Folia Mathematica
Vol. 18, No. 1, pp. 3–10
Acta Universitatis Lodziensis

FAMILIES OF INCREASING SEQUENCES POSSESSING THE HARMONIC SERIES PROPERTY

ROMAN WITUŁA, EDYTA HETMANIOK AND DAMIAN SŁOTA

Abstract. We say that family $\mathcal{W} \neq \emptyset$ of increasing sequences of positive real numbers possesses the harmonic series property (HSP – for the shortness of notation) if for any finite sequence of elements of $\mathcal{W}$, i.e. for any $k \in \mathbb{N}$, $\{a_n^{(i)}\}_{n=1}^{\infty} \in \mathcal{W}$, $i = 1, \ldots, k$, we have

$$\sum_{n=1}^{\infty} \Big(a_n^{(1)} + a_n^{(2)} + \ldots + a_n^{(k)} \Big)^{-1} = \infty$$

(the sequences $\{a_n^{(i)}\}_{n=1}^{\infty}$ and $\{a_n^{(j)}\}_{n=1}^{\infty}$ for different indices i and j can be the same). We prove in this paper that any maximal, with respect to inclusion, subset of $\mathcal{N}$ – the family of all increasing sequences of positive integers – possessing the harmonic series property has the cardinality of the continuum.

Moreover, we prove that for any countable (infinite) set $\mathcal{W} \subset \mathcal{N}$ there exists an "orthogonal" family $\mathcal{W}^{\perp} \subset \mathcal{N}$ such that

a) $\operatorname{card} \mathcal{W}^{\perp} = \mathfrak{c}$,

b) $\Big(\forall \{a_n\}, \{b_n\} \in \mathcal{W}^{\perp}\Big)\Big(\{a_n\} \neq \{b_n\} \Rightarrow \sum (a_n + b_n)^{-1} < \infty\Big)$ (this condition is a reason for using the word "orthogonal" – the value "0" or "$\neq 0$" of the scalar product is replaced here by the convergence or divergence, respectively, of the series),

c) $\Big(\forall \{a_n\} \in \mathcal{W}^{\perp}\Big)$ $\Big($ the family $\mathcal{W} \cup \{\{a_n\}_{n=1}^{\infty}\}$ possesses the harmonic series property $\Big)$.

All facts are proved constructively, by using the modified version of the classical Sierpiński family of increasing sequences having the cardinality of the continuum.

AMS Subject Classification. 40A05, 03E10.
Key words and phrases. Sierpiński family, harmonic series property.

1. Introduction

Aim of this paper is to give the answers for two fundamental questions

(1) How much one can extend the subsets of $\mathcal{N}$ – the family of all increasing sequences of positive integers – by preserving HSP?

It appears that each countable (finite or infinite) set of that kind is a subset of some set of the cardinality of the continuum, maximal with respect to inclusion.

(2) Given a countable set $\mathcal{W} \subset \mathcal{N}$ possessing HSP does there exists a set $\mathcal{W}^{\perp} \subset \mathcal{N}$ of the cardinality of the continuum "canalizing" the extensions of set $\mathcal{W}$ onto the sets possessing HSP by at most one element of $\mathcal{W}^{\perp}$? As "canalizing" we understand the fact that if $\mathcal{V}$ possesses HSP, $\mathcal{W} \subset \mathcal{V}$ and $\mathcal{W}^{\perp} \cap \mathcal{V} \neq \emptyset$, then $\operatorname{card}(\mathcal{W}^{\perp} \cap \mathcal{V}) = 1$.

Answer to this question is positive, as well.

2. Main results

Our main results are proved in three steps. At first let us prove the following lemma.

Lemma 1. *For each $\{a_n\} \in \mathcal{N}$ such that $\sum a_n^{-1} = +\infty$ there exists a countable infinite family $\mathcal{W} \subset \mathcal{N}$ including the sequence $\{a_n\}$ and possessing HSP.*

Proof. Let us define an auxiliary sequence $\{n_i\} \subset \mathbb{N}$ in the following way. We put $n_1 = 1$. Suppose that number $n_i \in \mathbb{N}$ is determined for some $i \in \mathbb{N}$. Then the number $n_{i+1} \in \mathbb{N}$ is selected so that $n_{i+1} > n_i + 1$ and so that

$$\sum_{n=n_i}^{n_{i+1}-1} \left[i(a_n + n_i^2 + n - n_i) \right]^{-1} \geqslant 1.$$

We use here the following property which is easy to justify: if $\{a_n\} \in \mathcal{N}$ and $\sum a_n^{-1} = +\infty$, then for each $\alpha > 0$ we have

$$\sum (a_n + \alpha n)^{-1} = +\infty.$$

$$\left(\begin{array}{l} \text{Indeed, we have} \\ (a_n \geqslant n,\ n \in \mathbb{N}) \Rightarrow \\ \Rightarrow \sum (a_n + \alpha n)^{-1} \geqslant \sum (a_n + \alpha a_n)^{-1} = (1+\alpha)^{-1} \sum a_n^{-1} = +\infty. \end{array} \right)$$

All we need now is to take

$$a_n^{(k)} = \begin{cases} a_n + n^2 & \text{for} \quad n < n_k, \\ a_n + n_i^2 + n - n_i & \text{for} \quad n :\ n_i \leqslant n < n_{i+1},\ i \geqslant k. \end{cases}$$

Certainly $\{a_n^{(k)}\}_n \in \mathcal{N}$, $\{a_n\} \neq \{a_n^{(k)}\}$ for each $k \in \mathbb{N}$ and $\{a_n^{(k)}\} \neq \{a_n^{(l)}\}$ whenever $k \neq l$.

Let $\mathcal{W} := \big\{\{a_n^{(k)}\}_n : \ k \in \mathbb{N}\big\} \cup \big\{\{a_n\}_n\big\}$.

One can easily verify that family $\mathcal{W}$ possess HSP.

Indeed, let us fix the sequences $\{a_n^{(k_1)}\}_n, \ldots, \{a_n^{(k_s)}\}_n$ from $\mathcal{W}$, not necessarily different, such that $k_1 \leqslant \ldots \leqslant k_s$. Then there exists $t \in \mathbb{N} \cup \{0\}$ such that $s \geqslant t$ and for every positive integer $i \geqslant \max\{k_s, s\}$ we get

$$\begin{aligned}\sum_{n=n_i}^{n_{i+1}-1} \big(a_n^{(k_1)} + \ldots + a_n^{(k_s)}\big)^{-1} &= \sum_{n=n_i}^{n_{i+1}-1} \big(ta_n + (s-t)(a_n + n_i^2 + n - n_i)\big)^{-1} \geqslant \\ &\geqslant \sum_{n=n_i}^{n_{i+1}-1} \big(sa_n + i(n_i^2 + n - n_i)\big)^{-1} \geqslant \\ &\geqslant \sum_{n=n_i}^{n_{i+1}-1} \big[i(a_n + n_i^2 + n - n_i)\big]^{-1} \geqslant 1.\end{aligned}$$

Hence we receive

$$\begin{aligned}\sum_{n\geqslant 1} \big(a_n^{(k_1)} + \ldots + a_n^{(k_s)}\big)^{-1} &\geqslant \sum_{i\geqslant \max\{k_s,s\}} \ \sum_{n=n_i}^{n_{i+1}-1} \big(a_n^{(k_1)} + \ldots + a_n^{(k_s)}\big)^{-1} \geqslant \\ &\geqslant \sum_{i\geqslant \max\{k_s,s\}} 1 = +\infty.\end{aligned}$$

□

Corollary 1 (from the proof of Lemma 1)**.** *For each finite family $\mathcal{W}' \subset \mathcal{N}$ possessing HSP there exists a countable infinite family $\mathcal{W} \subset \mathcal{N}$, possessing HSP as well, which contains the family $\mathcal{W}'$.*

Proof. Let $\mathcal{W}' = \big\{\{b_n^{(1)}\}_n, \ldots, \{b_n^{(s)}\}_n\big\}$. We set $a_n = \sum_{i=1}^{s} b_n^{(i)}$, $n \geqslant 1$. Let $\{a_n^{(k)}\}_n$, $k \in \mathbb{N}$ be sequences constructed in the same way as in the proof of Lemma 1 for the sequence $\{a_n\}$. Certainly $a_n^{(k)} > a_n > b_n^{(i)}$ for any $n, k, i \in \mathbb{N}$, $1 \leqslant i \leqslant s$. Thus, the family $\mathcal{W} = \big\{\{a_n^{(k)}\}_n : k \in \mathbb{N}\big\} \cup \mathcal{W}'$ possess HSP. □

In the second step we prove the following theorem.

Theorem 1. *For each countable infinite family $\mathcal{W}' \subset \mathcal{N}$ possessing HSP there exists a family $\mathcal{W} \subset \mathcal{N}$ with* $\operatorname{card} \mathcal{W} = \mathfrak{c}$*, possessing HSP as well, which contains $\mathcal{W}'$.*

Remark 1. *Family $\mathcal{W}$ can be obtained constructively. We intend here to use a concept of* ***the Sierpiński family of sequences of natural numbers****.*

By definition, family S of increasing sequences of natural numbers is called the Sierpiński family if $\operatorname{card} S = \mathfrak{c}$ *and any two different sequences* $\mathbb{a} = \{a_n\}_{n=1}^{\infty}$, $\mathbb{b} = \{b_n\}_{n=1}^{\infty} \in \mathcal{S}$ *possess almost disjoint sets of values, it means, a constant* $N \in \mathbb{N}$ *(depending only on* $\mathbb{a}$ *and* $\mathbb{b}$*) exists such that* $a_n \neq b_m$ *for any* $n, m \in \mathbb{N}$, $n, m \geqslant N$ *(we say also that* $\mathbb{a}$ *and* $\mathbb{b}$ *are almost disjoint sequences* [1]*).*

For technical reasons, in proof of Theorem 1 we will use the appropriate ***Sierpiński family $\mathcal{S}^*$ of zero-one sequences*** *associated with the fixed Sierpiński family* $\mathcal{S}$ *of sequences of natural numbers* ($\mathcal{S} \ni \{a_n\} \mapsto \{b_n\} \in \mathcal{S}^*$ *where* $b_k = 1$ *whenever* $k \in \{a_n\}$ *and* $b_k = 0$ *for other* $k \in \mathbb{N}$*. Thus, if* $\{b_n\}, \{c_n\} \in \mathcal{S}^*$ *and* $\{b_n\} \neq \{c_n\}$ *then there exists* $N \in \mathbb{N}$ *such that* $b_n + c_n \leqslant 1$ *for every* $n \geqslant N$ *and both equations* $b_n = 1$ *and* $c_n = 1$ *possess infinite sets of solutions).*

Proof. (of Therorem 1) Let $\mathcal{W}' = \big\{\{a_n^{(k)}\}_n : k \in \mathbb{N}\big\}$. We construct an auxiliary sequence $\{n_i\}$ of positive integers. Let $n_1 = 1$. If we have already the fixed $n_i \in \mathbb{N}$ for some $i \in \mathbb{N}$, then $n_{i+1} \in \mathbb{N}$ is selected so that $n_{i+1} > n_i + 1$ and so that the following inequality holds

$$\sum_{n=n_i}^{n_{i+1}-1} \left[i\left(2\sum_{k=1}^{i} a_n^{(k)} + n_i^2 + n - n_i \right) \right]^{-1} \geqslant 1.$$

Let S be a Sierpiński family of zero-one sequences. To each sequence $\{\varepsilon_i\} \in S$ we assign some sequence $\{e_n\} \subset \mathbb{N}$ defined in the following way
(1)

$$e_n = \begin{cases} \sum\limits_{k=1}^{i} a_n^{(k)} + n_i^2 + n - n_i & \text{for } n: \, n_i \leqslant n < n_{i+1} \text{ whenever } \varepsilon_i = 0, \\ \sum\limits_{k=1}^{i} a_n^{(k)} + n^2 & \text{for } n: \, n_i \leqslant n < n_{i+1} \text{ whenever } \varepsilon_i = 1. \end{cases}$$

Obviously $\{e_n\} \in \mathcal{N}$ and mapping $S \ni \{\varepsilon_i\} \overset{(1)}{\mapsto} \{e_n\} \in \mathcal{N}$ is one-to-one. Hence, if $\mathcal{R}$ is the family of all sequences $\{e_n\}$ defined by formula (1), then $\operatorname{card} \mathcal{R} = \operatorname{card} \mathcal{S} = \mathfrak{c}$. Moreover, let us notice that for any $k \in \mathbb{N}$ and $\{e_n\} \in \mathcal{R}$ we have $e_n > a_n^{(k)}$ for every $n \geqslant n_k$.

We prove that family $\mathcal{W} := \mathcal{W}' \cup \mathcal{R}$ possesses HSP. For this purpose let us fix the sequences $\{e_n^{(1)}\}, \ldots, \{e_n^{(s)}\} \in \mathcal{R}$ (not necessarily different, associated with sequences $\{\varepsilon_n^{(1)}\}, \ldots, \{\varepsilon_n^{(s)}\} \in S$, respectively) and sequences $\{a_n^{(k_1)}\}, \ldots, \{a_n^{(k_t)}\}$ (not necessarily different as well) where $k_1 \leqslant \ldots \leqslant k_t$.

If $i \in \mathbb{N}$, $i \geqslant \max\{k_t, s, t\}$ and $\varepsilon_i^{(1)} = \ldots = \varepsilon_i^{(s)} = 0$, then we get

$$\sum_{n=n_i}^{n_{i+1}-1} \left(a_n^{(k_1)} + \ldots + a_n^{(k_t)} + e_n^{(1)} + \ldots + e_n^{(s)}\right)^{-1} =$$

$$= \sum_{n=n_i}^{n_{i+1}-1} \left(a_n^{(k_1)} + \ldots + a_n^{(k_t)} + s\left(\sum_{k=1}^{i} a_n^{(k)} + n_i^2 + n - n_i\right)\right)^{-1} =$$

$$= \sum_{n=n_i}^{n_{i+1}-1} \left(t\sum_{k=1}^{i} a_n^{(k)} + s\left(\sum_{k=1}^{i} a_n^{(k)} + n_i^2 + n - n_i\right)\right)^{-1} \geqslant$$

$$\geqslant \sum_{n=n_i}^{n_{i+1}-1} \left(i\sum_{k=1}^{i} a_n^{(k)} + i\left(\sum_{k=1}^{i} a_n^{(k)} + n_i^2 + n - n_i\right)\right)^{-1} =$$

$$= \sum_{n=n_i}^{n_{i+1}-1} \left[i\left(2\sum_{k=1}^{i} a_n^{(k)} + n_i^2 + n - n_i\right)\right]^{-1} \geqslant 1.$$

All we need is to notice that the set $\{i \in \mathbb{N} : \varepsilon_i^{(1)} = \ldots = \varepsilon_i^{(s)} = 0\}$[1] is infinite which implies that

$$\sum_{n\geqslant 1} \left(a_n^{(k_1)} + \ldots + a_n^{(k_t)} + e_n^{(1)} + \ldots + e_n^{(s)}\right)^{-1} \geqslant$$

$$\geqslant \sum_{\substack{i\geqslant \max\{k_t,s\} \\ \varepsilon_i^{(1)}=\ldots=\varepsilon_i^{(s)}=0}} \sum_{n=n_i}^{n_{i+1}-1} \left(a_n^{(k_1)} + \ldots + a_n^{(k_t)} + e_n^{(1)} + \ldots + e_n^{(s)}\right)^{-1} \geqslant$$

$$\geqslant \sum_{\substack{i\geqslant \max\{k_t,s\} \\ \varepsilon_i^{(1)}=\ldots=\varepsilon_i^{(s)}=0}} 1 = +\infty.$$

□

[1]If $\mathcal{S}$ is the Sierpiński family (either the family of increasing sequences of positive integers or of the zero-one sequences), then for any two finite $A = \{\{a_n^{(k)}\}_n : 1 \leqslant k \leqslant K\} \subset \mathcal{S}$ and $B = \{\{b_n^{(l)}\}_n : 1 \leqslant l \leqslant L\} \subset \mathcal{S}$ if $A \cap B \neq \emptyset$ then the sequences $\left\{\sum_{k=1}^{K} a_n^{(k)}\right\}_n$ and $\left\{\sum_{l=1}^{L} b_n^{(l)}\right\}_n$ are either almost disjoint (in the case of increasing sequences of positive integers) or possesses the same property as two different zero-one sequences from family $\mathcal{W}^*$ described in Remark 1.

Corollary 2. *Each set $\mathcal{W}' \subset \mathcal{N}$ possessing HSP is a subset of some set of the cardinality of the continuum possessing HSP, maximal with respect to inclusion.*

Proof. Proof of the above corollary easily follows from the Kuratowski-Zorn Lemma. □

In the third step we focus on proving the extreme version of theorem proved in the second step. We prove the theorem in which the phenomenon of extension of the given family is subject to the phenomenon of "canalizing".

Theorem 2. *For each countable family $\mathcal{W}' \subset \mathcal{N}$ possessing HSP there exists family $\mathcal{W} \subset \mathcal{N}$ having cardinality $\mathfrak{c}$ such that*

a) $\forall \{a_n\} \in \mathcal{W}$ family $\mathcal{W}' \cup \{\{a_n\}_n\}$ possesses HSP;
b) $\forall \{a_n\}, \{b_n\} \in \mathcal{W} : (\{a_n\} \neq \{b_n\}) \Rightarrow \sum (a_n + b_n)^{-1} < +\infty$.

Proof. Let $\mathcal{W}' = \big\{\{a_n^{(k)}\}_n : k \in \mathbb{N}\big\}$. First, we construct an auxiliary sequence $\{n_i\} \subset \mathbb{N}$. We set $n_1 = 1$. If we have already defined the number n_i for some $i \in \mathbb{N}$, then the number $n_{i+1} \in \mathbb{N}$ is selected so that $n_{i+1} > n_i + 1$ and so that the following inequality holds

$$\sum_{n=n_i}^{n_{i+1}-1} \left[i\left(2\sum_{k=1}^{i} a_n^{(k)} + n_i^2 + n - n_i \right) \right]^{-1} \geqslant 1.$$

Let S be a Sierpiński family of zero-one sequences. To each sequence $\{\varepsilon_i\} \in S$ we assign some sequence $\{e_n\} \subset \mathbb{N}$ defined as follows

$$(2) \quad e_n = \begin{cases} \sum\limits_{k=1}^{i} a_n^{(k)} + n_i^2 + n - n_i & \text{for} \quad n_i \leqslant n < n_{i+1} \quad \text{whenever} \quad \varepsilon_i = 1, \\ \sum\limits_{k=1}^{i} a_n^{(k)} + n^2 & \text{for} \quad n_i \leqslant n < n_{i+1}, \quad \text{whenever} \quad \varepsilon_i = 0. \end{cases}$$

Certainly, the sequence $\{e_n\} \in \mathcal{N}$. From definition of family S we obtain that mapping (2) between family S and family $\mathcal{W}$ of sequences $\{e_n\}$, defined by formula (2), is one-to-one which means that $\operatorname{card} \mathcal{W} = \operatorname{card} \mathcal{S} = \mathfrak{c}$.

Let us fix the sequence $\{e_n\}$ belonging to $\mathcal{W}$ (assigned, as in (2), to the sequence $\{\varepsilon_i\} \in S$), number $t \in \mathbb{N}$ and sequences $\{a_n^{(k_1)}\}_n, \dots \{a_n^{(k_s)}\}_n$ (not necessarily different) where $k_1 \leqslant \dots \leqslant k_s$. Then if $i \in \mathbb{N}$, $i \geqslant \max\{k_s, s, t\}$

be such that $\varepsilon_i = 1$ the following estimation holds

$$\begin{aligned}
&\sum_{n=n_i}^{n_{i+1}-1} \big(te_n + a_n^{(k_1)} + \ldots + a_n^{(k_s)}\big)^{-1} \geqslant \\
&\qquad \geqslant \sum_{n=n_i}^{n_{i+1}-1} \bigg(t\bigg(\sum_{k=1}^{i} a_n^{(k)} + n_i^2 + n - n_i\bigg) + s\sum_{k=1}^{i} a_n^{(k)}\bigg)^{-1} \geqslant \\
&\qquad \geqslant \sum_{n=n_i}^{n_{i+1}-1} \bigg(i\bigg(\sum_{k=1}^{i} a_n^{(k)} + n_i^2 + n - n_i\bigg) + i\sum_{k=1}^{i} a_n^{(k)}\bigg)^{-1} = \\
&\qquad = \sum_{n=n_i}^{n_{i+1}-1} \bigg[i\bigg(2\sum_{k=1}^{i} a_n^{(k)} + n_i^2 + n - n_i\bigg)\bigg]^{-1} \geqslant 1
\end{aligned}$$

which, in view of infinity of the set $\{i \in \mathbb{N} : \varepsilon_i = 1\}$, implies

$$\begin{aligned}
\sum_{n\geqslant 1} \big(te_n + a_n^{(k_1)} + \ldots + a_n^{(k_s)}\big)^{-1} &\geqslant \sum_{\substack{i\geqslant 1\\ \varepsilon_i=1}} \sum_{n=n_i}^{n_{i+1}-1} \big(te_n + a_n^{(k_1)} + \ldots + a_n^{(k_s)}\big)^{-1} \geqslant \\
&\geqslant \sum_{i:\,\varepsilon_i=1} 1 = +\infty.
\end{aligned}$$

Let us fix now two different sequences $\{e_n\}$ and $\{f_n\} \in \mathcal{R}$ (assigned, as in (2), to sequences $\{\varepsilon_i\}$ and $\{\delta_i\} \in S$, respectively). There exists $i_0 \in \mathbb{N}$ such that for $i \geqslant i_0$ the inequality $\varepsilon_i + \delta_i \leqslant 1$ holds, it means that at least one of numbers ε_i or δ_i is equal to zero.

Hence we obtain

$$\begin{aligned}
\sum_{n\geqslant n_{i_0}} (e_n + f_n)^{-1} &= \sum_{i\geqslant i_0} \sum_{n=n_i}^{n_{i+1}-1} (e_n + f_n)^{-1} \leqslant \\
&\leqslant \sum_{i\geqslant i_0} \sum_{n=n_i}^{n_{i+1}-1} \bigg(\sum_{k=1}^{i} a_n^{(k)} + n_i^2 + n - n_i + \sum_{k=1}^{i} a_n^{(k)} + n^2\bigg)^{-1} \leqslant \\
&\leqslant \sum_{i\geqslant i_0} \sum_{n=n_i}^{n_{i+1}-1} n^{-2} = \sum_{n\geqslant n_{i_0}} n^{-2} < +\infty,
\end{aligned}$$

that is $\sum (e_n + f_n)^{-1} < +\infty$. The proof is finished. $\square$

References

[1] W. Sierpiński, *Cardinal and Ordinal Numbers*, PWN – Polish Scientific Publishers, Warsaw, 1965.

Roman Wituła
Institute of Mathematics, Silesian University of Technology
Kaszubska 23, Gliwice 44-100, Poland
E-mail: roman.witula@polsl.pl

Edyta Hetmaniok
Institute of Mathematics, Silesian University of Technology
Kaszubska 23, Gliwice 44-100, Poland
E-mail: edyta.hetmaniok@polsl.pl

Damian Słota
Institute of Mathematics, Silesian University of Technology
Kaszubska 23, Gliwice 44-100, Poland
E-mail: damian.slota@polsl.pl