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THE VERIFICATION OF THE MULTIVARIATE NORMAL DISTRIBUTION HYPOTHESIS IN A ONE-SAMPLE MODEL WITH THE METHOD OF ELIMINATION OF DISTURBING PARAMETERS

Abstract. In many statistical tasks a necessity of testing multivariate normality arises. In constructing multivariate normality tests there is a necessity of estimating unknown parameters μ and Σ from a given sample. The parameters are regarded as disturbing parameters.

The paper deals with some methods, by means of which unknown disturbing parameters are eliminated when the multivariate normality tests are applied.

In particular, the following methods are stressed: randomization method, reduction methods and conditional interval probability transformation method.

Key words: multivariate normality test, randomization method, reduction method, conditional interval probability transformation method.

1. INTRODUCTION

The assumption of a multivariate distribution of investigated random variables is very often made in multivariate analysis when we use statistical inference methods. This assumption may be verified with different tests for multivariate normality. A broad overview of them was given by Wagner (1990). Most of them are based on a suitable test statistics. Their distributions are searched when the hypothesis of multivariate normality is true. When the distributions are unknown the critical values are set by the Monte Carlo simulation.

One of the difficulties in constructing m.n.t. (multivariate normality tests) is the necessity of estimating unknown parameters of m.n.d. (multivariate

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normal distribution) from a given sample. In this case the problem may be reduced to the verification of a sample hypothesis of m.n.d. by eliminating unknown parameters. These issues are discussed in this paper.

2. TECHNICAL NOTATION

Let X be a p-dimensional random vector with a m.n.d. given by a cumulative function $F_p(x, \mu, \Sigma) = F_p(x)$, for $x \in R^p$, and $E(X) = \mu \in R^p$ and $D(X) = \Sigma \in I_p^>$ express the vector of expectations and covariance matrix respectively, while R^p and $I_p^>$ denote the real space of p-dimensional vectors and symmetric, positive definite matrices of order p. The fact that X has the given m.n.d. with the mentioned parameters we write by $N \sim N_p(\mu, \Sigma)$ (shortly $X \sim N_p$). The class of these distributions is expressed by the set $N_p = \{N_p(\mu, \Sigma): \mu \in R^p, \Sigma \in I_p^>\}$

Let vector X be divided into $X = (X_1', X_p)'$ where $X_1 = (X_1, ..., X_{p-1})'$ and, respectively, let $\mu = (\mu_1', \mu_p)'$ and

$$\Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \sigma_{pp} \end{bmatrix}\begin{matrix} p-1 \\ 1 \end{matrix}$$

where $\Sigma_{12} = \Sigma_{21}'$. The conditional distribution of vector X_1 with fixed $X_p = x_p$ is given by

$$X_1 \mid X_p \sim N_{p-1}(\mu_1 + \frac{x_p - \mu_p}{\sigma_{pp}} \Sigma_{12}, \ \Sigma_{11} - \frac{1}{\sigma_{pp}} \Sigma_{12}\Sigma_{21})$$

Let $U = (X_1, ..., X_n)$ be a sample of n independent p-dimensional observable random vectors. We assume that vectors X_j, $j = 1, ..., n$ are identically distributed according to given a cumulative function $G(x; V) = G_p(x)$, where V is a set belonging to the space of feasible parameters. The function $G_p(x)$ may be unknown, both with respect to the form and parameters, though we assume that for every $x \in R^p$ it is a continuous function. From sample U we find unbiased estimators of μ and Σ in the form of:

– vector of the arithmetic means

$$\bar{X} = \frac{1}{n} U1,$$

– covariance matrices

$$s = \frac{1}{n-1}(UU' - \frac{1}{n}\bar{X}\bar{X}').$$

and the matrix of the sum squares and products $A = (n-1)S$.

Assuming that $X_j \in N_p$ we can look at sample U as a model of one sample from the m.n.d. population. For this model we have the properties:

(i) $(\bar{X}, S)$ is a set of sufficient statistics,

(ii) $\bar{X}$ and S are independent,

(iii) $C'X \sim N_1(C'\mu, \sigma_C^2)$ and $C'SC \sim \sigma_C^2 \chi_m^2$,

where $C' \in R^p$, $C \neq 0$, $\sigma_C^2 = C'\Sigma C$ and $m = n - 1$,

(iv) $\bar{X} \sim N_p(\mu, \frac{1}{n}\Sigma)$, $S \sim W_p(\frac{1}{m}\Sigma, m)$ and $A \sim W_p(\Sigma, m)$, where $W_p(.,.)$ denotes the central p-dimensional Wishart distribution with changed arguments and $m = n - 1$,

(v) $csU \sim MN_p(1 \otimes \mu, \Sigma \otimes 1)$ where MN_p denotes the matrix-like normal distribution of p-dimensional independent random vectors with the distributions $N_p(\mu, \sigma)$ and csU denotes np-dimensional vector created from U by arranging in one column consecutive verse vector of matrix U.

The purpose of our analysis is investigating the equality of functions $G_p(x)$ and $F_p(x)$ i.e. we ask if function $G_p(x)$ may be considered identical with function $F_p(x)$, or, if the distributions of random vectors X_j belong with class N_p. We call this assumption a composite null hypothesis of m.n.d. and write it as

$HCM\colon G_p(x) = F_p(x)$,

against the alternative

$HCM_1\colon G_p(x) \neq F_p(x)$,

The HCM (hypothesis composite multivariate) may be expressed in the equivalent form

HCM: distributions $X_j \in N_p$, $j = 1,...,n$.

Verification of HCM hypothesis requires determining optimal s.t. for making a decision if sample U comes from the population with distribution MN_p. Parameters μ and Σ are unknown. We replace them with proper sample estimators. Such reasoning creates difficulties in determining a suitable test function and its distribution when HCM is true. Parameters μ and Σ are not of intrest for us and are treated as nuisance parameters.

Definition 2.1. Let N_p denote the class of p-dimensional n.d. and Δ the set of feasible decisions with parameters μ and σ:

a) if there exists subset $C_0 \subset \Delta \times R^p$, such, that $C = C_0 \times I_p^>$, then μ is called a disturbing parameter with respect to C;

b) if there exists subset $C_0 \subset \Delta \times I_p^>$, such, that $C = C_0 \times R^p$, then Σ is called a disturbing parameter with respect to C;

c) if there exists subset $C_0 \subset \Delta$, such, that $C = C_0 \times R^p \times I_p^>$, then μ and Σ are called disturbing parameters with respect to C, where $\times$ denotes the Cartesian product. We intend to construct a m.n.t. for verifying *HCM* after elimination of μ and Σ. We do that through a suitable transformation of sample U. Different methods of such transformation are possible. They allow to verify *HCM* by equivalent hypotheses about the standardized u.n.d. (*HN*) or uniform on the interval (0, 1) (*HJ*).

3. RANDOMIZATION METHOD

3.1. General remarks

In this method we use Durbin (1961) procedure generalized to the multivariate case. The idea of it is to join as many generated random variables as there exist disturbing parameters in *HCM* problem. In this case it will be generating a suitable symmetric matrix with the Wishart distribution and the vector of means from a population with a spherical p-dimensional normal distribution, and both these variables should be independent of each other and of observable matrix U.

3.2. Disturbing parameters μ and Σ

In the considered method of transforming matrix U we use the following lemmas:

Lemma 3.1. (Pearson, Sekar 1936) The product of independent random variables with the chi-square and beta distributions is identical with the distribution of the normally distributed variable.

Lemma 3.2. (Khatri 1959) If $S \sim W_p(\Sigma, n)$ then $U \sim MN_p(0, \Sigma \otimes I)$, then $\tilde{S} = S + UU' \sim W_p(\Sigma, 2n)$ and $Z = T^{-1}U \sim MB_p$ are independent, where $\tilde{S} = TT'$ and MB_p denotes the multivariate beta distribution given by density

$$f(Z) = c_n |I - ZZ|^{(n-p-1)/2},$$

and c is a constant.

According to lemma 3.2. we transform sample U, so that it would be a set of vectors from the MB_p population.

Let $U = (U_n, X_n)$ where $(U = (X_1, ..., X_m)$, for $m = n-1$, be matrix received from U after elimination of vector X_n. From the matrix U_n we determine $\bar{X}_n$ and A_n similarly as in point 2. We define vector $X_n^* = X_p - \bar{X}_n$. Its properties are contained in:

Lemma 3.3. $A = A_n + aX_n^*X_n^*$, where $a = m/n$.

Proof. Let's write $\bar{X} = \bar{X}_n + \frac{1}{n}X_n^*$, then $n\bar{X}\bar{X}' = (n-1)\bar{X}_n\bar{X}_n' - \frac{n}{n-1}X_n^*X_n^{*\prime} + X_nX_n'$ and for matrix A we have

$$A = \sum_{j=1}^{n-1} X_jX_j' + X_nX_n' - n\bar{X}\bar{X}' = U_nU_n' - (n-1)X_nX_n' + aX_n^*X_n^*.$$

Lemma 3.4. $X_n^* \sim N_p(0, \frac{1}{a}\Sigma)$ when HCM is true.

Proof. When HCM is true, then $X_j \sim N_p(\mu, \Sigma)$, so $\bar{X}_p \sim N_p(\mu, \frac{1}{n-1}\Sigma)$ and $E(X_n - \bar{X}_n) = 0$

and $D(X_n) + D(\bar{X}_n) - 2Cov(X_n, \bar{X}_n) = \Sigma + \frac{1}{n-1}\Sigma = a^{-1}\Sigma$.

Lemma 3.5. (Wagle 1968) The density of random vector $Z_n^* = L^{-1}X_n^*$ where $A = LL'$ has the form

$$g(Z_n^*) = c_n^* \,|I - aZ_n^*Z_n^{*\prime}|^{(n-p-2)/2}$$

where c_n^* is some constant.

Given lemmas relate to vector X_n. We transform them to the rest of the vectors $X_1, ..., X_{n-1}$, what results in $Z_j^* = L^{-1}X_j^*$, $j = 1, ..., n$ where X_j is the vector of means from sample U after excluding vector X_j. We create matrix $Z^* = (Z_1^*, ..., Z_n^*)$, whose distribution is given by lemma 3.2. when HCM is true. To matrix Z^* we apply a randomization procedure (Durbin 1961) the idea of which is to generate, in the problem considered, $p(p+3)/2$ additional random variables, as many as there are different disturbing parameters.

Let $B \sim W_p(I, n-1)$ and $D \sim N_p(0, \frac{1}{n}I)$ be the generated random matrix and random vector with the given distributions and independent of each other and of matrix A. We define matrix K and X^* such that $B = KK'$ and $X^* = (D, ..., D)$ then matrix $Y^* = X^* + \sqrt{a}KZ^* \sim MN_p(0, 1)$ when HCM is true. From the sphereness of the distribution of matrix Y^* we conclude that its elements are a random sample of np elements. Thus, the problem of verification of HCM was reduced to the equivalent problem of verification of HN.

3.3. Distrubing parameter μ

Let now $\mu = E(X)$ $\Sigma_0 = D(X)$ with known Σ_0 In the situation considered we have p disturbing parameters contained in vector μ. We define $X_j^* = \Sigma_0^{-1/2}(X_j - \bar{X})$ for $j = 1,..., n$, where $\Sigma_0^{1/2}$ is a symmetric square root of matrix Σ_0

Lemma 3.6. $X_j^* \sim N_p(0, aI)$ when HCM is true.

We generate matrix $\bar{X}^* \sim MN_p(0, \frac{1}{n}I)$ independent of matrix $U^* = (X_1^*,..., X_n^*)$ and we fix $Y^* = X^* + U^*$, which has the distribution $MN_p(0, I)$ when HCM is true. Thus the problem of verification of HCM was reduced to the verification of hypothesis HN.

3.4. Distrubing parameter Σ

Let U_1, U_n be as in 3.2. and let $\mu_0 = E(X)$ and $\Sigma = D(X)$, we know μ_0. We create matrix $A^0 = (U - \mu_0 1')(U - \mu_0 1')'$ and $A^0 = A_n^0 + X_n^0 X_n^{0\prime}$, where $X_n^0 = X_n - \mu_0$.

Lemma 3.7. We have $A_n^0 \sim W_p(\Sigma, n-1)$ and $X_n^0 \sim N_p(0, \Sigma)$, when $U \sim MN_p(1 \otimes \mu_0, \Sigma)$.

Let us find $Z_j^0 = L^0 X_j^0$ for $A^0 = L^0 L^{0\prime}$, for $j = 1,..., n$. Due to unknown matrix Σ we generate matrix $B^0 \sim W_p(I, n)$ independent of matrix $Z^0 = (Z_1^0,..., Z_n^0)$, and then we fix matrix $Y^0 = K^0 Z^0 \sim MN_p(0, I)$, where $B^0 = K^0 K^{0\prime}$ when HCM is true.

Again, the problem of HCM was reduced to the problem of verification of HN.

4. REDUCTION METHODS

4.1. General remarks

The reduction methods in univariate normality problem was suggested by S a r k a d i (1966). The idea is to eliminate from the sample considered as many variables as there are distrubing parameters. This is a converse approach to the one in section 3. In our case sample U is transformed according to Hensler's H e n s l e r et al. (1977) suggestion. We use the conditional distribution given in section 2.

4.2. Disturbing parameter μ

Let the assumption of sample U be as in 3.3. We define vector

$$X^* = b(U1 + \sqrt{n}X_n) = b(U_n 1 + (1 + \sqrt{n})X_n)$$

where $b = 1/(n + \sqrt{n})$. We transform sample U by reducing it by one vector, let us say, X_n. Without loosing generality we can choose any other column vector from sample U.

Lemma 4.1. For vector X^* we have

(a) $E(X^*) = \mu$,
(b) $D(X^*) = 2b\Sigma_0$,
(c) $Cov(X^*, X_j) = b\Sigma_0$,
(d) $X^* \sim N_p(\mu, 2b\Sigma_0)$ when HCM is true.

Proof. (a) $E(X^*) = b[mE(X_j) + (1 + \sqrt{n})]E(X_n) = b(n + \sqrt{n})\mu = \mu$ where $m = n - 1$

(b) $D(X^*) = b^2[mD(X_j) + (1 + \sqrt{n})^2 D(X_n)] = b^2(m + 1 + 2\sqrt{n} + \mathrm{n})\Sigma_0 = 2b\Sigma_0$;

(c) $Cov(X^*, X_j) = Cov(U1 + \sqrt{n}X_n, X_j) = b\sum_{k=1}^{n} Cov(X_k, X_j) + b\sqrt{n}\,Cov(X_n, X_j) = b\Sigma_0$;

(d) follows from the linear mapping $X^* = UB$, where

$$B = b\begin{bmatrix} 1_{n-1} \\ 1 + \sqrt{n} \end{bmatrix}.$$

To eliminate disturbing parameter μ we use transformation

$$Y_j = X_j - X^* \quad \text{for } j = 1, \ldots, n - 1.$$

Lemma 4.2. When HCM is true random vectors Y_j, $j = 1, \ldots, n - 1$ have independent distributions $N_p(0, \Sigma)$.

Proof. $E(Y_j) = 0$ what follows directly from lemma 4.1. To determine covariance matrix Y_j of vectors we present them in the form

$$Y_j = (1 - b)X_j - b\left[\sum_{\substack{k=1 \\ k \neq j}}^{m} X_k + (1 + \sqrt{n})X_n\right]$$

then $D(X_j) = [(1 - b)^2 + b^2(n - 2 + (1 + \sqrt{n})^2)]\Sigma_0 = (1 - 2b + b^2 + 2b^2(n + \sqrt{n}) - b^2)\Sigma_0 = \Sigma_0$. To prove independence of Y_j we determine covariance $Cov(Y_j, Y_{j'}) = Cov(X_j, X_{j'}) - 2\,Cov(X^*, X_{j'}) + D(X^*) = -2b\Sigma_0 + 2b\Sigma_0 = 0$ on strenght of lemma 4.1.

The normality of the distribution of vectors Y_j follows directly from the orthogonal transformation $y = UB$ where $B = (b_{jj})$ is a matrix with elements given in the form (Sarkadi and Tusnady 1977)

$$b_{jj'} = \begin{cases} 1 - b, & j = j' = 1,...,n-1 \\ -b, & j \neq j', \ j, j' = 1,...,n-1 \\ -(1+\sqrt{n})b, & j = n, \ j' = 1,...,n-1 \end{cases}$$

and the last n-th column is supplemented with Gram–Schmidt orthogonalization (OG-S) method.

The verification of the HCM with sample U was reduced to verification of HN with sample $Y_1,..., Y_{n-1}$ of $p(n-1)$ independent random variables.

4.3. Distrubing parameters μ and Σ

Now we have $q = p + p(p+1)/2 = p(p+3)/2$ parameters of noise.

Our reasoning needs $np - q = (2n - p - 3)/2$ independent random variables free from parameters μ and Σ. This means reducing e.g. the variables of matrix U by q.

In transforming matrix U we use the lemma about conditional distribution given in section 2 and the following lemma.

Lemma 4.3. (Hensler et al. 1977) If random vectors X_j, $j = 1,...,n$ are p-dimensional and independent and $D(X_j) = \Sigma$ and $Y^* = (Y_i,..., Y_n) = UA$ where A is a suitable orthogonal matrix, then vectors $Y_2,..., Y_n$ are independent with the distribution $N_p(0, \Sigma)$ if and only if $X_j \sim N_p(a_{j1}E(Y_1), \Sigma)$, $j = 1,...,n$, where a_{j1} are the elements of the first column of matrix A.

According to the given lemma, first we check the univariate normality of random variable X_p, then we use the orthogonal transformation to remove it and we start investigating the $(p-1)$ – dimensional normality provided that $X_p \sim N_1$. Earlier we eliminate parameter μ as in 4.2.

Let us consider the procedure for $p = 2$ assuming $E(X_j) = 0$ for $D(X_j) = \Sigma$ $j = 1,..., m$, where $m = n - 1$. We define verse vectors $U_{(1)}$, $U_{(2)}$ such that $U = (U_{(1)}, U_{(2)})'$ and an orthogonal matrix A with the first column $a_{j1} = X_{2j}/(U_{(2)}U'_{(2)})^{1/2}$, $j = 1,..., m$. Other columns are supplemented with the help of OG-S. Let us consider two n-element univariate random samples created of the elements of vectors $U_{(2)}$ and $W = (W_1,..., W_m) = U_{(1)}A$. According to lemma 4.3 the verification of HCM will be equivalent to verification of two HN stating that sample $X_{21}, X_{22},..., X_{2m}$ comes from the population $N(0, \sigma_{22})$, and sample $W_2, W_3,..., W_m$ comes from population $N(0, \sigma^2)$ where $\sigma^2 = (1 - \rho^2)\sigma_{11}$.

Instead of verifying two hypotheses we may verify the hypothesis of the joined normality combining Fisher tests.

According to Sarkadi method we transform variables $W_2, W_3, \dots, W_m$ to variables

$$R_{j-1} = \frac{W_j}{W^*}\psi_{m-2}\left[\frac{|W_M|\sqrt{n-2}}{W^*}\right], \quad j = 2, \dots, m-1,$$

where $W^* = (W_2^2 + \dots + W_2^2)^{1/2}$, and function $\psi_f(t)$ is given by the relation

$$Q_f([\psi_f(t)]^2) = 2P_f(t) - 1$$

where $P_f(t)$ and $1 - Q_f(t)$ are the cumulative functions of the t-Student and chi-square distributions respectively.

As a result, instead of verifying HN for sample $W_2, W_3, \dots, W_m$ we verify equivalent hypothesis HJ for sample $R_2, \dots, R_{m-2}$. Using this reasoning to sample $X_{21}, X_{22}, \dots, X_{2m}$ we reduce the problem of verification of HN to the problem of verifing HJ for sample $T_1, \dots, T_{m-1}$. Linking the two samples we get $(2n-3)$-element sample which, when HCM is true, is a sequence of independent random variables with the uniform distribution over the interval (0, 1). Let us remark that starting from sample U we reduced the problem to 5 variables connected with parameters $\mu_1, \mu_2, \sigma_{22}$ and $\sigma_{12} = \rho\sigma_{11}\sigma_{22}$.

We consider the case $p > 2$. We assume, according to the lemma given in section 2 about the conditional distribution, that after k iterations we have

$$U^{(k)} = (X_1^{(k)}, \dots, X_m^{(k)}) = (U^{*(k)}, U_{(q)}^{*(k)})',$$
$$U_{(q)}^{*(k)} = (X_{q,k}^{(k)}, \dots, X_{q,m}^{(k)})$$

where $m = n - 1$ and $q = p - k + 1$. We investigate the normality of variables $X_1, X_2, \dots, X_{p-k}$ provided that $X_p \sim N_1, X_{p-1} \sim N_1, \dots, X_q \sim N_1$, fixing elements $a_{j1}^{(k)} = X_{q,j}/(U_{(q)}^{*(k)}U_{(q)}^{*(k)\prime})^{1/2}$ of orthogonal matrix $A^{(k)}$ and $U_{(q)}^{(k+1)} = U^{*(k)}A^{(k)}$. Matrix $A^{(k)}$ has the first column $a_{ji}^{(k)}$ and the other elements are supplemented with the help of OG-S. As a result of consecutive iterations, where $U^{(1)} = U$, we get independent samples of sizes $n-1, n-2, \dots, n-p-1$. The problem of verification of HCM was reduced to verification of p independent samples. It is possible to apply procedures of tests for normality for many independent samples which were mentioned in the $p = 2$ case. A stronger action is the verification of HJ that the joined sample of size e.g. $m = p(2n - p - 3)/2$ comes from the population with the distribution $J(0, 1)$. This reasoning is analogous to the one given for $p = 2$.

5. CONDITIONAL INTERVAL PROBABILITY TRANSFORMATION METHOD

5.1. General remarks

We will now apply a transformation of sample U using the property of the characterization of the N_R distribution with the p-dimensional t-Student distribution $(t-S_p)$. It regards the conditional cumulative function. This function is determined by replacing unknown parameters μ and Σ of the N_p distributions with their sufficient statistics, which have the property of double transitiveness.

This means that random variables from sample U are transformed into a set of pq, where $q = n-p-1$, independent random variables with the distributions $J(0, 1)$. Our reasoning has two stages. In the first we determine the best, unbiased estimator of the density function of distribution N_p from sample U; in the second we use this density to determine the conditional density of the $t-S_p$ distribution. The last density refers to the set of $n-p-1$ vectors. The elements of these vectors are quantiles of the t-Student distribution with the suitable number of the degrees of freedom when HCM is true.

The notion of the conditional integral probability transformation will be understood according to the following definition.

Definition 5.1. Let $G(x, y)$ be the cumulative function of the bivariate distribution of (X, Y) and $G(x, y_0)$ its conditional cumultative function when $Y = y_0$.

The transformation $U = G(X, y_0)$ is called a conditional interval probability transformation.

With the transformation given in definition 5.1. is connected a familiar fact that if X is a continuous random variable then U is a random variable with the distribution $J(0, 1)$.

5.2. Disturbing parameters μ and Σ

Let $Q_n(X) = (X-\bar{X}_n)'A_n^{-1}(X-\bar{X}_n)$ be a quadratic form for random vector X, where $\bar{X}_n = U1/n$ and $A_n = UU' - n\bar{X}_n\bar{X}_n'$. Numbers $T_n = (\bar{X}_n, A_n)$ determined from the whole sample U are a system of sufficient statistics of parameters μ and Σ of distribution N_p. They have the property of double transitiveness. This means that if (T_n) is a sequence of sufficient statistics then each pair (T_n, X_{n+1}) and (T_{n+1}, X_{n+1}) may by computed from the other. For instance for $T_n = \bar{X}_n$ where $(T_{n+1}, X_{n+1}) = ((n\bar{X}_n + X_{n+1})/(n+1), X_{n+1})$ and $(T_n, X_{n+1}) = (((n+1)\bar{X}_n - X_{n+1})/n, X_{n+1})$.

Analogously, let us denote by $T_j = (\bar{X}_j, A_j)$ the system of sufficient statistics from sample $X_1,..., X_j$ Further, let $L_j' L_j = A_j^{-1}$ for $j = p+2,..., n$. Sets T_j are considered fixed.

Lemma 5.1. If $U \sim MN_p$ then the unbiased estimator with minimum variance for the density of distribution N_p is

$$f(x) = d_n |A_n|^{-1/2}(a - Q_n(x))^{(q-2)/2},$$

for $Q_n(x) < a$ $f(x) = 0$ for $Q_n(x) > a$, where d_n is a constant depending only on n, and $a = (n-1)/n$.

Lemma 5.2. (Rincon-Gallardo et al. 1979). Let the distribution of p-dimensional random vector Y with the fixed T_n be given by the conditional probability from lemma 5.1. Then random vector

$$Z = L_n(Y - \bar{X}_n)/(a - Q(Y))^{1/2}$$

has the conditional density of the form

$$g(z) = \tilde{d}_n(1 + z'z)^{-(n-1)/2}$$

respective to the distribution $t - S_p$ and $\tilde{d}_n$ is some constant.

Lemma 5.3. (Dickey 1967). The conditional cumulative function of the i-th component of random vector $Z = (Z_1,..., Z_p)'$ with the distribution given in lemma 5.2. is

$$\bar{P}(Z_i | Z_1,..., Z_{i-1}) = P_{q+i+1}\left[Z_i\left[\frac{q+i-1}{1 + \sum_{k=1}^{i-1} Z_k^3}\right]^{1/2}\right],$$

where $P_f(.)$ is given by the cummulative function of the distribution $t - S_f$.

Our results were given for sufficient statistics T_n. We can transform them to statistics $T_j, j = p+2,..., n$. As they also have the property of double transitiveness we can use lemmas 5.2 and 5.3 for vectors

$$Z_j = L_j(X_j - \bar{X}_j)/[(j-1)/j - Q_j(X_j)]^{1/2},$$

where $L_j' L_j = A_j^{-1}$, and then we define pq random variables

$$U_{ij} = P_{j-p+i-2}\left[Z_{ij}\left[\frac{j-p+1-2}{1 + \sum_{k=1}^{i-1} Z_k^2}\right]^{1/2}\right],$$

where $Z_j = (Z_{1j},..., Z_{pj})'$ for $i = 1,..., p$; $j = p+2,..., n$.

Thus, the problem of the verification of *HCM* has been reduced to the verification of *HJ*.

5.4. Disturbing parameters μ and σ

The results from 5.3. will be transformed to the case of unknown μ and $\Sigma = \sigma^2\Sigma_0$ where σ^2 is unknown, Σ_0-known. We define $L^*L^* = \Sigma_0^{-1}$, $g = tr(\Sigma_0^{-1}A_n)$ and $Z_j^* = L^*(X_j - \bar{X}_j)/[g(j-(X_j - \bar{X}_j)'\Sigma_0^{-1}(X_j - \bar{X}_j)]^{1/2}$ for $j = 3,..., n$ and $Z_j^* = (Z_{1j}^*,..., Z_{pj}^*)'$. Then, $p(n-2)$ random variables

$$U_{ij} = P_{p(j-2)+i-1}\left[Z_{ij}^*\left[\frac{p(j-2)+i-1}{1+\sum_{k=1}^{i-1} Z_k^{*2}}\right]^{1/2}\right],$$

have independent distributions $J(0, 1)$ for $i = 1,..., p$; $j = 3,..., n$. This means that again, the problem of the verification of hypothesis HCM has been reduced to *HI*.

6. FINAL REMARKS

All the given transformations have the property of identity. The results of testing *HCM* do not depend on the order of vectors in sample. The numerical side of the methods used is easy. There are many computer packages connected with the suitable decomposition of a positive definite symmetric matrix determining orthogonal matrix with the help of OG-S or computing the cummulative function of the t-Student distribution. The property of double transitiveness may also be programmed according to Rincon-Gallardo and Quesenberry (1982) algorithm.

The power properties of these procedures for different distributions of alternatives require separate analysis, as well as thier comparison with known t.m.n.

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SPRAWDZENIE HIPOTEZY O WIELOWYMIAROWYM ROZKŁADZIE NORMALNYM W MODELU JEDNEJ PRÓBY METODĄ ELIMINACJI PARAMETRÓW ZAKŁÓCAJĄCYCH

W wielu zagadnieniach statystycznych zachodzi potrzeba weryfikacji hipotezy o wielowymiarowym rozkładzie normalnym. Przy konstrukcji testów weryfikujących taką hipotezę istnieje konieczność oszacowania, na podstawie próby losowej, nieznanych parametrów rozkładu μ i Σ, które traktowane są jako parametry uboczne, zakłócające.

W pracy przedstawione zostały metody, za pomocą których eliminuje się nieznane, zakłócające parametry w testach służących do weryfikacji hipotezy o normalności. W szczególności zostały omówione następujące metody: randomizacji, redukcji oraz warunkowego całkowego przekształcenia probabilistycznego.