\boldsymbol{\beta}$ of the Sharpe style, obtained by OLS Method, is the solution to the quadratic programming problem (2) under the constraints imposed on the vector $\boldsymbol{\beta}$ in the form of the set (3):

$$\min_{\boldsymbol{\beta} \in \Theta} \quad \frac{1}{T} \sum_{t=1}^{T} (R_t - \mathbf{F}_t' \boldsymbol{\beta})^2 \tag{2}$$

$$\Theta = \{\boldsymbol{\beta} : \ \mathbf{1}' \boldsymbol{\beta} = 1, \ \boldsymbol{\beta} \geq 0\} \tag{3}$$

where $\mathbf{1} = (1,1,\dots,1)' \in \Re^k$. The vector of estimators $\hat{\boldsymbol{\beta}}_{\text{MNK}}$ is referred to as *vector of the OLS estimators of Sharpe-style weights.*

The restrictions imposed on the parameters of Sharpe style analysis result in the fact that the distribution of the OLS estimator is unknown. For this reason, the paper applies the Bayesian method for point estimation and construction of confidence intervals for the parameters of the Sharpe style model. Confidence intervals for style weights allow to verify whether a financial instrument represented by a given style index should belong to the set of the effective assets of the fund. As a result, the analysis allows to assess correctly the risk that a fund manager bears.

II. BAYESIAN ESTIMATORS AND CONFIDENCE INTERVALS FOR SHARPE STYLE WEIGHTS

The core of the Bayesian method is the assumption that the estimated parameter is a random variable of prior distribution $\pi(\boldsymbol{\beta})$. When estimating Sharpe style weights with the Bayesian method, it is assumed that the prior distribution representing the constraints of the vector of style coefficients is non-informative.

Since the Lebesque measure of the set $\boldsymbol{\Theta} = \{\boldsymbol{\beta}:\ \mathbf{1}'\boldsymbol{\beta} = 1,\ \boldsymbol{\beta} \geq 0\}$ equals zero, it is necessary to transform model (1) and as a consequence, the set $\boldsymbol{\Theta}$ into the set $\boldsymbol{\Theta}^*$ with the non-zero Lebesque measure. As a result of the transformation: $R_t^* = R_t - F_{tk}$, $F_{ti}^* = F_{ti} - F_{tk}$, $i = 1,2,...,k-1$ model (1) takes the form:

$$R_t^* = \beta_1 F_{t1}^* + \beta_2 F_{t2}^* + \ldots + \beta_{k-1} F_{t\,k-1}^* + \varepsilon_t,\ t = 1,2,\ldots,T \tag{4}$$

This model satisfies all assumptions of model (1). Let $\boldsymbol{\beta}^* = (\beta_1, \beta_2, \ldots, \beta_{k-1})'$. Since $\beta_k = 1 - \mathbf{1}^{*\prime}\boldsymbol{\beta}^*$, where $\mathbf{1}^* = (1,1,\ldots,1)' \in \Re^{k-1}$, we need to determine the vector $\hat{\boldsymbol{\beta}}^*_{\mathrm{MNK}}$ of the OLS estimators of the Sharpe-style vector $\boldsymbol{\beta}^*$. Problem (2) with constraints (3) is reduced to:

$$\min_{\boldsymbol{\beta}^* \in \boldsymbol{\Theta}^*} \quad \frac{1}{T}\sum_{t=1}^{T}(R_t^* - \mathbf{F}_t^{*\prime}\ \boldsymbol{\beta}^*)^2 \tag{5}$$

$$\Theta^* = \{\boldsymbol{\beta}^*\ :\ \mathbf{1}'\boldsymbol{\beta}^* \leq 1,\ \boldsymbol{\beta}^* \geq 0\} \tag{6}$$

where $\mathbf{F}_t^* = (F_{t1}^*, F_{t2}^*, \ldots, F_{t\,k-1}^*)'$, $t = 1,2,\ldots,T$.

Let $\mathbf{R}^* = (R_1^*, R_2^*, \ldots, R_T^*)'$, $\boldsymbol{\varepsilon} = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_T)'$ and $\mathbf{F}^* = \left(\mathbf{F}_1^*, \mathbf{F}_2^*, \ldots, \mathbf{F}_T^*\right)'$. $T \times (k-1)$-dimensional matrix of observations of explanatory variables of the model (4), $t = 1,2,\ldots,T$. Then, model (4) has the following matrix form: $\mathbf{R}^* = \mathbf{F}^*\boldsymbol{\beta}^* + \boldsymbol{\varepsilon}$, where the vector $\boldsymbol{\beta}^*$ satisfies the conditions (6). We assume that $\mathrm{rz}(\mathbf{F}^*) = k-1$, where $k-1 < T$. Moreover, we assume that $\boldsymbol{\beta}^*$ and σ^2 are random and vectors $\mathbf{F}_1^*, \mathbf{F}_2^*, \ldots, \mathbf{F}_T^*$ are independent of $\boldsymbol{\beta}^*, \sigma^2, \varepsilon_t$ for $t = 1,2,\ldots,T$. We assume that the prior distribution of parameters $(\boldsymbol{\beta}^*, \sigma)$ has the form $\pi(\boldsymbol{\beta}^*, \sigma) = \sigma^{-1}\chi_{[\sigma>0]}\, q(\boldsymbol{\beta}^*)$ where $q(\boldsymbol{\beta}^*) = \chi_{[\mathbf{1}^{*\prime}\boldsymbol{\beta}^* \leq 1,\ \boldsymbol{\beta}^* \geq 0]}$. Based on the above assumptions, the density function of the posteriori distribution of parameters $(\boldsymbol{\beta}^*, \sigma)$ is expressed in the following way:

$$\pi(\boldsymbol{\beta}^*, \sigma | \mathbf{R}^*, \mathbf{F}^*) \div \sigma^{-(T+1)} \exp\left\{-(2\sigma^2)^{-1}\left[\nu s^2 + (\boldsymbol{\beta}^* - \hat{\boldsymbol{\beta}}^*_{\mathrm{MNK}})'\mathbf{V}(\boldsymbol{\beta}^* - \hat{\boldsymbol{\beta}}^*_{\mathrm{MNK}})\right]^{-0,5(\nu+k-1)}\right\} q(\boldsymbol{\beta}^*)\ ,$$

where $\hat{\boldsymbol{\beta}}^*_{\mathrm{MNK}}$ is the vector of estimators which is the solution to problem (5) under constraints (6),[2] $\nu = T-(k-1)$, $s^2 = \nu^{-1}(\mathbf{R}^* - \mathbf{F}^*\hat{\boldsymbol{\beta}}^*_{\mathrm{MNK}})'(\mathbf{R}^* - \mathbf{F}^*\hat{\boldsymbol{\beta}}^*_{\mathrm{MNK}})$, $\mathbf{V} \equiv s^{-2}\mathbf{F}^{*\prime}\mathbf{F}^*$.

Since $\pi(\boldsymbol{\beta}^*|\mathbf{R}^*,\mathbf{F}^*) = \int_0^{+\infty}\pi(\boldsymbol{\beta}^*,\sigma|\mathbf{R}^*,\mathbf{F}^*)d\sigma$ and the analytical computation of the integral is not possible, the posterior distribution $\pi(\boldsymbol{\beta}^*|\mathbf{R}^*,\mathbf{F}^*)$ of the random vector $\boldsymbol{\beta}^*$ may only be generated by the Monte Carlo numerical integration (MCI). The key aspect of this procedure is to estimate of posterior moments of parameters $\boldsymbol{\beta}^*$. In order to do this, we consider the function $g(\boldsymbol{\beta}^*)$ of the random vector $\boldsymbol{\beta}^*$ of one- or multi-dimensional real values. The point estimator of $g(\boldsymbol{\beta}^*)$ has the form:

$$\mathrm{E}\left(g(\boldsymbol{\beta}^*)\,\middle|\,\mathbf{R}^*,\mathbf{F}^*\right) = \int\left(\frac{g(\boldsymbol{\beta}^*)\pi(\boldsymbol{\beta}^*|\mathbf{R}^*,\mathbf{F}^*)}{I(\boldsymbol{\beta}^*)}\right)I(\boldsymbol{\beta}^*)d\boldsymbol{\beta}^* \tag{7}$$

where $I(\boldsymbol{\beta}^*)$ is the *importance function* approximating the density function of the posteriori distribution $\pi(\boldsymbol{\beta}^*|\mathbf{R}^*,\mathbf{F}^*)$. Taking into account the selection criterion for the importance function, developed by Kloek T., Van Dijk H. K., (1978), the function $I(\boldsymbol{\beta}^*)$ is the density of the $(k-1)$-dimensional t-Student distribution with the expected value $\mathbf{0}$ and the covariance matrix $\nu(\nu-2)^{-1}\mathbf{V}^{-1}$, where $\mathbf{V} \equiv s^{-2}\mathbf{F}^{*\prime}\mathbf{F}^*$. For the N-element set of random vectors realizations $\{\boldsymbol{\beta}^*_1,\boldsymbol{\beta}^*_2,\ldots,\boldsymbol{\beta}^*_N\}$ generated from the distribution of density $I(\boldsymbol{\beta}^*)$, according to Chebyshev's law of great numbers the point estimator of $g(\boldsymbol{\beta}^*)$ has the form:

$$\mathrm{E}\left(g(\boldsymbol{\beta}^*)\,\middle|\,\mathbf{R}^*,\mathbf{F}^*\right) = \lim_{N\to\infty}\sum_{i=1}^{N}\frac{g(\boldsymbol{\beta}^*_i)\pi(\boldsymbol{\beta}^*_i|\mathbf{R}^*,\mathbf{F}^*)}{I(\boldsymbol{\beta}^*_i)} \tag{8}$$

Hence the estimator $\hat{g}(\boldsymbol{\beta}^*)$ of the parametric function is:

[2] Notation $\div$ means "proportional to".

$$\hat{g}(\boldsymbol{\beta}^*) = \frac{1}{N}\sum_{i=1}^{N} g(\boldsymbol{\beta}_i^*)q(\boldsymbol{\beta}_i^*) \tag{9}$$

The MCI method also allows to determine the Bayesian confidence intervals for the parameter β_i, $i=1,2,\ldots,k-1$, which is the *i*-th element of the random vector $\boldsymbol{\beta}^*$.

The Bayesian confidence interval for the parameter β_i, *where* $1-\alpha$ *is the confidence level,* is the interval $(\beta_{i(\frac{\alpha}{2})}, \beta_{i(1-\frac{\alpha}{2})})$ such that

$$\int_{\beta_{i(\frac{\alpha}{2})}}^{\beta_{i(1-\frac{\alpha}{2})}} \pi(\beta_i \big| \mathbf{R}^*, \mathbf{F}^*)\, d\beta_i = 1-\alpha \tag{10}$$

where $\pi(\beta_i \big| \mathbf{R}^*, \mathbf{F}^*)$ – marginal posterior density for the parameter β_i, $i=1,2,\ldots,k-1$.

The boundaries of the interval are defined by the conditions: $\Pi(\beta_{i(\frac{\alpha}{2})}\big|\mathbf{R}^*,\mathbf{F}^*)=\frac{\alpha}{2}$, $\Pi(\beta_{i(1-\frac{\alpha}{2})}\big|\mathbf{R}^*,\mathbf{F}^*)=1-\frac{\alpha}{2}$, where $\Pi(\beta_i\big|\mathbf{R}^*,\mathbf{F}^*)$ – marginal posterior cumulative distribution function of the parameter β_i. If density $\pi(\beta_i\big|\mathbf{R}^*,\mathbf{F}^*)$ is symmetrical unimodal, the above Bayesian confidence interval is also the highest posterior density interval (HPD), which has the form:

$$R(\pi_\alpha) = \{\beta_i^* : \pi(\beta_i^* \big| \mathbf{R}^*, \mathbf{F}^*) \geq \pi_\alpha\} \tag{11}$$

where π_α is the highest constant such that $P(\beta_i^* \in R(\pi_\alpha)) \geq 1-\alpha$. If the marginal posterior density function is continuous and unimodal, the HPD interval is $(\beta_{i\,L}^*, \beta_{i\,U}^*)$ where $\beta_{i\,L}^*, \beta_{i\,U}^*$ are the solutions to the optimization problem:

$$\min_{\beta_{i\,L}^* < \beta_{i\,U}^*} \left\{ \left| \pi(\beta_{i\,U}^* \big| \mathbf{R}^*, \mathbf{F}^*) - \pi(\beta_{i\,L}^* \big| \mathbf{R}^*, \mathbf{F}^*) \right| + \left| \Pi(\beta_{i\,U}^* \big| \mathbf{R}^*, \mathbf{F}^*) - \Pi(\beta_{i\,L}^* \big| \mathbf{R}^*, \mathbf{F}^*) - (1-\alpha) \right| \right\} \tag{12}$$

The computation of two types of the Bayesian intervals is possible with the use of the Markov chain Monte Carlo method. The estimation algorithms of these intervals are thoroughly discussed in Chen M.-H., Shao Q.-M. (2000).

III. EMPIRICAL RESULTS

The major categories of assets of the OPF investment portfolio, reflecting the strategic decisions on the assets allocation in subsequent months of the years 2003-2005, are shown in Chart 1. The portfolio also had a similar structure in the years 2006-2007. The main factors influencing the portfolio structure are character of the fund specifying statutory investment limits and the situation on the financial market.

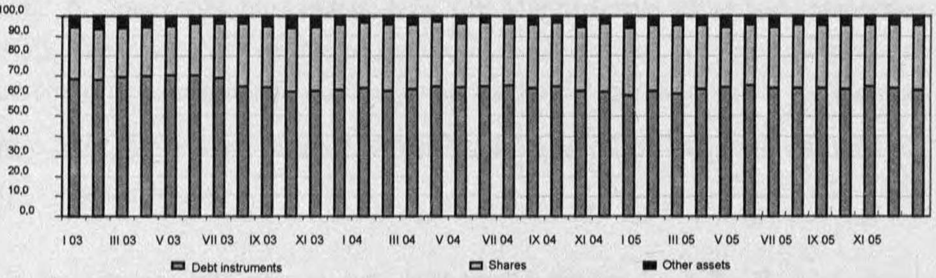

Chart 1. The structure of major investment categories of the OFE investment portfolio.
Source: Own compilation based on the data provided by the Insurance and Pension Funds Supervisory Commission.

We analyzed the influence of investment in the selected types of assets classes on the rates of returns of the OFE ING NN accounting unit in the studied period. In order to do this, we conducted point and interval estimation for the parameters of Sharpe style analysis with the Bayesian method. The real monthly rates of return from February 2002 to January 2007 were analysed. Monthly rates of return on the accounting unit of the fund were the dependent variable of the model.

During the first stage of the analysis, monthly rates of return based on WIG sector indices and the selected type of bonds constituted the set of explanatory variables of the model. The monthly rates of return on bonds were determined based on the accounting price. The selection criterion for the type of bonds was the largest share in the structure of the debt instruments portfolio of OFE ING NN in the studied time. Table 1 presents the description of the explanatory variables and the results of point and interval estimation of weights of the style analysis model for OFE ING NN.

Table 1. Results of point and interval estimation of weights for sector indices of the style analysis model for OFE ING NN

Sector index	Variable symbol	Value of Bayesian estimator	95% Bayesian confidence interval		95% Bayesian confidence HPD interval	
			L	U	L	U
WIG-finance	**F**	0.1764	0.1073	0.2491	0.1081	0.2496
WIG-retail and services	**HiU**	0.0688	0.0145	0.1200	0.0140	0.1186
WIG-manufacturing	**P**	0.0340	0.0039	0.0749	0.0016	0.0686
5-year fixed interest bonds	**O-PS**	0.7208	0.6622	0.7782	0.6606	0.7763
	TOTAL	1				

Source: Own computation.

With the probability of 0.95, asymptotic confidence intervals do not cover zero in the case of coefficients of all analysed indices, so the parameters are significantly different from zero, where $\alpha = 0.05$ is the significance level. We can conclude that within the studied period the investment in assets coming from these three sectors had a significant influence on the achieved rates of return at the significance level of 0.05.

We also analysed the model which explanatory variables were monthly rates of return of selected WIG sub-sector indices and treasury bonds.

Table 2. Results of point and interval estimation of weights for sub-sector indices of the style analysis model for OFEINGNN

Sub-sector index	Variable symbol	Value of Bayesian estimator	95% Bayesian confidence interval		95% Bayesian confidence HPD interval	
			L	U	L	U
WIG-banks	**FB**	0.2022	0.1409	0.2552	0.1425	0.2561
WIG-pharmaceuticals retail and distribution	**HHD**	0.0300	0.0025	0.0680	0.0001	0.0610
WIG-information technology	**HTI**	0.0385	0.0036	0.0813	0.0010	0.0747
WIG- wood industry	**PD**	0.0167	0.0006	0.0482	0.0002	0.0396
WIG- chemicals	**PH**	0.0248	0.0011	0.0683	0.0006	0.0599
5-year fixed interest bonds	**PS**	0.5265	0.2536	0.6917	0.2828	0.7043
Zero-coupon bonds	**OK**	0.1613	0.0062	0.4334	0.0008	0.3855
	TOTAL	1				

Source: Own computation.

With the probability of 0.95, asymptotic confidence intervals do not cover zero in the case of coefficients of all analysed sub-sector indices, so the parameters are significantly different from zero, where α=0.05 is the significance level. We can conclude that within the studied period the investment in assets coming from these sub-sectors had a significant influence on the rates of return achieved by the fund at the significance level of 0.05.

IV. CONCLUSION

The estimation results for the parameters of style analysis model for OFE ING NN, conducted with the use of the Bayesian method, are consistent with a general structure of major categories within the investment portfolio in terms of shares and bonds. Style analysis may complement effectiveness analyses based on classic tools for effectiveness measurement, for example, Sharpe, Treynor and Jansen's indices, offered by one-factor models (see Orwat A., Trzpiot G. (2004)). Interval estimation of the parameters for style analysis model is an important aspect of Sharpe style analysis method, which may be used to investigate style composition, style sensitivity and style change over time, which is under further research. The application of the results provided by these methods to particular market parameters allows to assess the effectiveness of the investment and classify the performance more precisely, which is also under further investigation.

REFERENCES

Chen M.-H., Shao Q.-M. (2000), *Monte Carlo Methods in Bayesian Computation*, Springer Series in Statistics, New York.

Kim T., White H., Stone D., (2005), *Asymptotic and Bayesian Confidence Intervals for Sharpe-Style Weights*, "Journal of Financial Econometric", Vol. 3, No 3, pp. 315–343.

Kloek T., Van Dijk H. K., (1978), *Bayesian Estimates of Equation System Parameters: An Application of Integration by Monte Carlo*, "Econometrica", No 46, pp. 1–19.

Orwat A., Trzpiot G., (2004), *Efektywność Inwestycji OFE na polskim rynku finansowym,* „Inwestycje finansowe i ubezpieczenia – Tendencje światowe a polski rynek", AE Wrocław, No 1037, pp. 295–303.

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METODA BAYESOWSKA W ESTYMACJI PRZEDZIAŁOWEJ WAG STYLI SHARPE'A W MODELU ANALIZY STYLI OFE

Polskie Otwarte Fundusze Emerytalne (OFE) dokonują inwestycji w pochodzące z różnych sektorów aktywa o odmiennych stopniach ryzyka zgodnie z Ustawą o Organizacji i Funkcjonowaniu Funduszy Emerytalnych z dnia 28 sierpnia 1997 roku. Ponieważ efektywność inwestycji w poszczególnych sektorach jest zróżnicowana, trudno jest odróżnić zyski wynikające z dokonanego wyboru między sektorami od zysków wynikających z wyboru konkretnych aktywów w ramach danego sektora. Analiza stylu opracowana przez Williama Sharpe'a, jest metodą przypisywania wyników zarządzania portfelem aktywów, umożliwiającą formułowanie wniosków na temat wpływów wywieranych na portfele inwestycyjne funduszy oraz stylów zarządzania nimi. Konsekwencją obecności ograniczeń nałożonych na parametry modelu analizy stylu Sharpe'a jest fakt, że rozkład estymatora metody najmniejszych kwadratów wag stylu nie jest znany. Celem pracy jest implementacja metody bayesowskiej w estymacji przedziałowej wag stylu Sharpe'a na przykładzie modelu analizy stylu zarządzania wybranego OFE.