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AN EVALUATION OF EFFICIENCY OF SOME ESTIMATORS FOR THE FIRST ORDER AUTOREGRESSIVE MODELS

1. Introduction

A Monte-Carlo study was done in response to calls of Chanda [6], Hendry, Trividi [9], Aigner [2], and Dent, Min [8] to compare and evaluate the efficiencies of the proposed estimators of unknown parameters using:

1) ordinary least squares (OLS),
2) modified least squares (MOD. LS),
3) approximate maximum likelihood (APR.ML),
4) exact maximum likelihood (EXACT ML),

for the stationary Markov model.

Monte Carlo experiments were conducted for N = 100, 200 and 1000 samples. The results are presented in 10 Tables and 16 Figures and the corresponding comments.

2. Estimation Procedures

The autoregressive scheme used in this paper in the discrete, time-stationary, first order autoregressive process with zero mean, i.e.

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(2.1) $$Z(t) = \phi_1 Z(t-1) + a(t),$$

where a(t) is a random shock with mean zero and variance σ_a^2 and the parameter ϕ_1 satisfies the stationarity condition $(-1<\phi_1<1)$. Given T observations on Z(t), it is very well known that

(i) the ordinary least squares estimate $\hat{\phi}_1$ of the parameter ϕ_1 can be determined by minimizing $\sum_{t=2}^{T} a^2(t)$ so that

(2.2) $$\hat{\phi}_1 = \frac{\sum_{t=2}^{T} Z(t)\, Z(t-1)}{\sum_{t=2}^{T} Z^2(t-1)}$$

(ii) The modified least squares estimate ϕ_1^* can be determined by dividing a time series into two halves, and estimating the autoregressive parameter for the whole series $(\hat{\phi}_1)$, for the first half $({}_1\hat{\phi}_1)$ and for the second half $({}_2\hat{\phi}_1)$ separately and so

(2.3) $$\phi_1^* = 2\,(\hat{\phi}_1) - \frac{{}_1\hat{\phi}_1 + {}_2\hat{\phi}_1}{2}$$

(see Quenouille [14]).

(iii) When a(t) is distributed normally, then the process (2.1) is called a Gaussian, and it is possible to consider maximum likelihood of the parameter ϕ_1, where the exact likelihood L can be written as

(2.4) $$L = (2\pi\sigma_a^2)^{-\frac{T}{2}} (1-\phi_1^2)^{\frac{1}{2}} \exp\left\{-\frac{1}{2\sigma_a^2}\left[(1-\phi_1^2)\; Z^2(1) + \sum_{t=z}^{T} (Z(t) - \phi_1 Z(t-1))^2\right]\right\}.$$

Differentiating log L with respect to ϕ_1 and setting the results equal to zero gives a cubic equation in ϕ_1. Therefore,

a) the exact maximum likelihood estimator of ϕ_1 is the solution of the cubic equation:

(2.5) $$g(\phi_1) = \phi_1^3 - u_2\phi_1^2 - u_1\phi_1 + u_0 = 0,$$

where

$$u_2 = \frac{T-2}{T-1} \cdot \frac{\sum_{t=2}^{T} Z(t)Z(t-1)}{\sum_{t=3}^{T} Z^2(t-1)}, \quad u_1 = \left[1 + \frac{1}{T-1}\left(1 + \frac{\sum_{t=1}^{T} Z^2(t)}{\sum_{t=3}^{T} Z^2(t-1)} \right) \right],$$

and

$$u_o = \frac{T}{T-1} \frac{\sum_{t=2}^{T} Z(t)Z(t-1)}{\sum_{t=3}^{T} Z^2(t-1)},$$

so, $g(\phi_1)$ in equation (2.5) can be rewritten as

$$(2.6) \qquad g(\phi_1) = \phi_1^3 + \frac{K_1}{2K_2}\left(1 - \frac{1}{T-1}\right)\phi_1^2 -$$

$$- \left[1 + \left(1 + \frac{K_o}{K_2}\right)\frac{1}{T-1}\right]\phi_1 - \frac{K_1}{2K_2}\left(1 + \frac{1}{T-1}\right) = 0,$$

where

$$K_o = \sum_{t=1}^{T} Z^2(t), \; K_1 = -2\sum_{t=2}^{T} Z(t)Z(t-1) \text{ and } K_2 = \sum_{t=3}^{T} Z^2(t-1)$$

(see A b d e l-R a z e k [1]).

The three roots of the polynomial $g(\phi_1) = 0$ may be located by considering

(i) $g(\infty) = \infty$,

(ii) $g(-\infty) = -\infty$,

(iii) $g(0) = -\frac{K_1}{2K_2}\left(1 + \frac{1}{T-1}\right)$,

(iv) $g(1) = -\frac{K_o + K_1 + K_2}{K_2(T-1)} = -\frac{\sum_{t=2}^{T}[Z(t) - Z(t-1)]^2}{(T-1)\sum_{t=3}^{T} Z^2(t-1)} < 0$, and

$$\text{(v)}\quad g(-1) = \frac{\sum_{t=2}^{T}[Z(t) - Z(t-1)]^2}{(T-1)\sum_{t=3}^{T} Z^2(t-1)} > 0,$$

so, there is one zero root if $K_1 = 0$, one root between -1 and zero if K_1 is positive, and one root between zero and 1 if K_1 is negative. Thus $g(\phi_1) = 0$ has three roots, namely $\tilde{\phi}_L$, $\tilde{\phi}_1$ and $\tilde{\phi}_U$ such that

$$(2.7)\qquad \tilde{\phi}_L < -1 < \tilde{\phi}_1 < 1 < \tilde{\phi}_U$$

and the exact maximum likelihood estimator of ϕ_1 is $\tilde{\phi}_1$, the unique root between -1 and +1;

b) for large values of T, equation (2.5) converges to

$$\tilde{g}(\phi_1) = \phi_1^3 - \frac{\sum_{t=2}^{T} Z(t)Z(t-1)}{\sum_{t=3}^{T} Z^2(t-1)}\phi_1^2 - \phi_1 + \frac{\sum_{t=2}^{T} Z(t)Z(t-1)}{\sum_{t=3}^{T} Z^2(t-1)} = 0$$

or

$$(\tilde{\phi}_1^2 - 1)\left(\tilde{\phi}_1 - \frac{\sum_{t=2}^{T} Z(t)Z(t-1)}{\sum_{t=3}^{T} Z^2(t-1)}\right) = 0.$$

So, we may write for large values of T the approximate maximum likelihood estimator

$$(2.8)\qquad \tilde{\phi}_1 = \frac{\sum_{t=2}^{T} Z(t)Z(t-1)}{\sum_{t=3}^{T} Z^2(t-1)}.$$

From (2.2) and (2.8) it is clear that the two estimators are asymptotically the same.

3. Low Order Moments

The least squares estimator $\hat{\phi}_1$ in (2.2) can be rewritten as

$$\hat{\phi}_1 = \xi_1 / \xi_0,$$

where

$$\xi_1 = Z'C_1Z \text{ and } \xi_0 = Z'C_0Z$$

$$\hat{\phi}_1 = Z'C_1Z/Z'C_0Z, \quad Z' = \{Z_0, \ldots, Z_T\},$$

$$C_1 = \begin{pmatrix} 0 & \frac{1}{2} & \cdots & 0 & 0 \\ \frac{1}{2} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & & \frac{1}{2} & 0 \end{pmatrix} \text{ and } C_0 = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & & & \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

Using theorem 6.7.3. of Anderson [3], it can be shown that the joint moment generating function of ξ_0, ξ_1 is

$$M(t_0, t_1) = |I - 2(t_0C_0 + t_1C_1)\Sigma|^{-\frac{1}{2}} \tag{3.1}$$

where Σ is the covariance matrix of Z with (i,j)-th element given by

$$\phi_1^{|i-j|}\sigma_a^2/(1-\phi_1^2)$$

i.e.

$$\Sigma = \frac{1}{1-\phi_1^2}\begin{pmatrix} 1 & \phi_1 & \cdots & \phi_1^{T-1} \\ \phi_1 & 1 & \cdots & \phi_1^{T-2} \\ \vdots & & & \\ \phi_1^{T-1} & \phi_1^{T-2} & \cdots & 1 \end{pmatrix}.$$

Assuming $\sigma_a^2 = 1$ we obtain

$$M(t_o, t_1) = (1 - \phi_1^2)^{\frac{1}{2}} C_T^{-\frac{1}{2}},$$

where:

$$C_T = \begin{vmatrix} 1 - 2t_o & -(\phi_1 + t_1) & 0 & & & \\ -(\phi_1 + t_1) & 1 - 2t_o + \phi_1^2 & -(\phi_1 + t_1), & & & \\ 0 & -(\phi_1 + t_1) & 1 - 2t_o + \phi_1^2 & & & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \\ & & -(\phi_1 + t_1) & 1 - 2t_o + \phi_1^2 & -(\phi_1 + t_1) & \\ & & & 0 & -(\phi_1 + t_1) & 1 \end{vmatrix}$$

Therefore

$$\left.\frac{\partial M(t_o, t_1)}{\partial t_1}\right|_{t_1=0} = -\frac{\sqrt{1 - \phi_1^2}}{2} \left(C_T\Big|_{t_1=0}\right)^{-\frac{3}{2}} \left.\frac{\partial C_T}{\partial t_1}\right|_{t_1=0} \tag{3.2}$$

and

$$\left.\frac{\partial^2 M(t_o, t_1)}{\partial t_1^2}\right|_{t_1=0} = -\frac{\sqrt{1 - \phi_1^2}}{2} \left(C_T\Big|_{t_1=0}\right)^{-\frac{3}{2}} \left[\left.\frac{\partial_2 C_T}{\partial t_1^2}\right|_{t_1=0} - \frac{3}{2}\left(C_T\Big|_{t_1=0}\right)^{-1} \left(\left.\frac{\partial C_T}{\partial t_1}\right|_{t_1=0}^{2}\right)\right]. \tag{3.3}$$

Using equations (3.2) and (3.3), the lower order moments of $\hat{\phi}_1$ are

$$E(\hat{\phi}_1) = \int_{-\infty}^{0} \left.\frac{\partial M(t_o, t_1)}{\partial t_1}\right|_{t_1=0} dt_o \text{ and } E(\phi_1^2) = \int_{-\infty}^{0}\left\{\int_{-\infty}^{v} \left.\frac{\partial_2 M(t_o, t_1)}{\partial t_1^2}\right|_{t_1=0} dt_o\right\} dv, \tag{3.4}$$

4. Simulation

Monte-Carlo experiments were designed to compare and evaluate the efficiencies of the estimators proposed in section 2. The experiments were conducted (on ODRA-1304 computer with the use of # MAR 2 own program), for N = 100, 200 and 1000 samples for the following values of sample size

$$T = 15, 20, 25, 30, 50 \text{ and } 100$$

and autocorrelation coefficient values

$$\phi_1 = 0.0, \pm 0.5 \text{ and } \pm 0.9.$$

Parameter choices are made in an attempt to provide representation throughout stationarity regions and the variance of noise is fixed at unity. Our findings are reported in 10 Tables (4.1--4.10) and 16 Figures (4.1-4.16).

5. Results and Conclusions

Generally, we can make the following observations from the tables and figures.

a. With respect to the bias of the estimate:

(i) ordinary least squares estimator clearly exhibits least absolute bias even for samples as small as 15 (except when $\phi_1 = 0.0$).

(ii) The modified least squares estimate based on Quenouille's formula [14] will improve the situation significantly (except when $\phi_1 = 0.0$ and 0.5).

(iii) The approximate maximum likelihood and exact maximum likelihood estimators exhibit approximately the same bias especially for intermediate and large values of T.

(iv) For $\phi_1 = 0.0$ and large values of T = 100, the OLS, approximate and exact maximum likelihood estimators exhibit the same bias.

(v) The absolute bias for the ordinary least squares, modified least squares, approximate and maximum likelihood estimators decreases and increases with $|\phi_1|$ and decreases as T increases; or we can say that the estimate $\hat{\phi}_1$ has a bias which increases linearly with the parameters ϕ_1 and $\frac{1}{T}$ (cf. Tables 4.1, 4.3, 4.5, 4.7, 4.9 and Figures 4.1, 4.2, 4.3, 4.7, 4.8, 4.9, 4.10, 4.11).

b. With respect to the mean square error of the estimate:

(i) The mean square errors (MSE) for the OLS estimates are considerably smaller than those for the modified ones.

(ii) On the basis of MSE the OLS estimator is best only for $\phi_1 = \pm 0.9$, while for other values of ϕ_1 the likelihood estimate has smaller MSE than the least squares.

(iii) For intermediate and large values of T the MSE for the approximate and exact likelihood estimates are approximately the same.

(iv) Ordinary, modified least squares, approximate and exact maximum likelihood estimators, each has a MSE which is smaller for large values of ϕ_1 and T, i.e. MSE is decreasing as $|\phi_1|$ and T are increasing, (cf. Tables 4.2, 4.4, 4.6, 4.8, 4.10 and Figures 4.4, 4.5, 4.6, 4.12, 4.13, 4.14, 4.15, 4.16).

c. The experiment is re-conducted for N = 100 and N = 200 replications. We notice that new information was elicited, with no changes in relative performance of the estimators on the criteria examined, and so the ranking of estimators never varied (cf. Tables 4.1, 4.2, 4.3, 4.4, 4.5 and 4.6).

Table 4.1

Relationship between the absolute value of the bias and values of ϕ_1 for small values of T for different replications

ϕ_1	T = 15 (small)											
	|BIAS|											
	OLS			MOD. OLS			APR. ML.			EXACT ML.		
	replications			replications			replications			replications		
	100	200	1000	100	200	1000	100	200	1000	100	200	1000
-0.9	0.1049	0.1341	0.1151	0.0174	0.0251	0.0034	0.1617	0.1888	0.1712	0.1704	0.1943	0.2013
-0.5	0.0733	0.0554	0.0186	0.0424	0.0301	0.0793	0.0759	0.0872	0.0530	0.0769	0.0861	0.0613
0.0	0.0477	0.0254	0.0131	0.0750	0.0415	0.0168	0.0443	0.0236	0.0122	0.0438	0.0237	0.0119
0.5	0.0439	0.0207	0.0357	0.1706	0.0738	0.0554	0.0051	0.0550	0.0689	0.0029	0.0760	0.0802
0.9	0.0724	0.1159	0.1229	0.0488	0.0198	0.0219	0.1315	0.1719	0.1784	0.2015	0.1926	0.1967

Table 4.2

Relationship between the mean squares error and the values of ϕ_1 for small values of T for different replications

ϕ_1	T = 15 (small)											
	MSE											
	OLS			MOD. OLS			APR. ML.			EXACT ML.		
	replications			replications			replications			replications		
	100	200	1000	100	200	1000	100	200	1000	100	200	1000
-0.9	0.1369	0.1621	0.1542	0.1663	0.1930	0.1892	0.1679	0.1947	0.1862	0.1743	0.2016	0.2066
-0.5	0.1929	0.2189	0.2071	0.2376	0.2683	0.2855	0.1861	0.2071	0.1945	0.1864	0.2080	0.1934
0.0	0.2129	0.2549	0.2261	0.2680	0.3158	0.2875	0.1977	0.2367	0.2099	0.1970	0.2364	0.2092
0.5	0.2039	0.2270	0.2070	0.2991	0.3083	0.2816	0.1853	0.2137	0.1961	0.1842	0.2128	0.1953
0.9	0.1415	0.1519	0.1539	0.2075	0.1740	0.1872	0.1606	0.1799	0.1875	0.2110	0.1989	0.2019

Table 4.3

Relationship between the absolute value of the bias and values of ϕ_1 for intermediate values of T for different replications

ϕ_1	T = 30 (intermediate)											
	|BIAS|											
	OLS			MOD. OLS			APR. ML.			EXACT ML.		
	replications			replications			replications			replications		
	100	200	1000	100	200	1000	100	200	1000	100	200	1000
-0.9	0.0707	0.0791	0.0622	0.0094	0.0173	0.0006	0.0992	0.1074	0.0911	0.0967	0.1156	0.0900
-0.5	0.0273	0.0381	0.0250	0.0192	0.0110	0.0248	0.0436	0.0541	0.0414	0.0434	0.0540	0.0412
0.0	0.0211	0.0206	0.0066	0.0282	0.0269	0.0063	0.0203	0.0201	0.0064	0.0204	0.0201	0.0064
0.5	0.0085	0.0097	0.0190	0.0679	0.0468	0.0356	0.0090	0.0266	0.0356	0.0088	0.0265	0.0366
0.9	0.0575	0.0692	0.0720	0.0067	0.0017	0.0132	0.0865	0.0978	0.1005	0.0843	0.0962	0.1034

Table 4.4

Relationship between the mean squares error and the values of ϕ_1 for intermediate values of T for different replications

ϕ_1	T = 30 (intermediate)											
	MSE											
	OLS			MOD. OLS			APR. ML.			EXACT ML.		
	replications			replications			replications			replications		
	100	200	1000	100	200	1000	100	200	1000	100	200	1000
-0.9	0.0894	0.1025	0.0866	0.1045	0.1100	0.1005	0.1034	0.1155	0.0986	0.1029	0.1238	0.0999
-0.5	0.1303	0.1406	0.1327	0.1529	0.1682	0.1579	0.1278	0.1389	0.1297	0.1278	0.1389	0.1299
0.0	0.1601	0.1551	0.1507	0.1854	0.1809	0.1738	0.1545	0.1498	0.1455	0.1545	0.1497	0.1455
0.5	0.1380	0.1390	0.1341	0.1716	0.1720	0.1645	0.1330	0.1346	0.1306	0.1335	0.1346	0.1306
0.9	0.0849	0.0895	0.0967	0.0987	0.0996	0.1050	0.0954	0.1056	0.1091	0.0961	0.1056	0.1130

Table 4.5

Relationship between the absolute value of the bias and values of ϕ_1 for large values of T for different replications

ϕ_1	T = 100 (large) |BIAS|											
	OLS replications			MOD. OLS replications			APR. ML. replications			EXACT ML. replications		
	100	200	1000	100	200	1000	100	200	1000	100	200	1000
-0.9	0.0196	0.0241	0.0235	0.0031	0.0020	0.0023	0.0285	0.0330	0.0323	0.0283	0.0326	0.0320
-0.5	0.0074	0.0082	0.0034	0.0093	0.0081	0.0148	0.0123	0.0131	0.0084	0.0123	0.0131	0.0084
0.0	0.0054	0.0000	0.0027	0.0063	0.0012	0.0022	0.0054	0.0000	0.0027	0.0054	0.0000	0.0027
0.5	0.0007	0.0025	0.0067	0.0158	0.0176	0.0107	0.0058	0.0076	0.0117	0.0058	0.0076	0.0117
0.9	0.0227	0.0215	0.0199	0.0028	0.0020	0.0018	0.0315	0.0304	0.0288	0.0314	0.0301	0.0285

Table 4.6

Relationship between the mean squares error and the values of ϕ_1 for large values of T for different replications.

ϕ_1	T = 100 (large)											
	MSE											
	OLS			MOD. OLS			APR. ML.			EXACT ML.		
	replications			replications			replications			replications		
	100	200	1000	100	200	1000	100	200	1000	100	200	1000
-0.9	0.0364	0.0445	0.0410	0.0407	0.0484	0.0445	0.0393	0.0471	0.0436	0.0395	0.0472	0.0437
-0.5	0.0683	0.0766	0.0696	0.0731	0.0800	0.0760	0.0679	0.0761	0.0690	0.0680	0.0761	0.0690
0.0	0.0820	0.0841	0.0823	0.0862	0.0871	0.0864	0.0812	0.0832	0.0815	0.0812	0.0832	0.0815
0.5	0.0732	0.0749	0.0709	0.0798	0.0830	0.0766	0.0727	0.0737	0.0704	0.0727	0.0737	0.0704
0.9	0.0995	0.0409	0.0381	0.0413	0.0466	0.0408	0.0425	0.0430	0.0407	0.0426	0.0430	0.0408

T a b l e 4.7

Relationship between the absolute value of the bias and values of ϕ_1 (1000 replications)

T	ϕ_1	|BIAS| (standard error)			
		OLS	MOD. OLS	APR. ML.	EXACT ML.
1	2	3	4	5	6
15	-0.9	0.1151	0.0034	0.1712	0.2013
		0.012	0.014	0.013	0.013
	-0.5	0.0186	0.0793	0.0530	0.0613
		0.014	0.017	0.017	0.014
	0.0	0.0131	0.0168	0.0122	0.0119
		0.015	0.017	0.015	0.015
	0.5	0.0357	0.0554	0.0689	0.0802
		0.014	0.017	0.014	0.014
	0.9	0.1229	0.0219	0.1784	0.1967
		0.012	0.014	0.013	0.013
30	-0.9	0.0622	0.0006	0.0911	0.0900
		0.009	0.010	0.010	0.010
	-0.5	0.0250	0.0248	0.0414	0.0412
		0.012	0.013	0.011	0.011
	0.0	0.0066	0.0063	0.0064	0.0064
		0.012	0.013	0.013	0.012
	0.5	0.0190	0.0356	0.0356	0.0366
		0.012	0.013	0.011	0.011
	0.9	0.0720	0.0132	0.1005	0.1034
		0.008	0.010	0.010	0.010
	-0.9	0.0235	0.0023	0.0323	0.0320
		0.006	0.006	0.007	0.006

Table 4.7. (contd.)

1	2	3	4	5	6
100	-0.5	0.0034	0.0148	0.0084	0.0084
		0.008	0.009	0.008	0.008
	0.0	0.0027	0.0022	0.0027	0.0027
		0.009	0.009	0.009	0.009
	0.5	0.0067	0.0107	0.0117	0.0117
		0.008	0.009	0.008	0.008
	0.9	0.0199	0.0018	0.0288	0.0285
		0.006	0.006	0.006	0.006

Table 4.8

Relationship between the mean squares error and the values of ϕ_1 (1000 replications)

T	ϕ_1	MSE			
		OLS	MOD. OLS	APR. ML.	EXACT ML.
15	-0.9	0.1542	0.1892	0.1862	0.2068
	-0.5	0.2071	0.2855	0.1945	0.1934
	0.0	0.2261	0.2875	0.2099	0.2092
	0.5	0.2070	0.2816	0.1961	0.1953
	0.9	0.1539	0.1872	0.1875	0.2019
30	-0.9	0.0866	0.1005	0.0986	0.0999
	-0.5	0.1327	0.1579	0.1297	0.1299
	0.0	0.1507	0.1738	0.1455	0.1455
	0.5	0.1341	0.1645	0.1306	0.1306
	0.9	0.0967	0.1050	0.1091	0.1130
100	-0.9	0.0410	0.0445	0.0436	0.0437
	-0.5	0.0696	0.0760	0.0690	0.0690
	0.0	0.0823	0.0864	0.0815	0.0815
	0.5	0.0709	0.0766	0.0704	0.0704
	0.9	0.0381	0.0408	0.0407	0.0408

Table 4.9

Relationship between the values of the absolute bias and the values of T (1000 replications)

ϕ_1	T	|BIAS| OLS	MOD. OLS	APR. ML.	EXACT ML.
-0.9	15	0.1151	0.0034	0.1712	0.2013
	20	0.0996	0.0156	0.1417	0.1488
	25	0.0779	0.0103	0.1121	0.1152
	30	0.0622	0.0006	0.0911	0.0900
	50	0.0423	0.0015	0.0598	0.0588
	100	0.0235	0.0023	0.0323	0.0320
-0.5	15	0.0186	0.0793	0.0530	0.0613
	20	0.0260	0.0484	0.0510	0.0532
	25	0.0183	0.0433	0.0384	0.0385
	30	0.0250	0.0248	0.0414	0.0412
	50	0.0146	0.0184	0.0245	0.0244
	100	0.0034	0.0148	0.0084	0.0084
0.0	15	0.0131	0.0168	0.0122	0.0119
	20	0.0136	0.0153	0.0129	0.0130
	25	0.0003	0.0044	0.0003	0.0003
	30	0.0066	0.0063	0.0064	0.0064
	50	0.0012	0.0025	0.0011	0.0011
	100	0.0027	0.0022	0.0027	0.0027
0.5	15	0.0357	0.0554	0.0689	0.0802
	20	0.0152	0.0577	0.0408	0.0452
	25	0.0318	0.0262	0.0513	0.0512
	30	0.0190	0.0356	0.0356	0.0366
	50	0.0020	0.0329	0.0121	0.0121
	100	0.0067	0.0107	0.0117	0.0117
0.9	15	0.1229	0.0219	0.1784	0.1967
	20	0.0898	0.0030	0.1324	0.1424
	25	0.0844	0.0175	0.1183	0.1198
	30	0.0720	0.0132	0.1005	0.1034
	50	0.0435	0.0026	0.0610	0.0600
	100	0.0199	0.0018	0.0288	0.0285

Table 4.10

Relationship between the mean squares error and the values of T (1000 replications)

ϕ_1	T	MSE			
		OLS	MOD. OLS	APR. ML.	EXACT ML.
-0.9	15	0.1542	0.1892	0.1862	0.2068
	20	0.1239	0.1412	0.1477	0.1555
	25	0.1043	0.1174	0.1211	0.1250
	30	0.0866	0.1005	0.0986	0.0999
	50	0.0622	0.0675	0.0693	0.0696
	100	0.0410	0.0445	0.0436	0.0437
-0.5	15	0.2071	0.2855	0.1945	0.1934
	20	0.1698	0.2194	0.1634	0.1631
	25	0.1466	0.1829	0.1418	0.1420
	30	0.1327	0.1579	0.1297	0.1299
	50	0.1018	0.1151	0.1003	0.1004
	100	0.0696	0.0760	0.0690	0.0690
0.0	15	0.2261	0.2875	0.2099	0.2092
	20	0.1878	0.2302	0.1780	0.1779
	25	0.1658	0.1923	0.1588	0.1588
	30	0.1507	0.1738	0.1455	0.1455
	50	0.1160	0.1282	0.1136	0.1136
	100	0.0823	0.0864	0.0815	0.0815
0.5	15	0.2070	0.2816	0.1961	0.1953
	20	0.1693	0.2214	0.1614	0.1613
	25	0.1531	0.1868	0.1488	0.1489
	30	0.1341	0.1645	0.1306	0.1306
	50	0.0984	0.1151	0.0962	0.0962
	100	0.0709	0.0766	0.0704	0.0704
0.9	15	0.1539	0.1872	0.1875	0.2019
	20	0.1186	0.1415	0.1398	0.1503
	25	0.1060	0.1167	0.1246	0.1276
	30	0.0967	0.1050	0.1091	0.1130
	50	0.0626	0.0672	0.0695	0.0697
	100	0.0381	0.0408	0.0407	0.0408

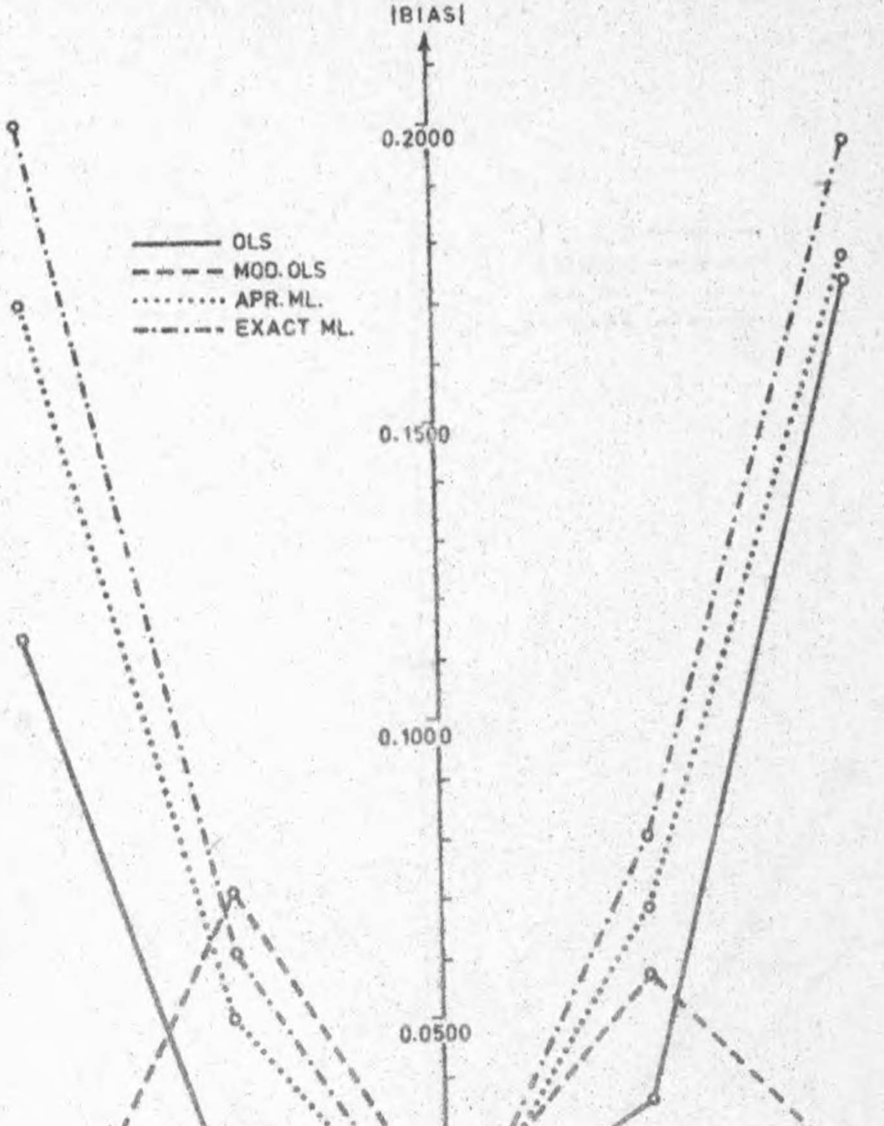

Fig. 4.1. Relationship between the absolute value of the bias and values of ϕ_1 (T = 15)

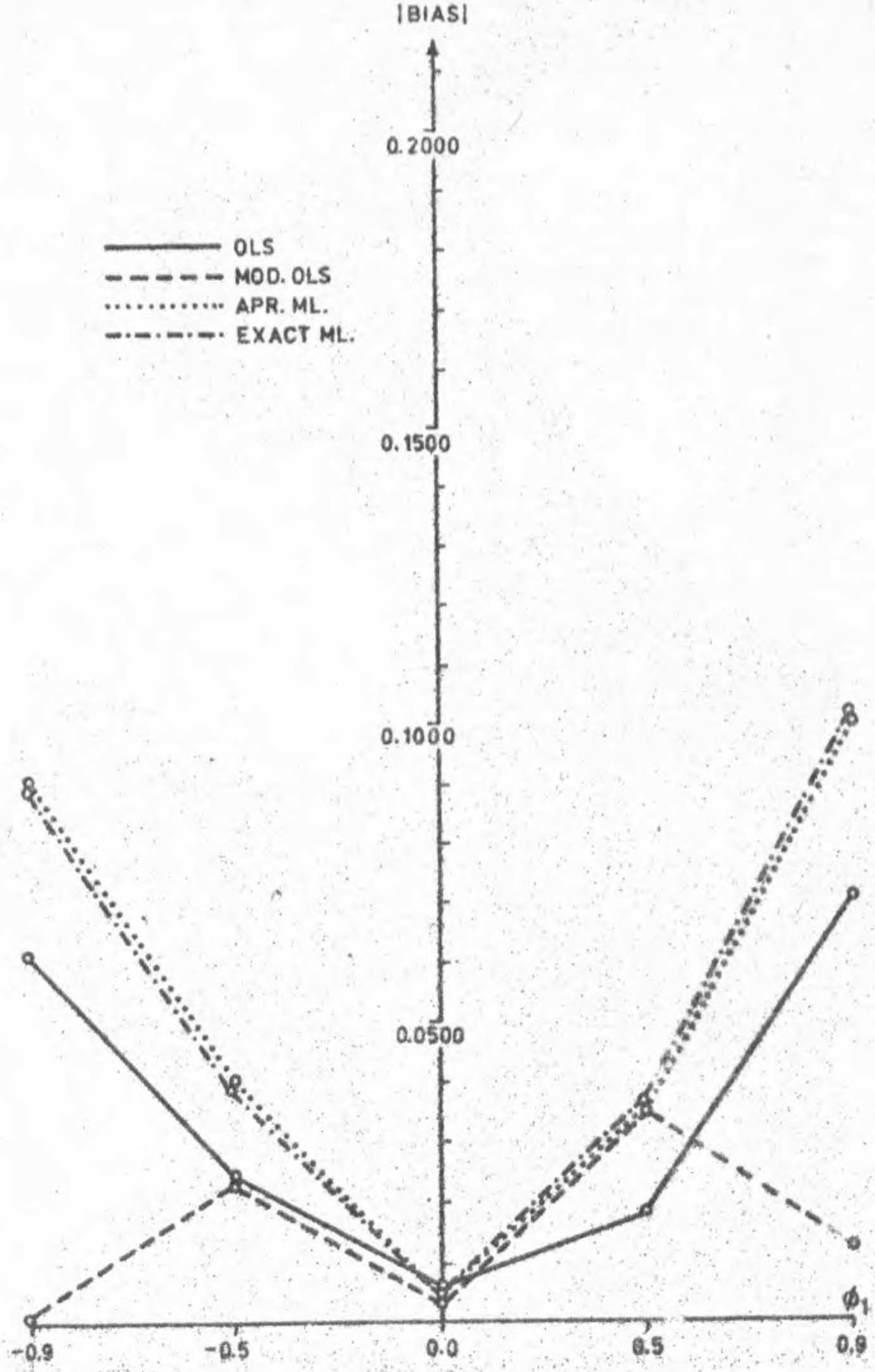

Fig. 4.2. Relationship between the absolute value of the bias and values of ϕ_1 (T = 30)

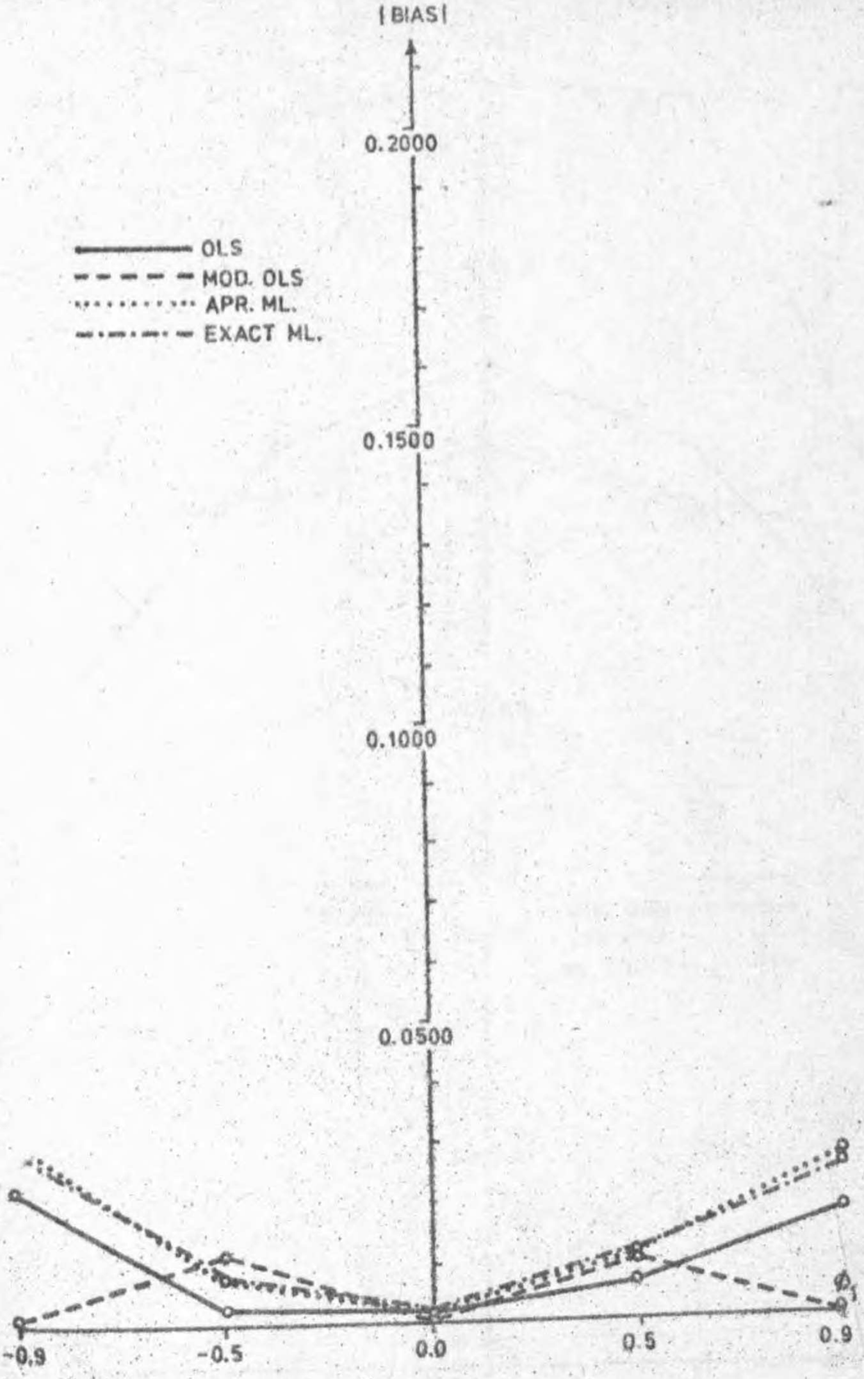

Fig. 4.3. Relationship between the absolute value of the bias and values of ϕ_1 (T = 100)

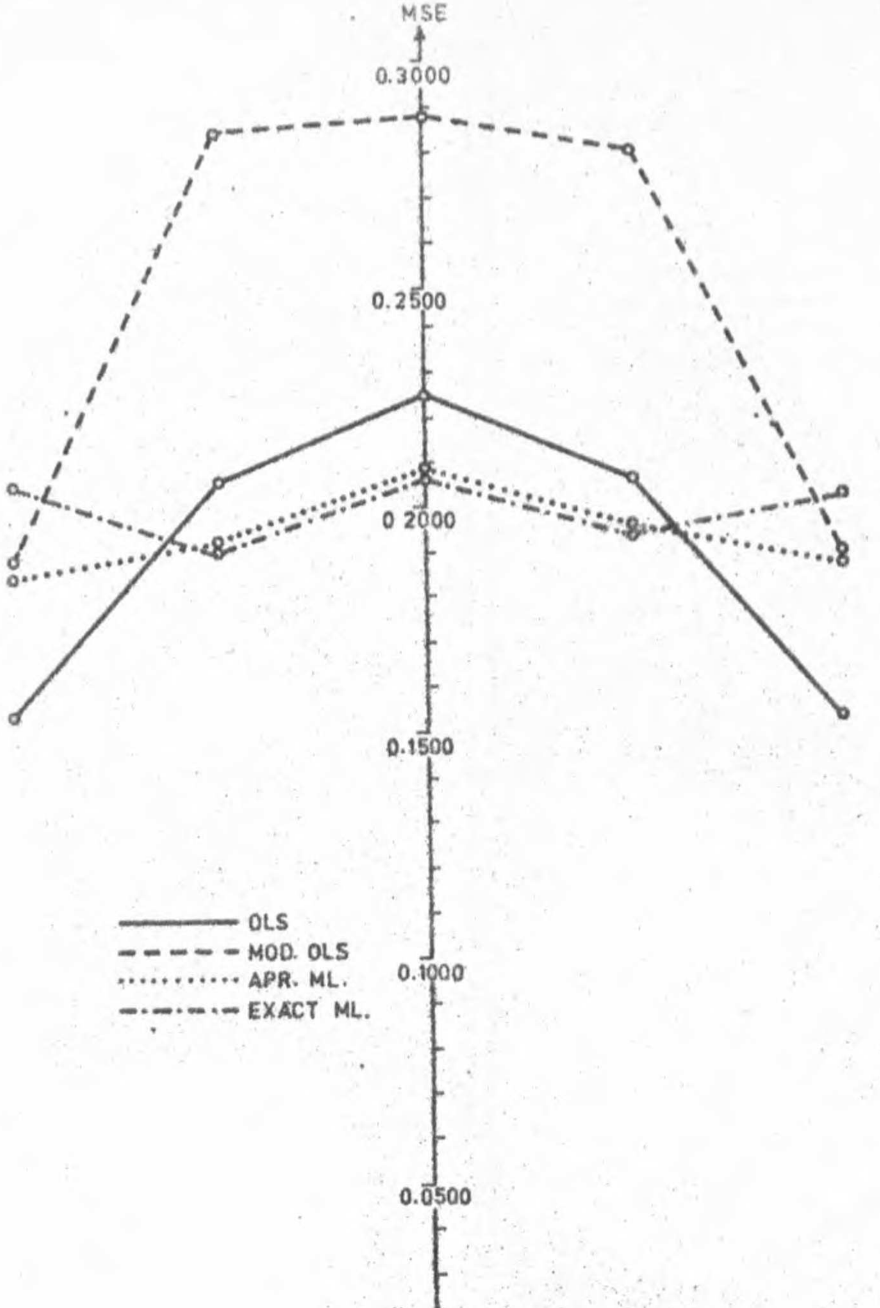

Fig. 4.4. Relationship between the mean squares error and the values of ϕ_1 (T = 15)

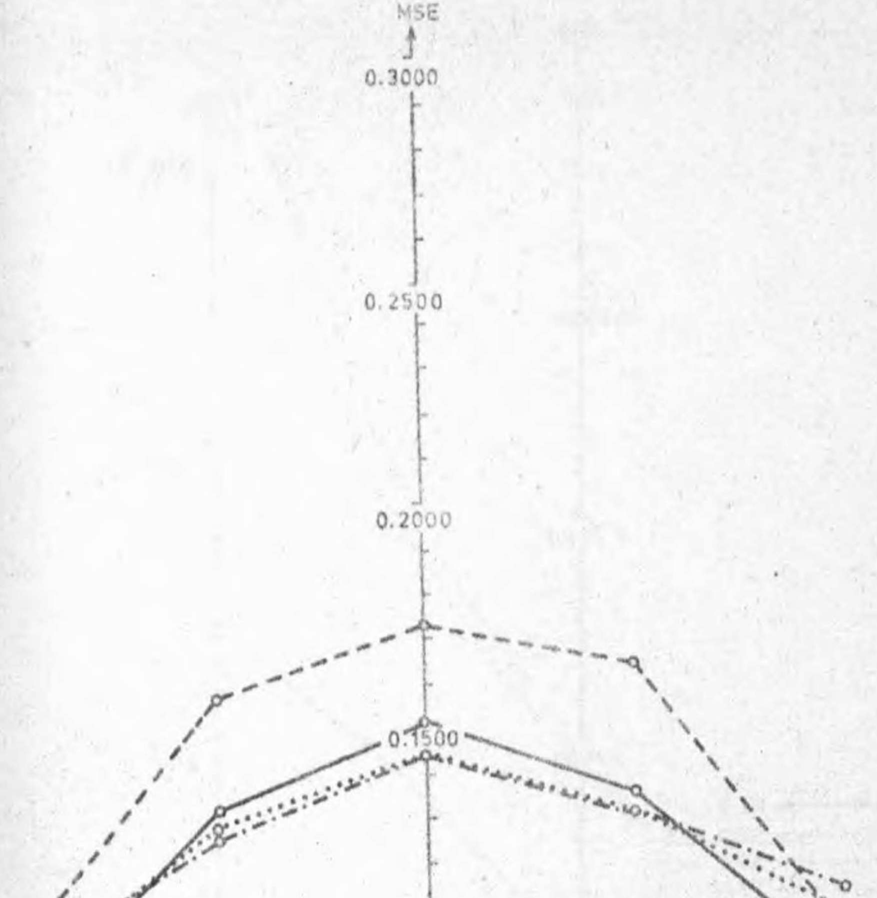

Fig. 4.5. Relationship between the mean squares error and the values of $\emptyset_1$ $(T = 30)$

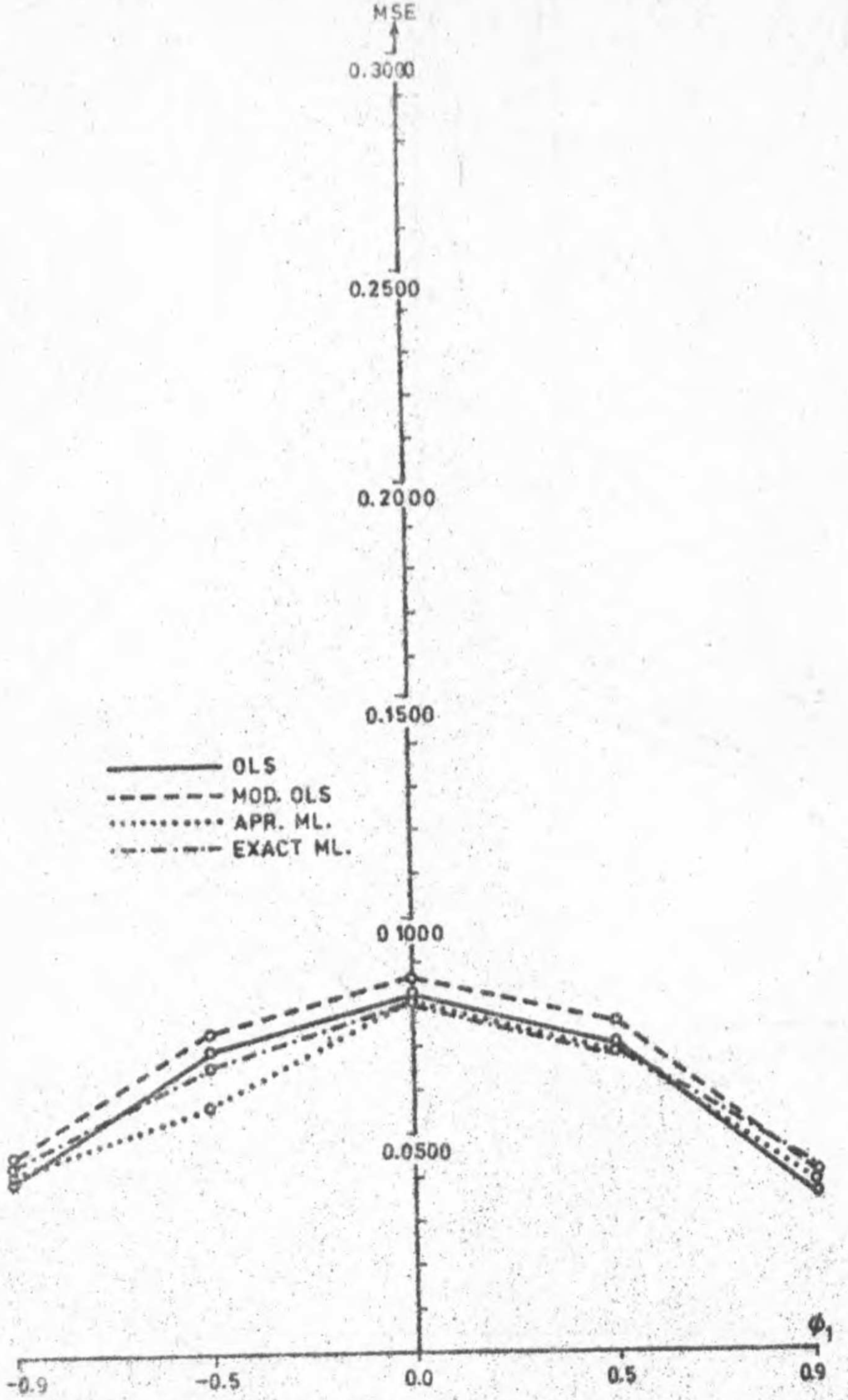

Fig. 4.6. Relationship between the mean squares error and the values of ϕ_1 (T = 100)

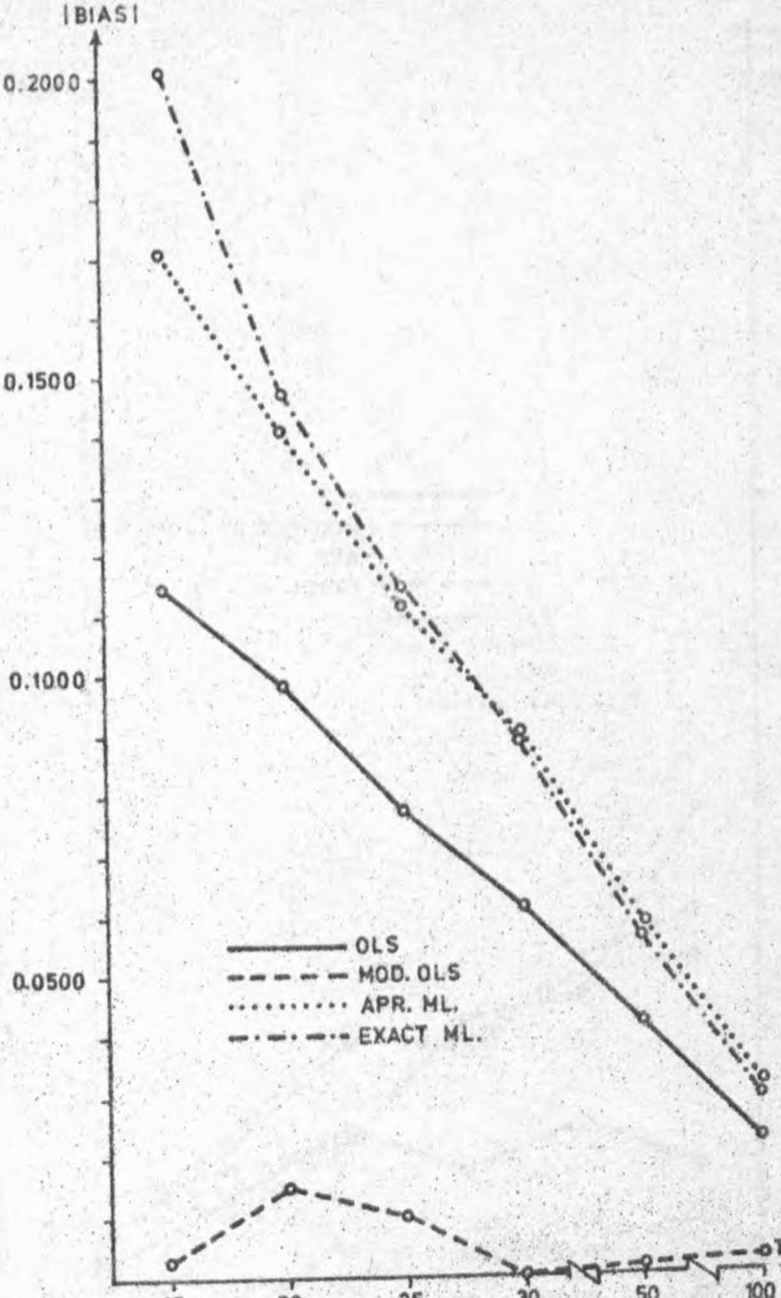

Fig. 4.7. Relationship between the values of the absolute bias and the values of T ($\emptyset_1$ = -0.9)

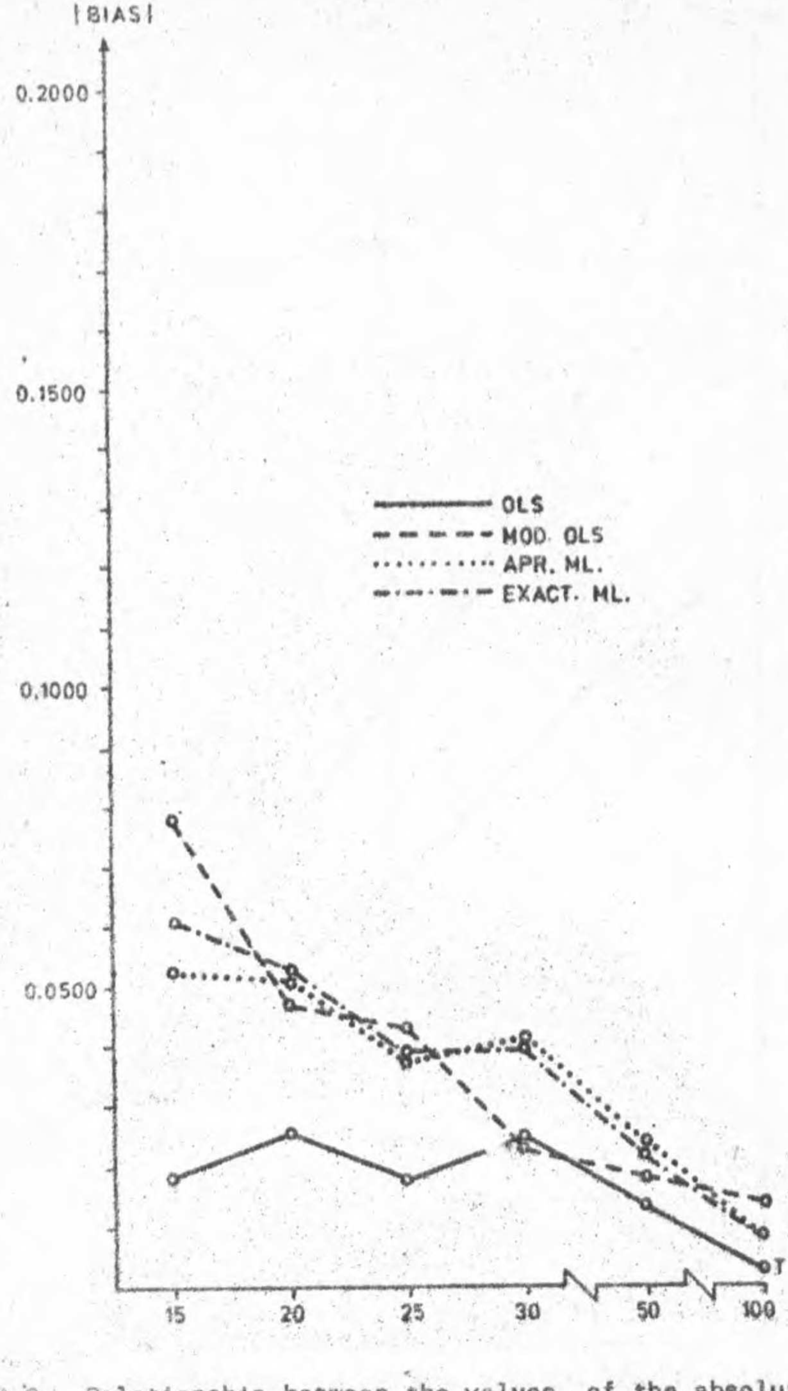

Fig. 4.8. Relationship between the values of the absolute bias and the values of T (ϕ_1 = -0.5)

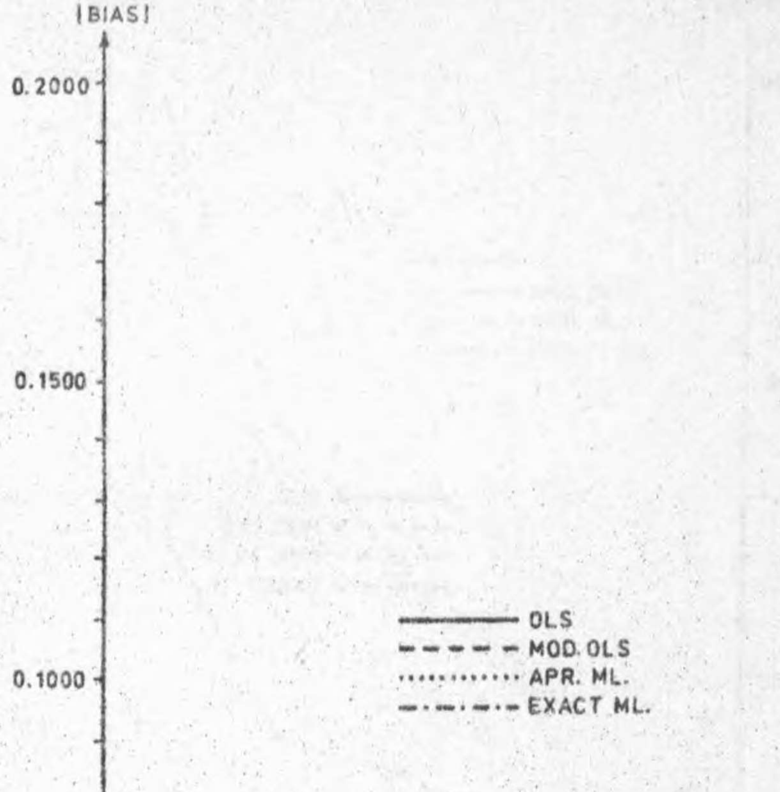

Fig. 4.9. Relationship between the values of the absolute bias and the values of T ($\emptyset_1$ = 0.0)

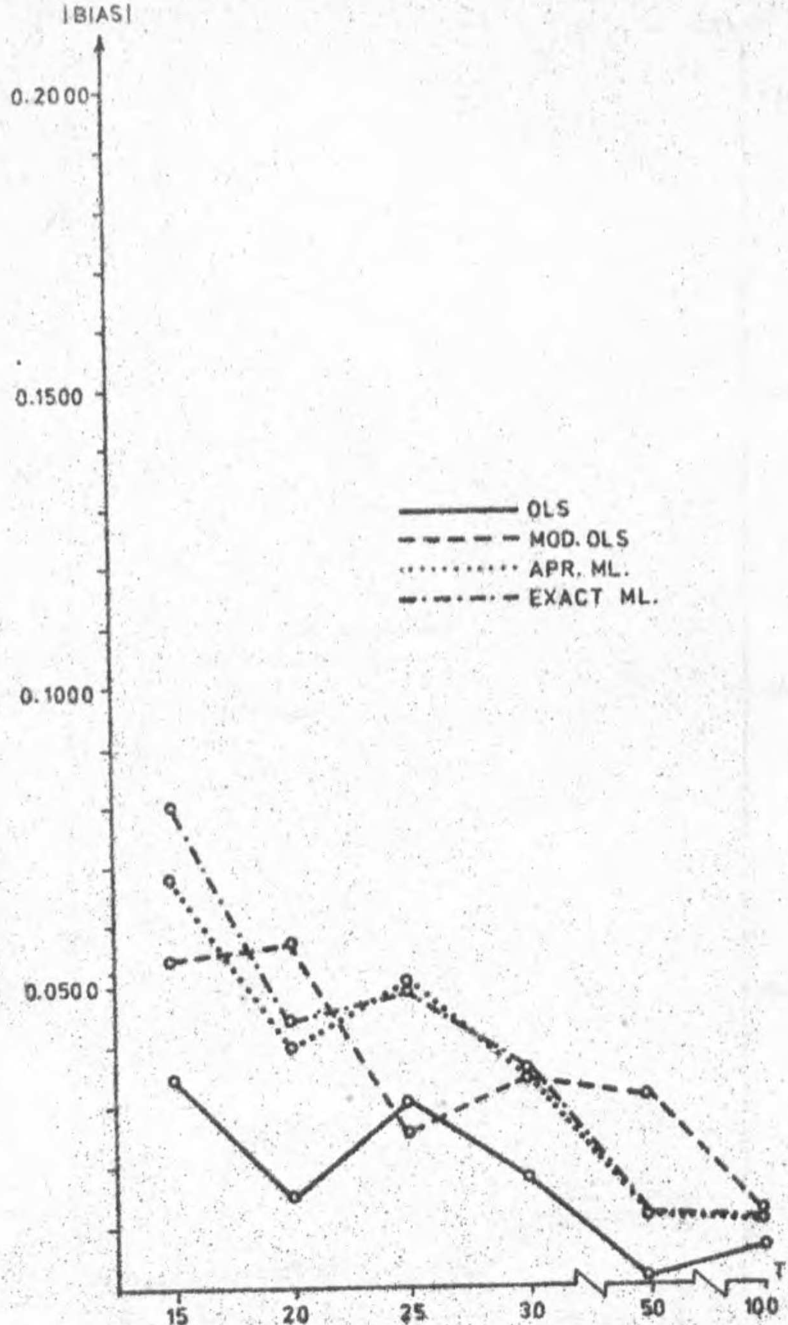

Fig. 4.10. Relationship between the values of the absolute bias and the values of T ($\phi_1 = 0.5$)

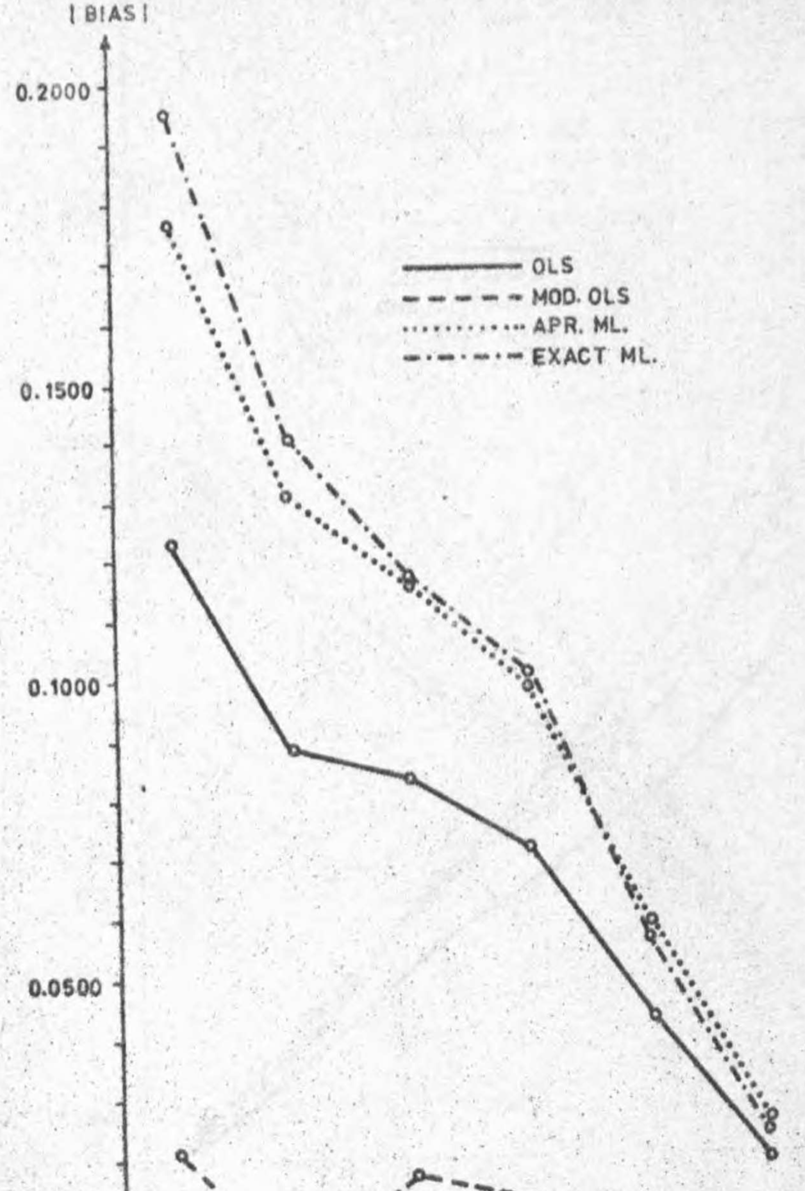

Fig. 4.11. Relationship between the values of the absolute bias and the values of T ($\emptyset_1 = 0.9$)

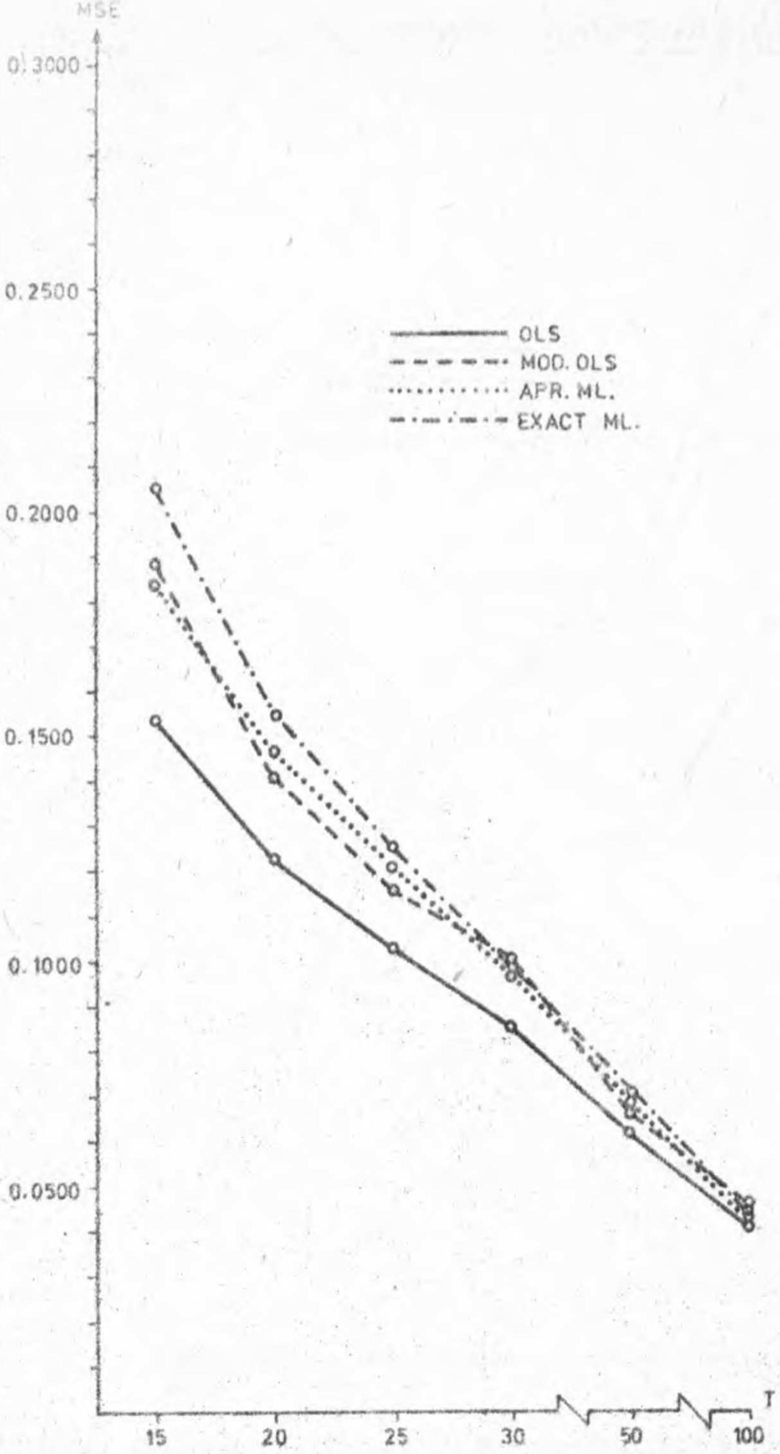

Fig. 4.12. Relationship between the mean squares error and the values of T (ϕ_1 = -0.9)

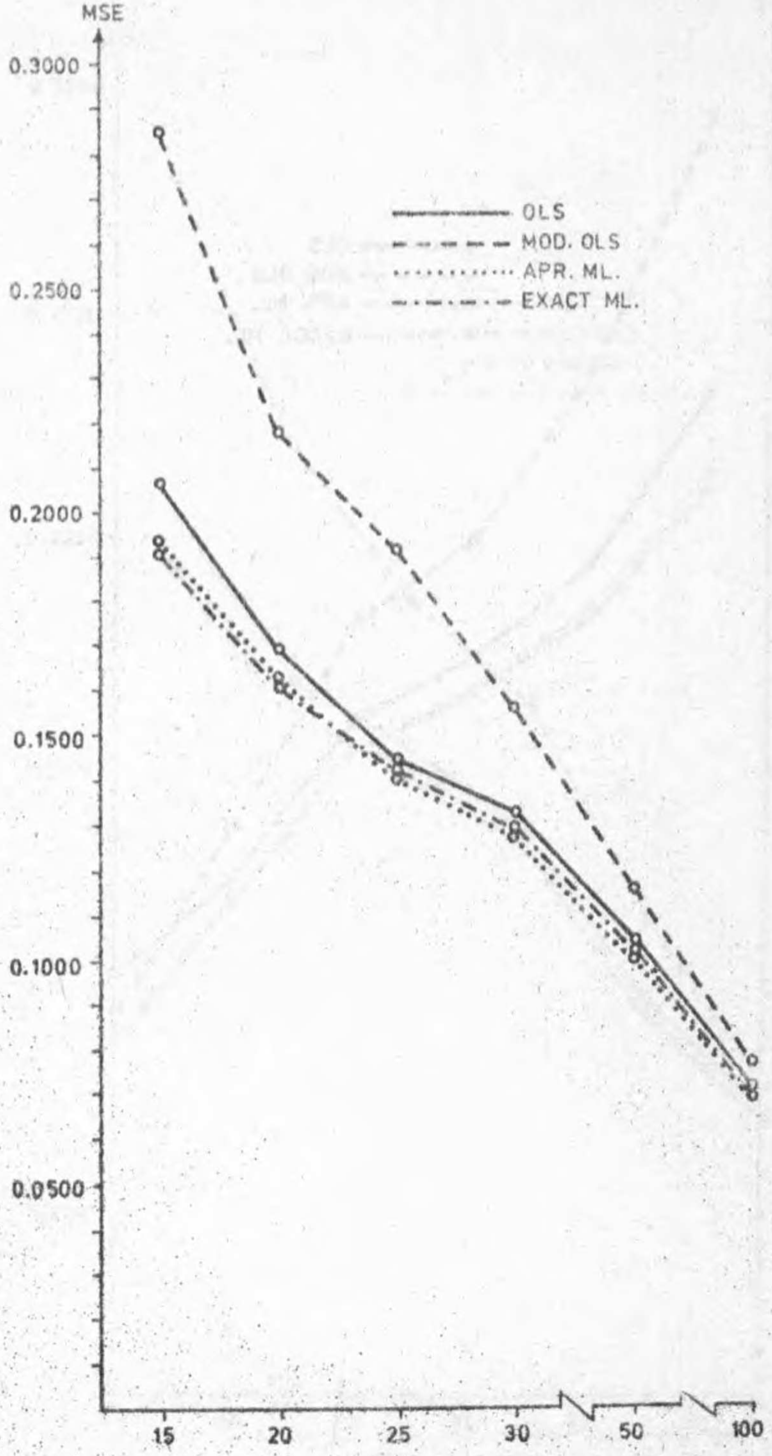

Fig. 4.13. Relationship between the mean squares error and the values of T ($\emptyset_1 = -0.5$)

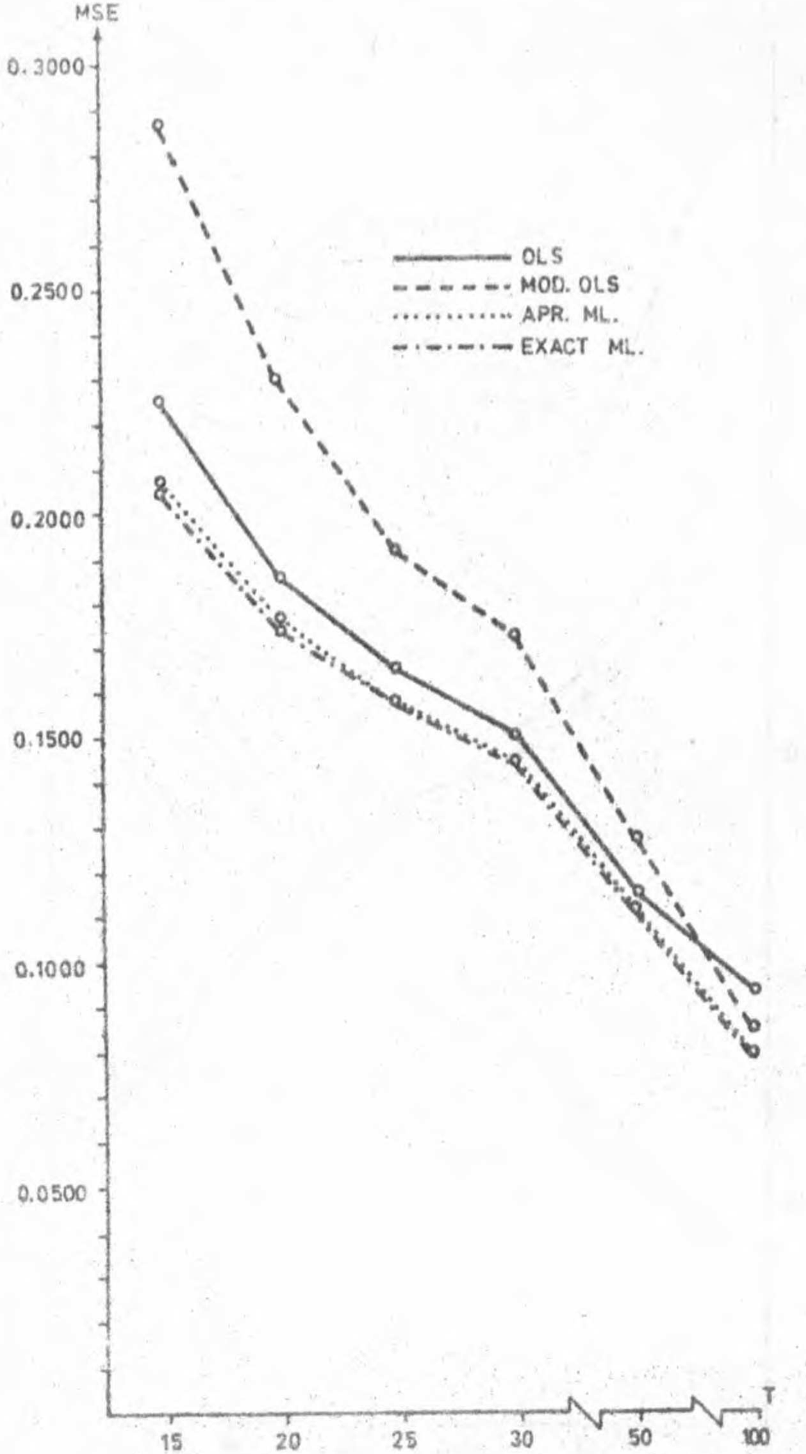

Fig. 4.14. Relationship between the mean squares error and the values of T ($\phi_1 = 0.0$)

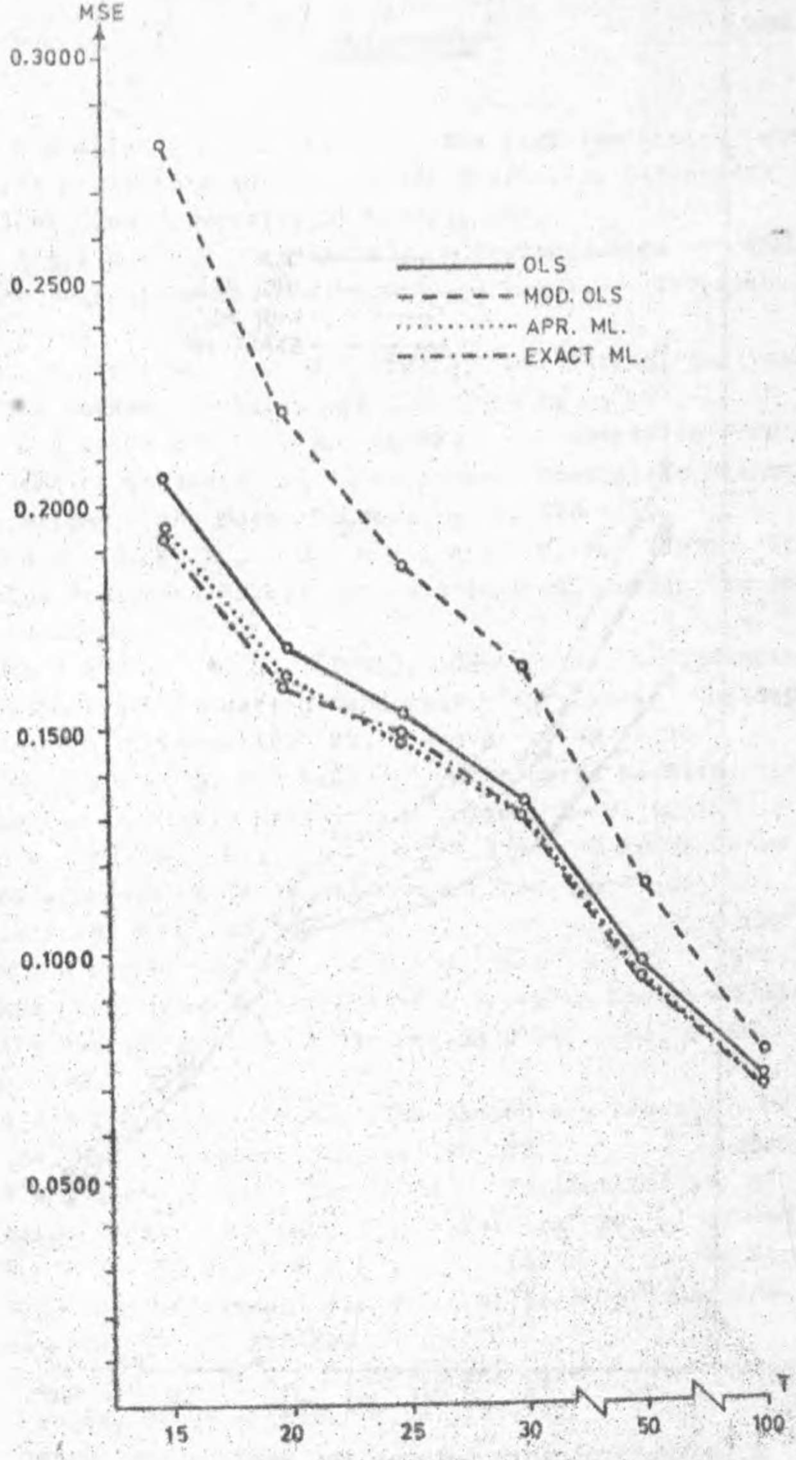

Fig. 4.15. Relationship between the mean squares error and the values of T ($\emptyset_1 = 0.5$)

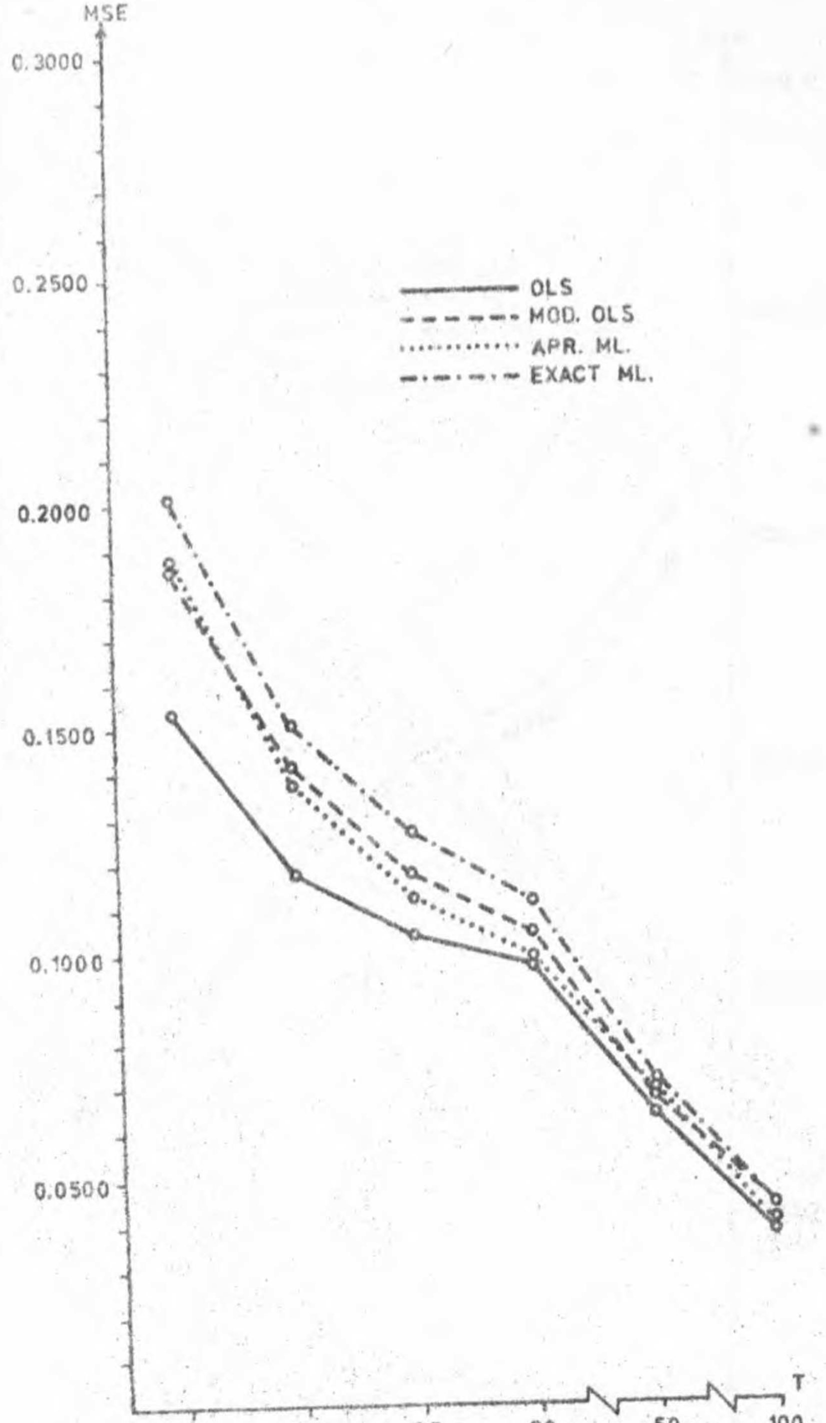

Fig. 4.16. Relationship between the mean squares error and the values of T ($\emptyset_1$ = 0.9)

References

[1] A b d e l-R a z e k (1972), The Distribution of Least Squares Estimators in First Order Stochastic Difference Equations, The University of Kansas, USA.

[2] A i g n e r D. J. (1971), A Compendium on Estimation of the Autoregressive-Moving Average Model, Internat. Econ. Rev. 12, p. 348-371.

[3] A n d e r s o n T. W. (1971), Statistical Analysis of Time Series, J. Wiley and Sons, New York.

[4] A n d e r s o n T. W. (1959), On Asymptotic Distributions of Estimates of Parameters of Stochastic Difference Equations, Ann. Math. Statis. 30, p. 676-687.

[5] B o x G. E. P., J e n k i n s G. M. (1970), Time Series Analysis, Forecasting and Control, Holden Day Inc., San Francisco.

[6] C h a n d a K. C. (1961), Comparative Efficiencies of Methods of Estimating Parameters in Linear Autoregressive Schemes, "Biometrika" 48, 3 and 4, p. 427-432.

[7] C o p a s J. B. (1966), Monte Carlo Results for Estimation in Stable Markov Time Seties, JRSS, A, 129, 1.

[8] D e n t W., M i n A. S. (1978), A Monte Carlo Study Autoregressive Integrated Moving Average Processes, "Econometrics" 7, p. 23-55.

[9] H e n d r y D. F., T r i v i d i P. K. (1971), Maximum Likelihood Estimation of Difference Equations with Moving Average Error: a Simulation Study, Rev. of Econom. Stud. p. 170.

[10] D o o b J. L. (1944), The Elementary Gaussian Processes, Ann. Math. Statist., 15, p. 229-282.

[11] M a l i n v a u d E. (1961), The Estimation of Distributed Lags. A Comment, "Econometrica" 29, p. 430-433.

[12] M a n n H. B., W a l d A. (1943), On the Statistical Treatment of Linear Stochastic Difference Equations, "Econometrica" 11, p. 173-220.

[13] N e l s o n C. R. (1973), The First Order Moving Average Process, "Econometrica" 2, p. 121-141.

[14] Q u e n o u i l l e M. H. (1949), Approximate Tests of Correlation in Time Series, JRSS, B. 11, p. 68.

[15] O r c u t t G. H., W i n o k u r H. S. (1969), First Order Autoregression: Inference, Estimation and Prediction, "Econometrica" 37, I, p. 1-14.

[16] S a w e T. (1978), The Exact Moments of the Least Squares Estimator for the Autoregressive Model, "Econometrics" 8, p. 159-172.

[17] S h e n t o n L. R., J o h n s o n W. L. (1965), Moments of a Serial Correlation Coefficient, JRSS, B. 27, 2, p. 308-320.

[18] T h r o n b e r H. (1967), Finite Sample Monte Carlo Studies: an Autoregressive Illustration, JASA, p. 801-818.

[19] W h i t e J. S. (1961), Asymptotic Expansions for the Mean and the Variance of the Serial Correlation Coefficient, "Biometrika" 48, p. 85-94.

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OCENA EFEKTYWNOŚCI PEWNYCH ESTYMATORÓW DLA MODELI AUTOREGRESJI PIERWSZEGO RZĘDU

Artykuł przedstawia porównanie efektywności następujących metod estymacji dla parametrów modeli autoregresji pierwszego rzędu:

1) zwykła metoda najmniejszych kwadratów (zmnk),
2) zmodyfikowana metoda najmniejszych kwadratów (mod mnk),
3) przybliżona metoda największej wiarygodności,
4) dokładna metoda największej wiarygodności.

Rezultaty eksperymentów Monte-Carlo, przedstawione w 10 tabelach i na 16 wykresach, wskazują, że

a) obciążenie rozważanych estymatorów jest podobne w przypadku małych wartości współczynników autokorelacji $\emptyset_1$; w przypadku $|\emptyset_1| > 0,5$ zmodyfikowana metoda najmniejszych kwadratów Quenouille'a jest lepsza;

b) błąd średniokwadratowy jest zwykle mniejszy dla zmnk niż dla mod mnk. Estymatory zmnk i mod mnk mają jednakże mniejszy błąd średniokwadratowy niż estymatory przybliżonej metody największej wiarygodności i dokładnej metody największej wiarygodności, której efektywność jest podobna.