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QUALIFIED DEFINITENESS

Abstract

According to Russell, the definite article ‘the’ in a definite description ‘the F ’ is used strictly in case there is a unique F and it is used loosely in case there is more than one F . Russell’s analysis of constructions of the form ‘the F is G ’ is concerned only with the strict use. We modify this analysis so as to allow also for the loose use. This is achieved essentially by replacing the usual undefined notion of identity in Russell’s uniqueness clause with the defined notion of qualified identity (i.e., ‘ a is the same as b in all Q -respects’, where Q is a subset of the set of predicate constants \mathcal{P}) proposed in earlier work. This modification gives us qualified notions of uniqueness and definiteness. A qualified definiteness statement ‘the Q -unique F is G ’ is strict in case $Q = \mathcal{P}$ and loose in case Q is a proper subset of \mathcal{P} . The account is made formally precise in terms of proof theory and proof-theoretic semantics. The framework is intended to be acceptable from a foundational intuitionistic point of view. It is applied to

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natural language constructions with complete, incomplete, and generic definite descriptions. Also constructions with nested and with predicatively used definite descriptions are considered as well as constructions involving possessives. This work incorporates and extends my *NCL'24*-paper ‘Incomplete descriptions and qualified definiteness’.

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1. Introduction

Sometimes we use the definite description ‘the F ’ in cases in which there is a unique F . According to Russell, the definite article ‘the’ is used strictly in such cases.¹ For example, speaking about Leo XIV, we use ‘the pope’ in (1) in this way.

- (1) The pope is bald.

Sometimes, as Russell notes, we use ‘the F ’ also in cases, in which there is more than one F . For example, ‘the bishop’ in (2) is used in this loose way (as would be ‘the pope’ during a schism).

- (2) The pope blesses the bishop.

According to Russell, such loose uses of ‘the F ’ should be avoided in favour of the indefinite description ‘an F ’. It seems, though, that Russell’s advice is not entirely adequate. For example, in case multiple F s are present and one intends to say something about a particular F (e.g., [20]).

In this paper, we propose a formal account of both uses of ‘the F ’ in terms of qualified definiteness. On a Russellian analysis, a construction of

¹Russell ([16]: 481): “Now *the*, when it is strictly used, involves uniqueness; we do, it is true, speak of “*the* son of So-and-so” even when So-and-so has several sons, but it would be more correct to say “*a* son of So-and-so”. Thus for our purposes we take *the* as involving uniqueness.”

the form ‘the F is G ’ is explained in terms of an existence, a uniqueness, and a predication clause:

(E) There is at least one F .

(U) There is at most one F .

(P) Every F is G .

We modify this analysis mainly by replacing the usual undefined notion of identity in the definition of uniqueness with the defined notion of *qualified identity* proposed in [24], i.e., ‘ a is the same as b in all \mathcal{Q} -respects’, where \mathcal{Q} is a subset of the set of predicate constants \mathcal{P} . The notion of *qualified uniqueness* that results from this replacement says:

(QU) For every x and y , if they are F , then they are identical with respect to every predicate in \mathcal{Q} .

Finally, a statement of *qualified definiteness* says, combining the three Russellian components:

(QD) The \mathcal{Q} -unique x which is F is G .

Qualified definiteness, unlike standard definiteness, allows for fine-tuning. Let \mathcal{Q}' be a proper subset of \mathcal{P} (i.e., $\mathcal{Q}' \subset \mathcal{P}$). If $\mathcal{Q} = \mathcal{P}$ in (QD), then we get the reading ‘the *only* x which is F is G ’. We may use this reading only in case there is a single x that is F . This is definiteness proper. If, on the other hand, we put $\mathcal{Q} = \mathcal{Q}'$, then we get: ‘the x which is F is G ’. We may use this reading only in case there are at least two things which are F that are indiscernible with respect to \mathcal{Q}' , but discernible with respect to $\mathcal{P} \setminus \mathcal{Q}'$. This is restricted definiteness. What is subject to restriction, on this account, is thus the set of \mathcal{Q} -respects (rather than, e.g., a domain of quantifiers [18]).

Below, we provide the details of this proposal. It will differ from competing semantic analyses of incomplete descriptions also in that it will be couched in a framework of proof-theoretic semantics (see [17] for an overview) rather than in some version of model-theoretic semantics. (For

an overview of the literature on incomplete descriptions see, e.g., [1, ch. 9], [12, sect. 5.3]. An elaborate model-theoretic account is [2].)

Sect. 2 defines the formal language. Sect. 3 recapitulates the relevant fragment of the intuitionistic bipredicational natural deduction systems defined in [24] and combines it with the rules for definiteness proposed in [3, 4] into proof systems for qualified definiteness, establishing normalization and the subexpression (incl. subformula) property for them. Sect. 4 defines an intuitionistic proof-theoretic semantics for qualified definiteness in terms of canonical derivations, and Sect. 5 applies this semantics to complete, incomplete, and generic definite descriptions in the manner suggested above. In Sect. 6 this account of definiteness is generalized so as to allow also for the analysis of natural language constructions with nested and with predicative uses of definite descriptions. Also constructions involving possessives are considered. This work incorporates [25] and extends that paper essentially with the generalizations described in Sect. 6.

2. The language

We extend the bipredicational language \mathcal{L} motivated and defined in [24] with contextually defined operators for qualified definiteness and call the extended language $\mathcal{L}\iota$.

\mathcal{L} is a first-order language. It is bipredicational, since it allows for both predication and predication failure. We first recapitulate those parts of its definition which are relevant for present purposes.

DEFINITION 2.1. \mathcal{C} is the set of individual (or nominal) constants (form: α_i) and \mathcal{P} is the set of n -ary predicate constants (form: φ_i^n) of \mathcal{L} . Moreover, Atm is the set of atomic sentences (form: $\varphi^n \alpha_1 \dots \alpha_n$) of \mathcal{L} . $Atm(\alpha) =_{def} \{A \in Atm : A \text{ contains at least one occurrence of } \alpha \in \mathcal{C}\}$ and $Atm(\varphi^n) =_{def} \{A \in Atm : A \text{ contains an occurrence of } \varphi^n \in \mathcal{P}\}$. A nominal term o_i is either a nominal constant or a nominal variable x_i . Atomic formulae have the form $\varphi^n o_1 \dots o_n$ and are used for predication. Negative predications (or predication failures) take the form $-\varphi^n o_1 \dots o_n$ (reading: ‘the ascriptive combination of φ^n with o_1, \dots, o_n fails’).

DEFINITION 2.2. Defined symbols of \mathcal{L} :

1. $\neg A =_{def} A \supset \perp$ (negation)
2. $A \leftrightarrow B =_{def} (A \supset B) \& (B \supset A)$ (equivalence)
3. Let φ^n be an n -ary predicate constant.

$$P_{\varphi^n}^n(o_1, o_2) =_{def} \\ \forall x_1 \dots \forall x_{n-1} \forall x_n ((\varphi^n o_1 x_2 \dots x_n \leftrightarrow \varphi^n o_2 x_2 \dots x_n) \\ \& (\varphi^n x_1 o_1 \dots x_n \leftrightarrow \varphi^n x_1 o_2 \dots x_n) \\ \& \dots \& (\varphi^n x_1 \dots x_{n-1} o_1 \leftrightarrow \varphi^n x_1 \dots x_{n-1} o_2))$$

$$N_{\varphi^n}^n(o_1, o_2) =_{def} \\ \forall x_1 \dots \forall x_{n-1} \forall x_n ((-\varphi^n o_1 x_2 \dots x_n \leftrightarrow -\varphi^n o_2 x_2 \dots x_n) \\ \& (-\varphi^n x_1 o_1 \dots x_n \leftrightarrow -\varphi^n x_1 o_2 \dots x_n) \\ \& \dots \& (-\varphi^n x_1 \dots x_{n-1} o_1 \leftrightarrow -\varphi^n x_1 \dots x_{n-1} o_2))$$

Let $\varphi_1^{k_1}, \dots, \varphi_m^{k_m}$ be all the predicate constants in \mathcal{Q} , where φ_i is k_i -ary and $\mathcal{Q} \subseteq \mathcal{P}$.

Positive qualified identity:

$$o_1 \stackrel{+}{=}_{\mathcal{Q}} o_2 =_{def} P_{\varphi_1}^{k_1}(o_1, o_2) \& \dots \& P_{\varphi_m}^{k_m}(o_1, o_2) \\ \text{('}o_1 \text{ is the same as } o_2 \text{ in all } \mathcal{Q}\text{-respects')}$$

Negative qualified identity:

$$o_1 \stackrel{-}{=}_{\mathcal{Q}} o_2 =_{def} N_{\varphi_1}^{k_1}(o_1, o_2) \& \dots \& N_{\varphi_m}^{k_m}(o_1, o_2) \\ \text{('}o_1 \text{ is the same as } o_2 \text{ in no } \mathcal{Q}\text{-respect')}$$

Remark 2.3. Note that, in contrast to \neg , the operator for predication failure $-$ is primitive. Moreover, unlike the former, it is sensitive to the internal structure of the formula to which it is prefixed. See [24] for the motivation of $-$.

Remark 2.4. Being a defined notion, qualified identity differs not only from standard identity, but also from the notion of relative identity introduced by Geach in [5]. For a more detailed comparison see [24].

Remark 2.5. If the stroke for predication failure were not present in the language, \mathcal{L} would be the language of plain first-order logic without standard identity. Importantly, the definition of qualified identity does not appeal to second-order quantification. In this respect our notion differs from the notion of identity considered, for example, by Read in [14]: $a = b =_{def} \forall F(Fa \leftrightarrow Fb)$.

\mathcal{L}_t extends \mathcal{L} with operators for qualified definiteness by adapting the definitions from [3, 4].

DEFINITION 2.6. We write $\varphi(x)$, suppressing the arity of φ , for atomic formulae $\varphi^n o_1 \dots o_n$ containing (possibly multiple occurrences of) x . Let $\mathcal{Q} \subseteq \mathcal{P}$.

1. *Positive qualified definiteness:*

$$\psi(\iota_{\mathcal{Q}} x \varphi(x)) =_{def} \exists x \varphi(x) \ \& \ \underbrace{\forall u \forall v ((\varphi(u) \ \& \ \varphi(v)) \supset u \stackrel{+}{=}_{\mathcal{Q}} v)}_{\text{Positive qualified uniqueness}}$$

$$\& \ \forall w (\varphi(w) \supset \psi(w))$$

(‘the \mathcal{Q} -unique x which is φ is ψ ’; simpler: ‘the \mathcal{Q} -unique φ is ψ ’)

2. *Negative qualified definiteness:*

$$\psi(\iota_{\mathcal{Q}} x -\varphi(x)) =_{def} \exists x -\varphi(x) \ \& \ \underbrace{\forall u \forall v ((-\varphi(u) \ \& \ -\varphi(v)) \supset u \stackrel{-}{=}_{\mathcal{Q}} v)}_{\text{Negative qualified uniqueness}}$$

$$\& \ \forall w (-\varphi(w) \supset \psi(w))$$

(‘the \mathcal{Q} -unique x which fails to be φ is ψ ’; simpler: ‘the \mathcal{Q} -unique $-\varphi$ is ψ ’)

Remark 2.7. The definition of positive qualified definiteness differs from the definition of definiteness proposed in [3, 4] in that it does not make use of the familiar primitive notion of identity in the uniqueness part. In this respect, it significantly departs also from the tradition.

Remark 2.8. The reading of the positive qualified uniqueness formula is ‘there is at most one \mathcal{Q} -qualified φ ’, that of the negative qualified uniqueness formula is ‘there is at most one \mathcal{Q} -qualified $-\varphi$ ’.

Qualified definiteness allows for degrees.

DEFINITION 2.9. Let $\mathcal{Q}' \subset \mathcal{P}$. Qualified definiteness has (i) the highest degree of definiteness, in case $\mathcal{Q} = \mathcal{P}$, and (ii) a lower degree, in case $\mathcal{Q} = \mathcal{Q}'$. Given $\mathcal{Q}' \subset \mathcal{P}$, we can make the following distinction:

1. *Maximal definiteness:*

- (a) $\psi(\iota_{\mathcal{P}}x\varphi(x))$: ‘the only x which is φ is ψ ’;
- (b) $\psi(\iota_{\mathcal{P}}x - \varphi(x))$: ‘the only x which fails to be φ is ψ ’.

2. *Restricted definiteness:*

- (a) $\psi(\iota_{\mathcal{Q}'}x\varphi(x))$: ‘the x which is φ is ψ ’;
- (b) $\psi(\iota_{\mathcal{Q}'}x - \varphi(x))$: ‘the x which fails to be φ is ψ ’.

A loosely used definite description ‘the F ’ is, thus, construed as a restriction of a strictly used ‘the F ’ (i.e., the maximally definite description ‘the only F ’).

DEFINITION 2.10. Negative predications with qualified definite descriptions take the following forms:

- 1. $-\psi(\iota_{\mathcal{Q}}x\varphi(x))$: ‘the \mathcal{Q} -unique x which is φ fails to be ψ ’;
- 2. $-\psi(\iota_{\mathcal{Q}}x - \varphi(x))$: ‘the \mathcal{Q} -unique x which fails to be φ fails to be ψ ’.

Remark 2.11. $\mathcal{L}\iota$ is an extension of \mathcal{L} only in the sense that it uses abbreviations which are not present in \mathcal{L} .

3. Proof systems

In order to obtain a proof system for reasoning with qualified definiteness, we enrich the intuitionistic bipredicational $\mathbf{IO}(\mathcal{S}_b^-)$ -systems defined in [24] with rules for qualified definiteness, by adapting the rules for definiteness presented in [3, 4]. We call the resulting systems $\mathbf{IO}(\mathcal{S}_b^-)\iota$ -systems.

3.1. Bipredictional natural deduction

We first repeat the parts of the definition of $\mathbf{IO}(\mathcal{S}_b^-)$ -systems from [24] which are relevant for present purposes.

3.1.1. Bipredictional subatomic systems

DEFINITION 3.1. A *bipredictional subatomic system* \mathcal{S}_b is a pair $\langle \mathcal{I}, \mathcal{R}_b \rangle$, where \mathcal{I} is a *subatomic base* and \mathcal{R}_b is a set of *introduction and elimination rules for atomic sentences and negative predications*. \mathcal{I} is a 3-tuple $\langle \mathcal{C}, \mathcal{P}, v \rangle$, where v is such that:

1. For any $\alpha \in \mathcal{C}$, $v : \mathcal{C} \rightarrow \wp(\text{Atm})$, where $v(\alpha) \subseteq \text{Atm}(\alpha)$.
2. For any $\varphi^n \in \mathcal{P}$, $v : \mathcal{P} \rightarrow \wp(\text{Atm})$, where $v(\varphi^n) \subseteq \text{Atm}(\varphi^n)$.

We let $\tau\Gamma \stackrel{\text{def}}{=} v(\tau)$ for any $\tau \in \mathcal{C} \cup \mathcal{P}$, and call $\tau\Gamma$ the set of *term assumptions* for τ . \mathcal{R}_b contains I/E-rules of the following form:

$$\frac{\mathcal{D}_0 \quad \mathcal{D}_1 \quad \dots \quad \mathcal{D}_n}{\varphi_0^n \Gamma \quad \alpha_1 \Gamma \quad \dots \quad \alpha_n \Gamma} (asI) \qquad \frac{\mathcal{D}_1}{\varphi_0^n \alpha_1 \dots \alpha_n} (asE_i)$$

$$\frac{\mathcal{D}_0 \quad \mathcal{D}_1 \quad \dots \quad \mathcal{D}_n}{\varphi_0^n \Gamma \quad -\varphi_0^n \alpha_1 \dots \alpha_n} (-asI) \qquad \frac{\mathcal{D}_1}{-\varphi_0^n \alpha_1 \dots \alpha_n} (-asE_i)$$

Side conditions:

1. asI : $\varphi_0^n \alpha_1 \dots \alpha_n \in \varphi_0^n \Gamma \cap \alpha_1 \Gamma \cap \dots \cap \alpha_n \Gamma$.
2. $-asI$: $\varphi_0^n \alpha_1 \dots \alpha_n \notin \varphi_0^n \Gamma \cap \alpha_1 \Gamma \cap \dots \cap \alpha_n \Gamma$.
3. asE_i and $-asE_i$: $i \in \{0, \dots, n\}$ and $\tau_i \in \{\varphi_0^n, \alpha_1, \dots, \alpha_n\}$.

Terminology: We say that $-\varphi_0^n \alpha_1 \dots \alpha_n$ is *negatively contained* in $\varphi_0^n \Gamma \cap \alpha_1 \Gamma \cap \dots \cap \alpha_n \Gamma$, in case the side condition on $-asI$ is satisfied.

DEFINITION 3.2 (Derivations in \mathcal{S}_b -systems).

Basic step. Any term assumption $\tau\Gamma$, any atomic sentence (resp. negative predication), i.e., a derivation from the open assumption of $\varphi_0^n\alpha_1\dots\alpha_n$ (resp. $-\varphi_0^n\alpha_1\dots\alpha_n$) is an \mathcal{S}_b -derivation.

Induction step. If \mathcal{D}_i , for $i \in \{0, \dots, n\}$, are \mathcal{S}_b -derivations, then an \mathcal{S}_b -derivation can be constructed by means of the I/E-rules for as and $-as$ displayed above.

Remark 3.3. The term assumptions are, so to speak, proof-theoretic semantic values of the non-logical constants. Applications of the subatomic introduction rules asI and $-asI$ serve to establish, on the basis of these values, the truth of atomic sentences and negative predications, respectively. Negative predication (or predication failure) is understood as subatomic derivation failure (cf. [24]).

3.1.2. Bipredicational subatomic identity systems

DEFINITION 3.4. Atomic sentences $\varphi(\alpha_1)$ and $\varphi(\alpha_2)$ are *mirror atomic sentences* if and only if they are exactly alike except that the former contains occurrences of α_1 at all the places at which the latter contains occurrences of α_2 , and vice versa.

DEFINITION 3.5. A *bipredicational subatomic identity system* \mathcal{S}_b^\pm is a 3-tuple $\langle \mathcal{I}, \mathcal{R}_b, \mathcal{R}_b^\pm \rangle$, which extends a bipredicational subatomic system with a set \mathcal{R}_b^\pm of I/E-rules for (positive/negative) qualified identity sentences, where $\mathcal{Q} \subseteq \mathcal{P}$.

1. $\stackrel{\pm}{=}_{\mathcal{Q}}$:

$$\begin{array}{ccccccc}
 [\varphi_1(\alpha_1)]^{(1_1)} & [\varphi_1(\alpha_2)]^{(1_2)} & & [\varphi_k(\alpha_1)]^{(k_1)} & & [\varphi_k(\alpha_2)]^{(k_2)} & \\
 \mathcal{D}_{1_1} & \mathcal{D}_{1_2} & & \mathcal{D}_{k_1} & & \mathcal{D}_{k_2} & \\
 \varphi_1(\alpha_2) & \varphi_1(\alpha_1) & \dots & \varphi_k(\alpha_2) & & \varphi_k(\alpha_1) & \\
 \hline
 & & & & & & \stackrel{(\pm)_{\mathcal{Q}I}}{=}_{1_1, \dots, k_2} \\
 & & & \alpha_1 \stackrel{\pm}{=}_{\mathcal{Q}} \alpha_2 & & &
 \end{array}$$

$$\frac{\mathcal{D}_1 \quad \mathcal{D}_{i_1}}{\alpha_1 \stackrel{\pm}{=}_{\mathcal{Q}} \alpha_2 \quad \frac{\varphi_i(\alpha_1)}{\varphi_i(\alpha_2)}} \quad (\stackrel{\pm}{=}_{\mathcal{Q}}\text{E}_i1) \qquad \frac{\mathcal{D}_1 \quad \mathcal{D}_{i_2}}{\alpha_1 \stackrel{\pm}{=}_{\mathcal{Q}} \alpha_2 \quad \frac{\varphi_i(\alpha_2)}{\varphi_i(\alpha_1)}} \quad (\stackrel{\pm}{=}_{\mathcal{Q}}\text{E}_i2)$$

where $\varphi_i \in \mathcal{Q}$, $i \in \{1, \dots, k\}$, and $\varphi_i(\alpha_1)$ and $\varphi_i(\alpha_2)$ are mirror atomic sentences.

2. $\bar{=}_{\mathcal{Q}}$:

$$\frac{\begin{array}{cccc} [-\varphi_1(\alpha_1)]^{(1_1)} & [-\varphi_1(\alpha_2)]^{(1_2)} & [-\varphi_k(\alpha_1)]^{(k_1)} & [-\varphi_k(\alpha_2)]^{(k_2)} \\ \mathcal{D}_{1_1} & \mathcal{D}_{1_2} & \mathcal{D}_{k_1} & \mathcal{D}_{k_2} \\ -\varphi_1(\alpha_2) & -\varphi_1(\alpha_1) & \dots & -\varphi_k(\alpha_2) & -\varphi_k(\alpha_1) \end{array}}{\alpha_1 \bar{=}_{\mathcal{Q}} \alpha_2} \quad (\bar{=}_{\mathcal{Q}}\text{I}, 1_1, \dots, k_2)$$

$$\frac{\mathcal{D}_1 \quad \mathcal{D}_{i_1}}{\alpha_1 \bar{=}_{\mathcal{Q}} \alpha_2 \quad \frac{-\varphi_i(\alpha_1)}{-\varphi_i(\alpha_2)}} \quad (\bar{=}_{\mathcal{Q}}\text{E}_i1) \qquad \frac{\mathcal{D}_1 \quad \mathcal{D}_{i_2}}{\alpha_1 \bar{=}_{\mathcal{Q}} \alpha_2 \quad \frac{-\varphi_i(\alpha_2)}{-\varphi_i(\alpha_1)}} \quad (\bar{=}_{\mathcal{Q}}\text{E}_i2)$$

where $\varphi_i \in \mathcal{Q}$, $i \in \{1, \dots, k\}$, and $\varphi_i(\alpha_1)$ and $\varphi_i(\alpha_2)$ are mirror atomic sentences.

Remark 3.6. In contrast to the standard I-rules for identity, the I-rules for qualified identity allow one to introduce formulae in which the identity predicate is not necessarily flanked by two occurrences of the same constant.

DEFINITION 3.7. It will sometimes be convenient to use the notation $\{\mathcal{D}\}$ for the set of the subderivations $\mathcal{D}_{2_1}, \mathcal{D}_{2_2}, \dots, \mathcal{D}_{k_1}, \mathcal{D}_{k_2}$ in applications of I-rules for qualified identity.

Remark 3.8. The I-rules for qualified identity reflect the definitions of the qualified identity predicates stated in Definition 2.2(3). They absorb the logical operators at work in the definienda into the metalanguage. As mentioned, second-order quantifiers are not among these operators. It can be argued that the rule ($=\text{I}$) proposed in [14] and its refined version ($=\text{I}'$) proposed in [15] do indeed reflect the second-order definition of identity

mentioned in Remark 2.5, that is, $a = b =_{def} \forall F(Fa \leftrightarrow Fb)$. Read’s rules, with F a predicate variable that ranges over monadic (*sic!*) predicate letters ([15, p. 415, fn. 20]), are as follows:

$$\frac{[Fa]}{\mathcal{D}_1} \quad \frac{[Fa] \quad [Fb]}{\mathcal{D}_1 \quad \mathcal{D}_2} \\ \frac{Fb}{a = b} (=I) \quad \frac{Fb \quad Fa}{a = b} (=I')$$

Side condition on (=I):

“provided ‘ F ’ does not occur (as a predicate variable) in any assumption other than Fa ” ([14, p. 116]).

Side condition on (=I’):

“where the predicate variable ‘ F ’ does not occur in any parametric assumptions” ([15, p. 415]).

Read’s rules involve no second-order universal quantifier in their body. However, their side conditions are reminiscent of the usual condition on $\forall I$, except for dealing with predicate variables. In this way, the second-order universal quantifier of the aforementioned definition appears to be absorbed into the side conditions. In [14], Read argues that $(=I')$ can be simplified to $(=I)$. We note that, from a foundational intuitionistic perspective, Read’s argument is not acceptable as it appeals to classical *reductio ad absurdum* (cf. [14, p. 116]). In [15], Read adopts general elimination rules for $=$ and argues in favour of $(=I')$.

3.1.3. Bipredicational subatomic natural deduction systems

DEFINITION 3.9 (Derivations in $\mathbf{IO}(\mathcal{S}_b^=)$ -systems).

Basic step. Any derivation in an $\mathcal{S}_b^=$ -system and any formula A (i.e., a derivation from the open assumption of A) is a derivation in an $\mathbf{IO}(\mathcal{S}_b^=)$ -system.

Induction step. If \mathcal{D}_1 , \mathcal{D}_2 , and \mathcal{D}_3 are derivations in an $\mathbf{IO}(\mathcal{S}_b^=)$ -system, and C possibly a term assumption, then a derivation in an $\mathbf{IO}(\mathcal{S}_b^=)$ -system can be constructed by means of the rules:

$$\begin{array}{c}
\begin{array}{ccc}
\mathcal{D}_1 & \mathcal{D}_2 & \\
\frac{A}{A \& B} & \frac{B}{A \& B} & (\&I) \\
\mathcal{D}_1 & & \\
\frac{A \& B}{A} & & (\&E1) \\
\mathcal{D}_1 & & \\
\frac{A \& B}{B} & & (\&E2)
\end{array} \\
\\
\begin{array}{cc}
\mathcal{D}_1 & \mathcal{D}_1 \\
\frac{A}{A \vee B} & \frac{B}{A \vee B} \\
(\vee I1) & (\vee I2)
\end{array} \\
\\
\begin{array}{ccc}
\mathcal{D}_1 & \mathcal{D}_2 & \mathcal{D}_3 \\
\frac{A \vee B}{C} & \frac{[A]^{(u)}}{C} & \frac{[B]^{(v)}}{C} \\
(\vee E), u, v & & \\
\mathcal{D}_1 & & \\
\frac{B}{A \supset B} & & (\supset I), u \\
\mathcal{D}_1 & \mathcal{D}_2 & \\
\frac{A \supset B}{B} & \frac{A}{B} & (\supset E)
\end{array} \\
\\
\begin{array}{ccc}
\mathcal{D}_1 & \mathcal{D}_1 & \mathcal{D}_1 \\
\frac{A(x/o)}{\forall x A} & \frac{A(x/o)}{\forall x A} & \frac{A(x/o)}{\exists x A} \\
(\forall I) & (\forall E) & (\exists I) \\
\mathcal{D}_1 & \mathcal{D}_2 & \\
\frac{\exists x A}{C} & \frac{[A(x/o)]^{(u)}}{C} & (\exists E), u
\end{array} \\
\\
\begin{array}{c}
\mathcal{D}_1 \\
\frac{\perp}{A} \\
(\perp i)
\end{array}
\end{array}$$

Side conditions:

1. In $\forall I$: (i) if o is a proper variable y , then $o \equiv x$ or o is not free in A , and o is not free in any assumption of a formula which is open in the derivation of $A(x/o)$; (ii) if o is a nominal constant, then o does neither occur in an undischarged assumption of a formula, nor in $\forall x A$, nor in a term assumption leaf $o\Gamma$; (iii) o is nominal constant and $\frac{\mathcal{D}_1}{A(x/o)}$ for all $o \in \mathcal{C}$.
2. In $\forall E$: o is free for x in A .
3. In $\exists E$: (i) if o is a proper variable y , then $o \equiv x$ or o is not free in A , and o is not free in C nor in any assumption of a formula which is open in the derivation of the upper occurrence of C other than

$[A(x/o)]^{(u)}$; (ii) if o is a nominal constant, then o does neither occur in an undischarged assumption of a formula, nor in $\exists xA$, nor in C , nor in a term assumption leaf $o\Gamma$.

4. In $\exists\text{I}$: o is free for x in A .

Minimal bipredicational subatomic natural deduction systems, $\mathbf{MO}(\mathcal{S}_b^-)$ -systems, result from $\mathbf{IO}(\mathcal{S}_b^-)$ -systems, in case $\perp\text{i}$ is removed.

In case we employ the $\forall\text{I}$ -rule according to the provisos for it given in (i) [(ii), (iii)], we use the labels $\forall\text{I.i}$ [$\forall\text{I.ii}$, $\forall\text{I.iii}$]. Similarly, for the $\exists\text{E}$ -rule and the labels $\exists\text{E.i}$ and $\exists\text{E.ii}$.

We mention the main results obtained for $\mathbf{IO}(\mathcal{S}_b^-)$ -systems in [24] making use of the methods developed in [13]; see also [21].

THEOREM 3.10 (Normalization). *Any derivation \mathcal{D} in an $\mathbf{IO}(\mathcal{S}_b^-)$ -system can be transformed into a normal $\mathbf{IO}(\mathcal{S}_b^-)$ -derivation.*

Importantly, $\mathbf{IO}(\mathcal{S}_b^-)$ -systems enjoy the subformula property as a special case of the subexpression property. The latter property deals with units and expressions. Roughly, a unit is either a formula or a term assumption $\tau\Gamma$, and an expression is either a formula or the non-logical constant τ of $\tau\Gamma$.

THEOREM 3.11 (Subexpression property). *If \mathcal{D} is a normal derivation of a unit U from a set of units Γ in an $\mathbf{IO}(\mathcal{S}_b^-)$ -system, then each unit in \mathcal{D} is a subexpression of an expression in $\Gamma \cup \{U\}$.*

COROLLARY 3.12 (Subformula property). *If \mathcal{D} is a normal $\mathbf{IO}(\mathcal{S}_b^-)$ -derivation of formula A from a set of formulae Γ , then each formula in \mathcal{D} is a subformula of a formula in $\Gamma \cup \{A\}$.*

These results guarantee, e.g., the consistency of the systems and simplify proof search in them.

Remark 3.13. Digression: $\mathbf{IO}(\mathcal{S}_b^-)$ -systems are special cases of the $\mathbf{I}(\mathcal{S}_b^-)$ -systems studied in [24]. The ‘**O**’ indicates the lack of rules which handle predication conflicts (c -rules; cf. [24]: 116). The terminology of negative containment (Definition 3.1) introduced in [25] becomes relevant, in case c -rules are present in an $\mathbf{I}(\mathcal{S}_b^-)$ -system. For such systems the following

formulation of the subexpression property is more adequate than the one used in [24]: ‘(...) then each unit in \mathcal{D} is a subexpression of an expression in $\Gamma \cup \{U\}$ or negatively contained’. Similarly, for the formulation of the subformula property: ‘(...) then each formula in \mathcal{D} is a subformula of a formula in $\Gamma \cup \{A\}$ or negatively contained’. The original formulation which treats *as*-formulae as “subexpressions” of term assumptions for the non-logical constants in U (resp. A) constitutes an abuse of terminology.

3.2. Bipredicational natural deduction for qualified definiteness

We now add rules for the introduction and elimination of qualified definiteness to $\mathbf{IO}(\mathcal{S}_b^-)$ -systems in order to obtain $\mathbf{IO}(\mathcal{S}_b^-)_{\iota}$ -systems which are sufficient to define a proof-theoretic semantics for the simplest possible constructions involving definite descriptions.

DEFINITION 3.14. Let $\mathcal{Q} \subseteq \mathcal{P}$. In the $\iota_{\mathcal{Q}}\mathbf{I}$ -rule below, the conclusion of \mathcal{D}_1 [\mathcal{D}_2 , \mathcal{D}_3] corresponds to the E- [QU-, P-] clause. Likewise for $\iota_{\mathcal{Q}}\mathbf{I}$.

1. *Rules for positive qualified definiteness:*

$$\frac{\mathcal{D}_1 \quad \exists x\varphi(x) \quad \mathcal{D}_2 \quad \forall u\forall v((\varphi(u) \ \& \ \varphi(v)) \supset u \stackrel{\pm}{=}_{\mathcal{Q}} v) \quad \mathcal{D}_3 \quad \forall w(\varphi(w) \supset \psi(w))}{\psi(\iota_{\mathcal{Q}}x\varphi(x))} (\iota_{\mathcal{Q}}\mathbf{I})$$

$$\frac{\mathcal{D}_1 \quad \psi(\iota_{\mathcal{Q}}x\varphi(x))}{\exists x\varphi(x)} (\iota_{\mathcal{Q}}\mathbf{E1}) \quad \frac{\mathcal{D}_1 \quad \psi(\iota_{\mathcal{Q}}x\varphi(x))}{\forall u\forall v((\varphi(u) \ \& \ \varphi(v)) \supset u \stackrel{\pm}{=}_{\mathcal{Q}} v)} (\iota_{\mathcal{Q}}\mathbf{E2})$$

$$\frac{\mathcal{D}_1 \quad \psi(\iota_{\mathcal{Q}}x\varphi(x))}{\forall w(\varphi(w) \supset \psi(w))} (\iota_{\mathcal{Q}}\mathbf{E3})$$

The $\iota_{\mathcal{Q}}\mathbf{I}/\mathbf{E}$ -rules for $-\psi(\iota_{\mathcal{Q}}x\varphi(x))$ are analogous.

2. Rules for negative qualified definiteness:

$$\begin{array}{c}
 \mathcal{D}_1 \qquad \qquad \qquad \mathcal{D}_2 \qquad \qquad \qquad \mathcal{D}_3 \\
 \frac{\exists x - \varphi(x) \quad \forall u \forall v ((-\varphi(u) \ \& \ -\varphi(v)) \supset u \bar{=}_{\mathcal{Q}} v) \quad \forall w (-\varphi(w) \supset \psi(w))}{\psi(\iota_{\mathcal{Q}}x - \varphi(x))} \ (\iota_{\mathcal{Q}}-I) \\
 \\
 \mathcal{D}_1 \qquad \qquad \qquad \mathcal{D}_1 \\
 \frac{\psi(\iota_{\mathcal{Q}}x - \varphi(x))}{\exists x - \varphi(x)} \ (\iota_{\mathcal{Q}}-E1) \quad \frac{\psi(\iota_{\mathcal{Q}}x - \varphi(x))}{\forall u \forall v ((-\varphi(u) \ \& \ -\varphi(v)) \supset u \bar{=}_{\mathcal{Q}} v)} \ (\iota_{\mathcal{Q}}-E2) \\
 \\
 \mathcal{D}_1 \\
 \frac{\psi(\iota_{\mathcal{Q}}x - \varphi(x))}{\forall w (-\varphi(w) \supset \psi(w))} \ (\iota_{\mathcal{Q}}-E3)
 \end{array}$$

The $\iota_{\mathcal{Q}}-I/E$ -rules for $-\psi(\iota_{\mathcal{Q}}x - \varphi(x))$ are analogous.

Example 3.15. Let $\mathcal{Q} = \{\varphi_1, \dots, \varphi_k\}$, $\mathcal{Q} \subseteq \mathcal{P}$, and $\varphi_i, \varphi_j \in \mathcal{Q}$, where $i, j \in \{1, \dots, k\}$ and $i \neq j$.

$$\mathcal{D}_1 = \frac{\varphi_i \Gamma \quad \dots \quad \alpha \Gamma}{\exists x \varphi_i(x)} \quad (3.1)$$

$$\begin{array}{c}
 \frac{\frac{[\varphi_1(\alpha)]^{(1_1)}}{\varphi_1 \Gamma} \quad \dots \quad \frac{[\varphi_i(\alpha) \ \& \ \varphi_i(\beta)]^{(1)}}{\beta \Gamma}}{\varphi_1(\beta)} \quad \frac{[\varphi_1(\beta)]^{(1_2)}}{\varphi_1 \Gamma} \quad \dots \quad \frac{[\varphi_i(\alpha) \ \& \ \varphi_i(\beta)]^{(1)}}{\alpha \Gamma}}{\varphi_1(\alpha)} \quad \frac{\{\mathcal{D}\}}{1_1, \dots, k_2} \\
 \frac{\alpha \bar{=}_{\mathcal{Q}} \beta}{(\varphi_i(\alpha) \ \& \ \varphi_i(\beta)) \supset \alpha \bar{=}_{\mathcal{Q}} \beta} \quad 1 \\
 \frac{\forall v ((\varphi_i(\alpha) \ \& \ \varphi_i(v)) \supset \alpha \bar{=}_{\mathcal{Q}} v)}{\forall u \forall v ((\varphi_i(u) \ \& \ \varphi_i(v)) \supset u \bar{=}_{\mathcal{Q}} v)} \quad \text{iii} \\
 \mathcal{D}_2 = \frac{\quad \text{iii}}{\quad \text{iii}} \quad (3.2)
 \end{array}$$

$$\mathcal{D}_3 = \frac{\frac{\varphi_j \Gamma \quad \dots \quad \frac{[\varphi_i(\alpha)]^{(2)}}{\alpha \Gamma}}{\varphi_j(\alpha)} \quad \frac{\varphi_i(\alpha) \supset \varphi_j(\alpha)}{2}}{\forall w(\varphi_i(w) \supset \varphi_j(w))} \text{iii} \quad (3.3)$$

$$\frac{\mathcal{D}_1 \quad \forall u \forall v((\varphi_i(u) \& \varphi_i(v)) \supset u \stackrel{\pm}{\mathcal{Q}} v) \quad \mathcal{D}_2 \quad \forall w(\varphi_i(w) \supset \varphi_j(w)) \quad \mathcal{D}_3}{\varphi_j(\iota_{\mathcal{Q}} x \varphi_i(x))} (\iota_{\mathcal{Q}} \text{I}) \quad (3.4)$$

3.3. Normalization and the subformula property

In order to prove normalization for $\mathbf{IO}(\mathcal{S}_b^=)_{\iota}$ -systems, we make use of the following conversions.

DEFINITION 3.16. The *conversions (detour, permutation, simplification)* for $\mathbf{IO}(\mathcal{S}_b^=)_{\iota}$ -systems comprise those for $\mathbf{IO}(\mathcal{S}_b^=)$ -systems (see [24]) and the following detour conversions:

1. $\iota_{\mathcal{Q}}$ -Conversions:

$$\frac{\mathcal{D}_1 \quad \forall u \forall v((\varphi(u) \& \varphi(v)) \supset u \stackrel{\pm}{\mathcal{Q}} v) \quad \mathcal{D}_2 \quad \forall w(\varphi(w) \supset \psi(w))}{\frac{\psi(\iota_{\mathcal{Q}} x \varphi(x))}{\exists x \varphi(x)} (\iota_{\mathcal{Q}} \text{E1})} (\iota_{\mathcal{Q}} \text{I})$$

conv

$$\mathcal{D}_1 \\ \exists x \varphi(x)$$

$$\frac{\mathcal{D}_1 \quad \forall u \forall v((\varphi(u) \& \varphi(v)) \supset u \stackrel{\pm}{\mathcal{Q}} v) \quad \mathcal{D}_2 \quad \forall w(\varphi(w) \supset \psi(w))}{\frac{\psi(\iota_{\mathcal{Q}} x \varphi(x))}{\forall u \forall v((\varphi(u) \& \varphi(v)) \supset u \stackrel{\pm}{\mathcal{Q}} v)} (\iota_{\mathcal{Q}} \text{E2})} (\iota_{\mathcal{Q}} \text{I})$$

$$\begin{array}{c}
 \text{conv} \\
 \mathcal{D}_2 \\
 \forall u \forall v ((\varphi(u) \ \& \ \varphi(v)) \supset u \stackrel{\pm}{\mathcal{Q}} v) \\
 \\
 \frac{\begin{array}{ccc}
 \mathcal{D}_1 & \mathcal{D}_2 & \mathcal{D}_3 \\
 \exists x \varphi(x) & \forall u \forall v ((\varphi(u) \ \& \ \varphi(v)) \supset u \stackrel{\pm}{\mathcal{Q}} v) & \forall w (\varphi(w) \supset \psi(w))
 \end{array}}{\frac{\psi(\iota_{\mathcal{Q}} x \varphi(x))}{\forall w (\varphi(w) \supset \psi(w))} \ (\iota_{\mathcal{Q}} \text{E3})} \ (\iota_{\mathcal{Q}} \text{I}) \\
 \text{conv} \\
 \mathcal{D}_3 \\
 \forall w (\varphi(w) \supset \psi(w))
 \end{array}$$

2. $\iota_{\mathcal{Q}}$ -Conversions: analogous.

Remark 3.17. Unlike the ιE2 -rules in [3, 4], the above E2 -rules have a single premiss and invert directly.

THEOREM 3.18 (Normalization). *Any derivation \mathcal{D} in an $\mathbf{IO}(\mathcal{S}_b^=)$ -system can be transformed into a normal $\mathbf{IO}(\mathcal{S}_b^=)$ -derivation.*

PROOF: We repeat the corresponding proof for $\mathbf{IO}(\mathcal{S}_b^=)$ -systems in [24], taking also the detour conversions for qualified definiteness into account. As a result, all detours can be eliminated from derivations in these systems. \square

THEOREM 3.19 (Subexpression property). *If \mathcal{D} is a normal derivation of a unit U from a set of units Γ in an $\mathbf{IO}(\mathcal{S}_b^=)$ -system, then each unit in \mathcal{D} is a subexpression of an expression in $\Gamma \cup \{U\}$.*

PROOF: We proceed like in the corresponding proof for $\mathbf{IO}(\mathcal{S}_b^=)$ -systems in [24]. As a result, all expressions in \mathcal{D} are subexpressions of either the root or the leaves of \mathcal{D} . \square

COROLLARY 3.20 (Subformula property). *If \mathcal{D} is a normal $\mathbf{IO}(\mathcal{S}_b^=)$ -derivation of formula A from a set of formulae Γ , then each formula in \mathcal{D} is a subformula of a formula in $\Gamma \cup \{A\}$.*

Remark 3.21. Since the identity predicates used in the proof systems [3, 4], are primitive, such a subformula result is not available for these systems. This remark also applies to other available intuitionistic natural deduction systems for definiteness (e.g., [11, 19]).

COROLLARY 3.22 (Internal completeness). Internal completeness of $\mathbf{IO}(\mathcal{S}_b^=)\iota$ -systems in the sense of [6] (adapted to natural deduction) is given by Corollary 3.20.² To establish internal completeness for them in the sense of [24, p. 127], we proceed like described therein.

4. A proof-theoretic semantics

On the basis of the results obtained, we may formulate a subatomic proof-theoretic semantics for qualified definiteness. For this purpose, we adjust the corresponding definitions from [24] to the present systems.

DEFINITION 4.1.

1. A derivation \mathcal{D} of a formula A in an $\mathbf{IO}(\mathcal{S}_b^=)\iota$ -system is a *canonical derivation* iff it derives A by means of an application of an I-rule (in the last step of \mathcal{D}).
2. A canonical derivation \mathcal{D} of A in an $\mathbf{IO}(\mathcal{S}_b^=)\iota$ -system is a *canonical proof* of A in that system iff there are no applications of *as*-rules or *–as*-rules in \mathcal{D} and all assumptions of \mathcal{D} have been discharged.
3. The conclusions of canonical $\mathbf{IO}(\mathcal{S}_b^=)\iota$ -derivations are $\mathbf{IO}(\mathcal{S}_b^=)\iota$ -theses and the conclusions of $\mathbf{IO}(\mathcal{S}_b^=)\iota$ -derivations which are also proofs are $\mathbf{IO}(\mathcal{S}_b^=)\iota$ -theorems.

²Cf. Girard ([6, pp. 139–140]): “If we consider cut-free proofs, then all possible proofs are already there, there is no way to produce new ones. In other terms, the calculus is complete—nothing is missing. Observe that this completeness does not refer to any sort of model, it is an internal property of syntax. Such a property cannot be an accident, it should be given its real place, the first: *The subformula property is the actual completeness.*”

DEFINITION 4.2 (Meaning). Let I be an $\mathbf{IO}(\mathcal{S}_b^-)_\iota$ -system.

1. The meaning of a *non-logical constant* τ is given by the term assumptions $\tau\Gamma$ for τ which are determined by the subatomic base of the \mathcal{S}_b^- -system of I .
2. The meaning of a *formula* A of \mathcal{L}_ι is given by the set of canonical derivations of A in I .

Remark 4.3. The rules for qualified identity (Definition 3.5) allow not only for reductions in terms of detour conversions, but also for expansions (cf. [22, p. 256]). This is a further point, in which they differ from the standard natural deduction rules for identity (cf. [24, p. 104]). For an overview of the structural proof theory of identity see [9].

Remark 4.4. Note that this formal account of meaning does not make use of a semantic ontology (e.g., individuals, possible worlds), something essential to model-theoretic semantics. Specifically, the meaning of $\exists xA$ reads: ‘For at least one x , A ’, where x is a nominal variable ranging over \mathcal{C} . This feature of the present semantics makes it particularly natural for the analysis of constructions which involve non-denoting (or empty) terms (e.g., ‘Pegasus’, ‘the captive unicorn’).

5. On incomplete descriptions

Qualified uniqueness allows for fine-tuning.

Remark 5.1. Let $\{\varphi_i\} \subset \mathcal{Q}' \subset \mathcal{P}$ and $\varphi_i \in \mathcal{P}$, where $i \in \{1, \dots, k\}$. We consider the following cases: (i) $\mathcal{Q} = \mathcal{P}$, (ii) $\mathcal{Q} = \mathcal{Q}'$, and (iii) $\mathcal{Q} = \{\varphi_i\}$.

Case (i): Like (3.2), but with \mathcal{Q} replaced by \mathcal{P} . This case gives us the maximal degree of qualified uniqueness. For every x and y , if they are φ_i , then they are identical with respect to every predicate (i.e., they are indiscernible in every respect).

Case (ii): Like case (i), but with \mathcal{P} replaced by \mathcal{Q}' and with $\{\mathcal{D}\}$ replaced by $\{\mathcal{D}'\}$, where $\{\mathcal{D}'\}' \subset \{\mathcal{D}\}$. This case gives us an intermediate

degree of qualified uniqueness. For every x and y , if they are φ_i , then they are identical with respect to every predicate in \mathcal{Q}' (i.e., they are indiscernible with respect to \mathcal{Q}' , but discernible with respect to $\mathcal{P} \setminus \mathcal{Q}'$).

Case (iii):

$$\begin{array}{c}
 \frac{\frac{[\varphi_i(\alpha)]^{(1_1)}}{\varphi_i\Gamma} \quad \dots \quad \frac{\frac{[\varphi_i(\alpha)\&\varphi_i(\beta)]^{(1)}}{\varphi_i(\beta)}}{\beta\Gamma}}{\varphi_i(\beta)} \quad \frac{[\varphi_i(\beta)]^{(1_2)}}{\varphi_i\Gamma} \quad \dots \quad \frac{\frac{[\varphi_i(\alpha)\&\varphi_i(\beta)]^{(1)}}{\varphi_i(\alpha)}}{\alpha\Gamma}}{\varphi_i(\alpha)} \quad 1_{1,2} \\
 \frac{\alpha \stackrel{\pm}{=}_{\{\varphi_i\}} \beta}{(\varphi_i(\alpha)\&\varphi_i(\beta)) \supset \alpha \stackrel{\pm}{=}_{\{\varphi_i\}} \beta} \quad 1 \\
 \frac{\forall y((\varphi_i(\alpha)\&\varphi_i(y)) \supset \alpha \stackrel{\pm}{=}_{\{\varphi_i\}} y)}{\forall x\forall y((\varphi_i(x)\&\varphi_i(y)) \supset x \stackrel{\pm}{=}_{\{\varphi_i\}} y)} \quad \text{iii} \\
 \text{iii}
 \end{array} \tag{5.1}$$

This case gives us the minimal degree of qualified uniqueness. For every x and y , if they are φ_i , then they are identical with respect to every predicate in the singleton $\{\varphi_i\}$ (i.e., they are indiscernible with respect to the predicate φ_i , but discernible with respect to at least one predicate in $\mathcal{P} \setminus \{\varphi_i\}$). (Likewise for negative qualified uniqueness.)

Qualified definiteness allows for fine-tuning, since it involves qualified uniqueness.

Remark 5.2. Let $\{\varphi_i\} \subset \mathcal{Q}' \subset \mathcal{P}$, let $P = \varphi_i$, and $B = \varphi_j$ for $\varphi_i, \varphi_j \in \mathcal{Q}'$, where $i, j \in \{1, \dots, k\}$ and $i \neq j$. P : ‘... is a pope’; B : ‘... is bald’. And let $\mathcal{D}_2(i)$ [$\mathcal{D}_2(ii)$, $\mathcal{D}_2(iii)$] refer to the derivation for case (i) [(ii), (iii)] mentioned in the previous remark. We may, then, distinguish three general cases of qualified definiteness.

Case (i). Maximal qualified definiteness:

$$\frac{\mathcal{D}_1 \quad \frac{\mathcal{D}_2(i) \quad \forall u\forall v((\varphi_i(u)\&\varphi_i(v)) \supset u \stackrel{\pm}{=}_{\mathcal{P}} v)}{\varphi_j(\iota_{\mathcal{P}}x\varphi_i(x))} \quad \mathcal{D}_3 \quad \forall w(\varphi_i(w) \supset \varphi_j(w))}{\varphi_j(\iota_{\mathcal{P}}x\varphi_i(x))} \quad (\iota_{\mathcal{P}}I) \tag{5.2}$$

The premisses of the $\iota_{\mathcal{P}}$ I-application say that there is at least one thing which is φ_i , that any two things which are φ_i are the same in any respect, and that everything that is φ_i is φ_j . The conclusion $\varphi_j(\iota_{\mathcal{P}}x\varphi_i(x))$ can be read: ‘the \mathcal{P} -unique x which is φ_i is φ_j ’, or, simplifying the reading of Definition 2.9(1) further, ‘the only φ_i is φ_j ’. We may use these readings only in case there is a single x that is φ_i . This is definiteness proper. We use it for the analysis of (1), in case there is no schism.

Case (ii). Intermediate qualified definiteness:

$$\frac{\mathcal{D}_1 \quad \mathcal{D}_{2(ii)} \quad \mathcal{D}_3}{\frac{\exists x\varphi_i(x) \quad \forall u\forall v((\varphi_i(u)\&\varphi_i(v)) \supset u \stackrel{\pm}{=}_{\mathcal{Q}'} v) \quad \forall w(\varphi_i(w) \supset \varphi_j(w))}{\varphi_j(\iota_{\mathcal{Q}'}x\varphi_i(x))} (\iota_{\mathcal{Q}'}\text{I})} \quad (5.3)$$

The premisses of the $\iota_{\mathcal{Q}'}$ I-application say that there is at least one thing which is φ_i , that any two things which are φ_i are the same (only) in any \mathcal{Q}' -respect, and that everything that is φ_i is φ_j . The conclusion $\varphi_j(\iota_{\mathcal{Q}'}x\varphi_i(x))$ can be read: ‘the \mathcal{Q}' -unique x which is φ_i is φ_j ’, or simply ‘the φ_i is φ_j ’. We may use these readings only in case there are at least two things that are φ_i which are discernible with respect to $\mathcal{P} \setminus \mathcal{Q}'$. It will be natural to use this restricted kind of definiteness for the analysis of (1) in times of schism.

Case (iii). Minimal qualified definiteness:

$$\frac{\mathcal{D}_1 \quad \mathcal{D}_{2(iii)} \quad \mathcal{D}_3}{\frac{\exists x\varphi_i(x) \quad \forall u\forall v((\varphi_i(u)\&\varphi_i(v)) \supset u \stackrel{\pm}{=}_{\{\varphi_i\}} v) \quad \forall w(\varphi_i(w) \supset \varphi_j(w))}{\varphi_j(\iota_{\{\varphi_i\}}x\varphi_i(x))} (\iota_{\{\varphi_i\}}\text{I})} \quad (5.4)$$

The premisses of the $\iota_{\{\varphi_i\}}$ I-application say that there is at least one thing which is φ_i , that any two things which are φ_i are the same only with respect to $\{\varphi_i\}$, and that everything that is φ_i is φ_j . The conclusion $\varphi_j(\iota_{\{\varphi_i\}}x\varphi_i(x))$ can be read: ‘the $\{\varphi_i\}$ -unique x which is φ_i is φ_j ’. We may use this reading only in case there are at least two things that are φ_i which are discernible with respect to at least one predicate in $\mathcal{P} \setminus \{\varphi_i\}$. In a sense, this minimal degree of definiteness (a special case of restricted definiteness; cf. Definition 2.9(2)) comes close to generic definiteness: ‘the

generic φ_i is φ_j '. Similarly for negative qualified definiteness.

Example 5.3. It is straightforward to construct canonical derivations for simple constructions such as those given below. For reasons of convenience, the predicates in the symbolizations are written out.

- (1) The pope is bald. $Bald(\iota_{\mathcal{P}}xPope(x))$
- (3) The king of France is not real. $-Real(\iota_{\mathcal{P}}x(King-of^2(x, France)))$
- (4) The bishop is bald. $Bald(\iota_{\mathcal{Q}}xBishop(x))$
- (5) The Englishman is brave. $Brave(\iota_{\{Englishman\}}x(Englishman(x)))$
- (6) The non-smoker is healthy. $Healthy(\iota_{\{Smoker\}}x(-Smoker(x)))$

Concerning (3): cf. Remark 4.4. The symbolization of (6) in terms of negative predication is not entirely direct. The use of subatomic negation (cf. [23]) should be more adequate here.

6. Generalizations

Building on [3, 4], we now generalize the $\mathbf{IO}(\mathcal{S}_b^-)\iota$ -systems for qualified definiteness described above ($\iota_{\mathcal{Q}}$ -systems, for short) in order to obtain proof systems which are suitable for the analysis of constructions such as, e.g., (2) and:

- (7) The dog descends from the wolf. (Cf. [12, (33)].)
- (8) The pope puts the zucchetto on the zucchetto. (Cf. [12, (38)].)
- (9) The king of the jungle loves the queen of the desert.
- (10) Leo XIV is the bishop of Rome. (Cf. [12, (61)].)
- (11) The rabbit in the box looks at the rabbit in the hat. (Cf. [8, p. 661].)
- (12) The man wearing the beret with the button is French. ([10, p. 450].)

- (13) The man wearing the beret and carrying the newspaper is French. ([10, p. 451].)
- (14) The man wearing the beret and carrying the newspaper walks his dog.

We proceed in three generalization steps.

6.1. Generalization A: Parallel qualified definiteness

First, we turn the rules for parallel definiteness from [3, 4] into rules for parallel qualified definiteness. These rules will allow us to analyse constructions such as (2), (7), and (8).

DEFINITION 6.1. *Contextual definitions* (\mathcal{L}^A). We write $\varphi(x_1, \dots, x_m)$, suppressing the arity of φ , for atomic formulae $\varphi^{n_1 \dots n_m}$ containing (possibly multiple occurrences of) x_i , where $i \in \{1, \dots, m\}$. We generalize the contextual definitions for qualified definiteness (Definition 2.6) by replacing them with contextual definitions for parallel qualified definiteness. Let $\mathcal{Q}_1, \dots, \mathcal{Q}_n \subseteq \mathcal{P}$.

1. *Parallel positive qualified definiteness:*

$$\begin{aligned} \psi(\iota_{\mathcal{Q}_1} x_1 \varphi_1(x_1), \dots, \iota_{\mathcal{Q}_n} x_n \varphi_n(x_n)) =_{def} \\ (\exists x_1 \varphi_1(x_1) \ \& \ \dots \ \& \ \exists x_n \varphi_n(x_n)) \ \& \\ (\forall u_1 \forall v_1 ((\varphi_1(u_1) \ \& \ \varphi_1(v_1)) \supset u_1 \overset{\pm}{\mathcal{Q}_1} v_1) \ \& \ \dots \ \& \\ \forall u_n \forall v_n ((\varphi_n(u_n) \ \& \ \varphi_n(v_n)) \supset u_n \overset{\pm}{\mathcal{Q}_n} v_n)) \ \& \\ (\forall w_1 \dots \forall w_n ((\varphi_1(w_1) \ \& \ \dots \ \& \ \varphi_n(w_n)) \supset \psi(w_1, \dots, w_n))) \end{aligned}$$

2. *Parallel negative qualified definiteness:*

$$\begin{aligned} \psi(\iota_{\mathcal{Q}_1} x_1 - \varphi_1(x_1), \dots, \iota_{\mathcal{Q}_n} x_n - \varphi_n(x_n)) =_{def} \\ (\exists x_1 - \varphi_1(x_1) \ \& \ \dots \ \& \ \exists x_n - \varphi_n(x_n)) \ \& \\ (\forall u_1 \forall v_1 ((-\varphi_1(u_1) \ \& \ -\varphi_1(v_1)) \supset u_1 \overset{-}{\mathcal{Q}_1} v_1) \ \& \ \dots \ \& \\ \forall u_n \forall v_n ((-\varphi_n(u_n) \ \& \ -\varphi_n(v_n)) \supset u_n \overset{-}{\mathcal{Q}_n} v_n)) \ \& \\ (\forall w_1 \dots \forall w_n ((-\varphi_1(w_1) \ \& \ \dots \ \& \ -\varphi_n(w_n)) \supset \psi(w_1, \dots, w_n))) \end{aligned}$$

(Likewise with $-\psi$.)

Example 6.2. Symbolizations:

- (2) The pope blesses the bishop.
 $Blesses^2(\iota_{\mathcal{P}}x(Pope(x)), \iota_{\mathcal{Q}}y(Bishop(y)))$
- (7) The dog descends from the wolf. (Cf. [12, (33)].)
 $Descends-from^2(\iota_{\{Dog\}}x(Dog(x)), \iota_{\{Wolf\}}y(Wolf(y)))$
- (8) The pope puts the zucchetto on the zucchetto. (Cf. [12, (38)].)
 $Puts-on^3(\iota_{\mathcal{P}}x(Pope(x)), \iota_{\mathcal{Q}}y(Zucchetto(y)), \iota_{\mathcal{Q}'}z(Zucchetto(z)))$

Next, we generalize $\iota_{\mathcal{Q}}$ -systems by replacing the rules for qualified definiteness (Definition 3.14) with rules for parallel qualified definiteness. We call the resulting systems $\iota_{\mathcal{Q}}A$ -systems. To present the generalized rules in a more compact form we make use of abbreviations.

DEFINITION 6.3. *Abbreviations ($\iota_{\mathcal{Q}}A$ -systems).* Let $\mathcal{Q}_k \subseteq \mathcal{P}$ with $k \in \{1, \dots, n\}$.

1. *Abbreviations for parallel positive qualified definiteness:*

- (a) $E_k: \exists x_k \varphi_k(x_k)$
- (b) $QU_k: \forall u_k \forall v_k ((\varphi_k(u_k) \& \varphi_k(v_k)) \supset u_k \stackrel{\pm}{=}_{\mathcal{Q}_k} v_k)$
- (c) $P: \forall w_1 \dots \forall w_n ((\varphi_1(w_1) \& \dots \& \varphi_n(w_n)) \supset \psi(w_1, \dots, w_n))$

2. *Abbreviations for parallel negative qualified definiteness:*

- (a) $-E_k: \exists x_k -\varphi_k(x_k)$
- (b) $-QU_k: \forall u_k \forall v_k ((-\varphi_k(u_k) \& -\varphi_k(v_k)) \supset u_k \stackrel{-}{=}_{\mathcal{Q}_k} v_k)$
- (c) $-P: \forall w_1 \dots \forall w_n ((-\varphi_1(w_1) \& \dots \& -\varphi_n(w_n)) \supset \psi(w_1, \dots, w_n))$

(Likewise with $-\psi$.)

DEFINITION 6.4. *Derivations ($\iota_{\mathcal{Q}}A$ -systems).* The following rules replace the rules for qualified definiteness in Definition 3.14.

1. *Rules for parallel positive qualified definiteness:*

$$\frac{\mathcal{D}_{1_1} \quad \mathcal{D}_{1_n} \quad \mathcal{D}_{2_1} \quad \mathcal{D}_{2_n} \quad \mathcal{D}_3}{E_1 \dots E_n \quad QU_1 \dots QU_n \quad P} (\iota_{\mathcal{Q}} I_i^A)$$

$$\frac{\psi(\iota_{\mathcal{Q}_1} x_1 \varphi_1(x_1), \dots, \iota_{\mathcal{Q}_n} x_n \varphi_n(x_n))}{\psi(\iota_{\mathcal{Q}_1} x_1 \varphi_1(x_1), \dots, \iota_{\mathcal{Q}_n} x_n \varphi_n(x_n))} (\iota_{\mathcal{Q}} I_i^A)$$

$$\frac{\mathcal{D}_1}{\psi(\iota_{\mathcal{Q}_1} x_1 \varphi_1(x_1), \dots, \iota_{\mathcal{Q}_n} x_n \varphi_n(x_n))} (\iota_{\mathcal{Q}} E_k^A 1)$$

$$\frac{E_k}{\mathcal{D}_1}$$

$$\frac{\psi(\iota_{\mathcal{Q}_1} x_1 \varphi_1(x_1), \dots, \iota_{\mathcal{Q}_n} x_n \varphi_n(x_n))}{QU_k} (\iota_{\mathcal{Q}} E_k^A 2)$$

$$\frac{\mathcal{D}_1}{\psi(\iota_{\mathcal{Q}_1} x_1 \varphi_1(x_1), \dots, \iota_{\mathcal{Q}_n} x_n \varphi_n(x_n))} (\iota_{\mathcal{Q}} E_i^A 3)$$

$$\frac{P}{P}$$

where $i \in \{1, \dots, n\}$ (arity of ψ), $k \in \{1, \dots, n\}$

2. Rules for parallel negative qualified definiteness:

$$\frac{\mathcal{D}_{1_1} \quad \mathcal{D}_{1_n} \quad \mathcal{D}_{2_1} \quad \mathcal{D}_{2_n} \quad \mathcal{D}_3}{-E_1 \dots -E_n \quad -QU_1 \dots -QU_n \quad -P} (\iota_{\mathcal{Q}-} I_i^A)$$

$$\frac{\psi(\iota_{\mathcal{Q}_1} x_1 - \varphi_1(x_1), \dots, \iota_{\mathcal{Q}_n} x_n - \varphi_n(x_n))}{\psi(\iota_{\mathcal{Q}_1} x_1 - \varphi_1(x_1), \dots, \iota_{\mathcal{Q}_n} x_n - \varphi_n(x_n))} (\iota_{\mathcal{Q}-} I_i^A)$$

$$\frac{\mathcal{D}_1}{\psi(\iota_{\mathcal{Q}_1} x_1 - \varphi_1(x_1), \dots, \iota_{\mathcal{Q}_n} x_n - \varphi_n(x_n))} (\iota_{\mathcal{Q}-} E_k^A 1)$$

$$\frac{-E_k}{-E_k}$$

$$\frac{\mathcal{D}_1}{\psi(\iota_{\mathcal{Q}_1} x_1 - \varphi_1(x_1), \dots, \iota_{\mathcal{Q}_n} x_n - \varphi_n(x_n))} (\iota_{\mathcal{Q}-} E_k^A 2)$$

$$\frac{-QU_k}{-QU_k}$$

$$\frac{\mathcal{D}_1}{\psi(\iota_{\mathcal{Q}_1} x_1 - \varphi_1(x_1), \dots, \iota_{\mathcal{Q}_n} x_n - \varphi_n(x_n))} (\iota_{\mathcal{Q}-} E_i^A 3)$$

$$\frac{-P}{-P}$$

where $i \in \{1, \dots, n\}$ (arity of ψ), $k \in \{1, \dots, n\}$

(Likewise with $-\psi$.)

DEFINITION 6.5 (Detour conversions ($\iota_{\mathcal{Q}}A$ -systems)). Like Definition 3.16, except that the conversions for qualified definiteness are replaced by the following ones.

1. *Detour conversions for parallel positive qualified definiteness:*

$$\frac{\frac{\mathcal{D}_{1_1} \quad \mathcal{D}_{1_n} \quad \mathcal{D}_{2_1} \quad \mathcal{D}_{2_n} \quad \mathcal{D}_3}{E_1 \dots E_n \quad QU_1 \dots QU_n \quad P} (\iota_{\mathcal{Q}}I_i^A) \quad \text{conv} \quad \mathcal{D}_{1_k}}{\psi(\iota_{\mathcal{Q}_1}x_1\varphi_1(x_1), \dots, \iota_{\mathcal{Q}_n}x_n\varphi_n(x_n))} (\iota_{\mathcal{Q}}E_k^A 1) E_k$$

$$\frac{\frac{\mathcal{D}_{1_1} \quad \mathcal{D}_{1_n} \quad \mathcal{D}_{2_1} \quad \mathcal{D}_{2_n} \quad \mathcal{D}_3}{E_1 \dots E_n \quad QU_1 \dots QU_n \quad P} (\iota_{\mathcal{Q}}I_i^A) \quad \text{conv} \quad \mathcal{D}_{2_k}}{\psi(\iota_{\mathcal{Q}_1}x_1\varphi_1(x_1), \dots, \iota_{\mathcal{Q}_n}x_n\varphi_n(x_n))} (\iota_{\mathcal{Q}}E_k^A 2) QU_k$$

$$\frac{\frac{\mathcal{D}_{1_1} \quad \mathcal{D}_{1_n} \quad \mathcal{D}_{2_1} \quad \mathcal{D}_{2_n} \quad \mathcal{D}_3}{E_1 \dots E_n \quad QU_1 \dots QU_n \quad P} (\iota_{\mathcal{Q}}I_i^A) \quad \text{conv} \quad \mathcal{D}_3}{\psi(\iota_{\mathcal{Q}_1}x_1\varphi_1(x_1), \dots, \iota_{\mathcal{Q}_n}x_n\varphi_n(x_n))} (\iota_{\mathcal{Q}}E_i^A 3) P$$

2. *Detour conversions for parallel negative qualified definiteness: mutatis mutandis.*

Remark 6.6. Normalization, the subexpression property, the subformula property, and internal completeness for $\iota_{\mathcal{Q}}A$ -systems are obtained in a manner analogous to Theorem 3.18, Theorem 3.19, and the corollaries of the latter. Specifically, for normalization also the detour conversions for parallel qualified definiteness have to be used. These involve no complications. This makes it rather straightforward to establish the results. Meaning is then explained like in Definition 4.2.

Example 6.7. Consider (8). We use the following symbolization:

$$P^3(\iota_{\mathcal{P}}x(P^1(x)), \iota_{\mathcal{Q}'}y(Z^1(y)), \iota_{\mathcal{Q}''}z(Z^1(z)))$$

Let $\mathcal{Q}', \mathcal{Q}'' \subset \mathcal{P}$ such that $\mathcal{Q}' \neq \mathcal{Q}''$. For reasons of illustration, let the derivations for the E-clauses have the form of (3.1).

$$\begin{array}{ccc} \mathcal{D}_{1(E)} & \mathcal{D}_{2(E)} & \mathcal{D}_{3(E)} \\ \exists xP^1(x) & \exists yZ^1(y) & \exists zZ^1(z) \end{array} \tag{6.1}$$

Let the derivations for the QU-clauses have the form of (3.2).

$$\begin{aligned}
 & \mathcal{D}_{1(QU)} \\
 & \forall u_1 \forall v_1 ((P^1(u_1) \& P^1(v_1)) \supset u_1 \stackrel{\pm}{=}_{\mathcal{P}} v_1) \\
 & \mathcal{D}_{2(QU)} \\
 & \forall u_2 \forall v_2 ((Z^1(u_2) \& Z^1(v_2)) \supset u_2 \stackrel{\pm}{=}_{\mathcal{Q}'} v_2) \\
 & \mathcal{D}_{3(QU)} \\
 & \forall u_3 \forall v_3 ((Z^1(u_3) \& Z^1(v_3)) \supset u_3 \stackrel{\pm}{=}_{\mathcal{Q}''} v_3)
 \end{aligned} \tag{6.2}$$

The conclusion of $\mathcal{D}_{1(QU)}$ says that there is at most one \mathcal{P} -qualified pope, that of $\mathcal{D}_{2(QU)}$ says that there is at most one \mathcal{Q}' -qualified zucchetto, and that of $\mathcal{D}_{3(QU)}$ says that there is at most one \mathcal{Q}'' -qualified zucchetto. Accordingly, given the derivations of the E-clauses, there is a unique pope, but there is more than one zucchetto. Finally, let the derivation for the P-clause be an adjustment of (3.3).

$$\begin{aligned}
 & \mathcal{D}_{(P)} \\
 & \forall w_1 \forall w_2 \forall w_3 ((P^1(w_1) \& Z^1(w_2) \& Z^1(w_3)) \supset P^3(w_1, w_2, w_3))
 \end{aligned} \tag{6.3}$$

Let $\{\mathcal{D}_{(E)}\} = \{\mathcal{D}_{1(E)}, \mathcal{D}_{2(E)}, \mathcal{D}_{3(E)}\}$ and $\{\mathcal{D}_{(QU)}\} = \{\mathcal{D}_{1(QU)}, \mathcal{D}_{2(QU)}, \mathcal{D}_{3(QU)}\}$. We combine these derivations by means of the I-rule for parallel positive qualified definiteness into a canonical derivation:

$$\frac{\{\mathcal{D}_{(E)}\} \quad \{\mathcal{D}_{(QU)}\} \quad \mathcal{D}_{(P)}}{P^3(\iota_{\mathcal{P}}x(P^1(x)), \iota_{\mathcal{Q}'}y(Z^1(y)), \iota_{\mathcal{Q}''}z(Z^1(z)))} (\iota_{\mathcal{Q}}I_3^A) \tag{6.4}$$

6.2. Generalization B: Parallel nested qualified definiteness

In a second generalization step, we adapt the rules for nested definiteness defined in [3, 4] to parallel nested qualified definiteness. This will allow us to analyse constructions like (9) and (10).

DEFINITION 6.8 (Contextual definitions ($\mathcal{L}\iota^B$)). We generalize the definitions for parallel qualified definiteness in Definition 6.1 by replacing them with definitions for parallel nested qualified definiteness. Let $\mathcal{Q}_{11}, \dots, \mathcal{Q}_{n_m} \subseteq \mathcal{P}$.

1. *Parallel nested positive qualified definiteness:*

$$\begin{aligned}
& \psi(\iota_{\mathcal{Q}_{n_1}} x_{n_1} \varphi_{n_1}(x_{n_1}, \dots, \iota_{\mathcal{Q}_{2_1}} x_{2_1} \varphi_{2_1}(x_{2_1}, \dots, \iota_{\mathcal{Q}_{1_1}} x_{1_1} \varphi_{1_1}(x_{1_1})) \dots), \dots, \\
& \iota_{\mathcal{Q}_{n_m}} x_{n_m} \varphi_{n_m}(x_{n_m}, \dots, \iota_{\mathcal{Q}_{2_m}} x_{2_m} \varphi_{2_m}(x_{2_m}, \dots, \iota_{\mathcal{Q}_{1_m}} x_{1_m} \varphi_{1_m}(x_{1_m})) \dots)) =_{def} \\
& (\exists x_{n_1} \varphi_{n_1}(x_{n_1}, \dots, \iota_{\mathcal{Q}_{2_1}} x_{2_1} \varphi_{2_1}(x_{2_1}, \dots, \iota_{\mathcal{Q}_{1_1}} x_{1_1} \varphi_{1_1}(x_{1_1}))) \& \dots \& \\
& \exists x_{n_m} \varphi_{n_m}(x_{n_m}, \dots, \iota_{\mathcal{Q}_{2_m}} x_{2_m} \varphi_{2_m}(x_{2_m}, \dots, \iota_{\mathcal{Q}_{1_m}} x_{1_m} \varphi_{1_m}(x_{1_m})))) \& \\
& (\forall u_{n_1} \forall v_{n_1} ((\varphi_{n_1}(u_{n_1}, \dots, \iota_{\mathcal{Q}_{2_1}} x_{2_1} \varphi_{2_1}(x_{2_1}, \dots, \iota_{\mathcal{Q}_{1_1}} x_{1_1} \varphi_{1_1}(x_{1_1}))) \& \\
& \varphi_{n_1}(v_{n_1}, \dots, \iota_{\mathcal{Q}_{2_1}} x_{2_1} \varphi_{2_1}(x_{2_1}, \dots, \iota_{\mathcal{Q}_{1_1}} x_{1_1} \varphi_{1_1}(x_{1_1})))) \supset u_{n_1} \stackrel{\pm}{\mathcal{Q}_{n_1}} v_{n_1}) \& \dots \& \\
& \forall u_{n_m} \forall v_{n_m} ((\varphi_{n_m}(u_{n_m}, \dots, \iota_{\mathcal{Q}_{2_m}} x_{2_m} \varphi_{2_m}(x_{2_m}, \dots, \iota_{\mathcal{Q}_{1_m}} x_{1_m} \varphi_{1_m}(x_{1_m}))) \& \\
& \varphi_{n_m}(v_{n_m}, \dots, \iota_{\mathcal{Q}_{2_m}} x_{2_m} \varphi_{2_m}(x_{2_m}, \dots, \iota_{\mathcal{Q}_{1_m}} x_{1_m} \varphi_{1_m}(x_{1_m})))) \\
& \supset u_{n_m} \stackrel{\pm}{\mathcal{Q}_{n_m}} v_{n_m})) \& \\
& (\forall w_{n_1} \dots \forall w_{n_m} (\varphi_{n_1}(w_{n_1}, \dots, \iota_{\mathcal{Q}_{2_1}} x_{2_1} \varphi_{2_1}(x_{2_1}, \dots, \iota_{\mathcal{Q}_{1_1}} x_{1_1} \varphi_{1_1}(x_{1_1}))) \& \dots \& \\
& \varphi_{n_m}(w_{n_m}, \dots, \iota_{\mathcal{Q}_{2_m}} x_{2_m} \varphi_{2_m}(x_{2_m}, \dots, \iota_{\mathcal{Q}_{1_m}} x_{1_m} \varphi_{1_m}(x_{1_m})))) \\
& \supset \psi(w_{n_1}, \dots, w_{n_m}))
\end{aligned}$$

2. *Parallel nested negative qualified definiteness:*

$$\begin{aligned}
& \psi(\iota_{\mathcal{Q}_{n_1}} x_{n_1} - \varphi_{n_1}(x_{n_1}, \dots, \iota_{\mathcal{Q}_{2_1}} x_{2_1} - \varphi_{2_1}(x_{2_1}, \dots, \iota_{\mathcal{Q}_{1_1}} x_{1_1} - \varphi_{1_1}(x_{1_1})) \dots), \dots, \\
& \iota_{\mathcal{Q}_{n_m}} x_{n_m} - \varphi_{n_m}(x_{n_m}, \dots, \iota_{\mathcal{Q}_{2_m}} x_{2_m} - \varphi_{2_m}(x_{2_m}, \dots, \iota_{\mathcal{Q}_{1_m}} x_{1_m} - \varphi_{1_m}(x_{1_m})) \dots)) \\
& =_{def} \\
& (\exists x_{n_1} - \varphi_{n_1}(x_{n_1}, \dots, \iota_{\mathcal{Q}_{2_1}} x_{2_1} - \varphi_{2_1}(x_{2_1}, \dots, \iota_{\mathcal{Q}_{1_1}} x_{1_1} - \varphi_{1_1}(x_{1_1}))) \& \dots \& \\
& \exists x_{n_m} - \varphi_{n_m}(x_{n_m}, \dots, \iota_{\mathcal{Q}_{2_m}} x_{2_m} - \varphi_{2_m}(x_{2_m}, \dots, \iota_{\mathcal{Q}_{1_m}} x_{1_m} - \varphi_{1_m}(x_{1_m})))) \& \\
& (\forall u_{n_1} \forall v_{n_1} ((-\varphi_{n_1}(u_{n_1}, \dots, \iota_{\mathcal{Q}_{2_1}} x_{2_1} - \varphi_{2_1}(x_{2_1}, \dots, \iota_{\mathcal{Q}_{1_1}} x_{1_1} - \varphi_{1_1}(x_{1_1}))) \& \\
& -\varphi_{n_1}(v_{n_1}, \dots, \iota_{\mathcal{Q}_{2_1}} x_{2_1} - \varphi_{2_1}(x_{2_1}, \dots, \iota_{\mathcal{Q}_{1_1}} x_{1_1} - \varphi_{1_1}(x_{1_1})))) \supset u_{n_1} \stackrel{-}{\mathcal{Q}_{n_1}} v_{n_1}) \& \dots \& \\
& \forall u_{n_m} \forall v_{n_m} ((-\varphi_{n_m}(u_{n_m}, \dots, \iota_{\mathcal{Q}_{2_m}} x_{2_m} - \varphi_{2_m}(x_{2_m}, \dots, \iota_{\mathcal{Q}_{1_m}} x_{1_m} - \varphi_{1_m}(x_{1_m}))) \& \\
& -\varphi_{n_m}(v_{n_m}, \dots, \iota_{\mathcal{Q}_{2_m}} x_{2_m} - \varphi_{2_m}(x_{2_m}, \dots, \iota_{\mathcal{Q}_{1_m}} x_{1_m} - \varphi_{1_m}(x_{1_m})))) \\
& \supset u_{n_m} \stackrel{-}{\mathcal{Q}_{n_m}} v_{n_m})) \& \\
& (\forall w_{n_1} \dots \forall w_{n_m} (-\varphi_{n_1}(w_{n_1}, \dots, \iota_{\mathcal{Q}_{2_1}} x_{2_1} - \varphi_{2_1}(x_{2_1}, \dots, \iota_{\mathcal{Q}_{1_1}} x_{1_1} - \varphi_{1_1}(x_{1_1}))) \& \dots \& \\
& -\varphi_{n_m}(w_{n_m}, \dots, \iota_{\mathcal{Q}_{2_m}} x_{2_m} - \varphi_{2_m}(x_{2_m}, \dots, \iota_{\mathcal{Q}_{1_m}} x_{1_m} - \varphi_{1_m}(x_{1_m})))) \\
& \supset \psi(w_{n_1}, \dots, w_{n_m}))
\end{aligned}$$

(Likewise with $-\psi$.)

Example 6.9. Symbolizations:

- (9) The king of the jungle loves the queen of the desert.

$$\text{Loves}^2(\iota_{\mathcal{Q}_2} x(\text{King-of}^2(x, \iota_{\mathcal{Q}_1} y(\text{Jungle}(y))))), \\ \iota_{\mathcal{Q}_2} z(\text{Queen-of}^2(z, \iota_{\mathcal{Q}_1} u(\text{Desert}(u))))))$$

- (10) Leo XIV is the bishop of Rome.

$$\text{Holds}^2(\text{LeoXIV}, \iota_{\mathcal{P}} x(\text{Office-of}^2(x, \iota_{\mathcal{P}} y(\text{Bishop-of}^2(y, \text{Rome}))))))$$

The symbolization of (10) does apparently not rest on an interpretation of that sentence which takes ‘is’ to express identity (cf. [16, p. 483]). It rests on an interpretation which takes the description in (10) to be predicative. However, it does not construe that description as a predicate (cf. [7]; for overview see [12, sect. 7.1]). Rather it takes the ‘is’ to indicate a relational predicate that needs to be specified.

In order to obtain $\iota_{\mathcal{Q}}B$ -systems, we generalize $\iota_{\mathcal{Q}}A$ -systems by replacing the rules for parallel qualified definiteness (Definition 6.4) with rules for parallel nested qualified definiteness.

DEFINITION 6.10 (Abbreviations ($\iota_{\mathcal{Q}}B$ -systems)).

1. Abbreviations for parallel nested positive qualified definiteness:

- (a) *E*-abbreviations:

$\{E_{1_k}\}$:

$$\underbrace{\exists x_{1_1} \varphi_{1_1}(x_{1_1})}_{E_{1_1}}, \dots, \underbrace{\exists x_{1_m} \varphi_{1_m}(x_{1_m})}_{E_{1_m}}$$

$\{E_{2_k}\}$:

$$\underbrace{\exists x_{2_1} \varphi_{2_1}(x_{2_1}, \dots, \iota_{\mathcal{Q}_{1_1}} x_{1_1} \varphi_{1_1}(x_{1_1}))}_{E_{2_1}}, \dots,$$

$$\underbrace{\exists x_{2_m} \varphi_{2_m}(x_{2_m}, \dots, \iota_{\mathcal{Q}_{1_m}} x_{1_m} \varphi_{1_m}(x_{1_m}))}_{E_{2_m}}$$

\vdots

$\{E_{n_k}\}$:

$$\underbrace{\exists x_{n_1} \varphi_{n_1}(x_{n_1}, \dots, \iota_{Q_{2_1}} x_{2_1} \varphi_{2_1}(x_{2_1}, \dots, \iota_{Q_{1_1}} x_{1_1} \varphi_{1_1}(x_{1_1})))}_{E_{n_1}}, \dots,$$

$$\underbrace{\exists x_{n_m} \varphi_{n_m}(x_{n_m}, \dots, \iota_{Q_{2_m}} x_{2_m} \varphi_{2_m}(x_{2_m}, \dots, \iota_{Q_{1_m}} x_{1_m} \varphi_{1_m}(x_{1_m})))}_{E_{n_m}}$$

(b) *QU-abbreviations*: see Figure 1.

(c) *P-abbreviations*:

P_1 :

$$\forall w_{1_1} \dots \forall w_{1_m} ((\varphi_{1_1}(w_{1_1}) \& \dots \& \varphi_{1_m}(w_{1_m})) \supset \psi_1(w_{1_1}, \dots, w_{1_m}))$$

P_2 :

$$\forall w_{2_1} \dots \forall w_{2_m} ((\varphi_{2_1}(w_{2_1}, \dots, \iota_{Q_{1_1}} x_{1_1} \varphi_{1_1}(x_{1_1})) \& \dots \& \varphi_{2_m}(w_{2_m}, \dots, \iota_{Q_{1_m}} x_{1_m} \varphi_{1_m}(x_{1_m}))) \supset \psi_2(w_{2_1}, \dots, w_{2_m}))$$

\vdots

P_n :

$$\forall w_{n_1} \dots \forall w_{n_m} (((\varphi_{n_1}(w_{n_1}, \dots, \iota_{Q_{2_1}} x_{2_1} \varphi_{2_1}(w_{2_1}, \dots, \iota_{Q_{1_1}} x_{1_1} \varphi_{1_1}(x_{1_1}))) \& \dots \& \varphi_{n_m}(w_{n_m}, \dots, \iota_{Q_{2_m}} x_{2_m} \varphi_{2_m}(w_{2_m}, \dots, \iota_{Q_{1_m}} x_{1_m} \varphi_{1_m}(x_{1_m})))) \supset \psi_n(w_{n_1}, \dots, w_{n_m}))$$

(Likewise with $-\psi_j$.)

(d) *QD-abbreviations*:

QD_1 :

$$\psi_1(\iota_{Q_{1_1}} x_{1_1} \varphi_{1_1}(x_{1_1}), \dots, \iota_{Q_{1_m}} x_{1_m} \varphi_{1_m}(x_{1_m}))$$

QD_2 :

$$\psi_2(\iota_{Q_{2_1}} x_{2_1} \varphi_{2_1}(x_{2_1}, \dots, \iota_{Q_{1_1}} x_{1_1} \varphi_{1_1}(x_{1_1})), \dots, \iota_{Q_{2_m}} x_{2_m} \varphi_{2_m}(x_{2_m}, \dots, \iota_{Q_{1_m}} x_{1_m} \varphi_{1_m}(x_{1_m})))$$

\vdots

QD_n :

$$\psi_n(\iota_{Q_{n_1}} x_{n_1} \varphi_{n_1}(x_{n_1}, \dots, \iota_{Q_{2_1}} x_{2_1} \varphi_{2_1}(x_{2_1}, \dots, \iota_{Q_{1_1}} x_{1_1} \varphi_{1_1}(x_{1_1}))) \dots, \dots, \iota_{Q_{n_m}} x_{n_m} \varphi_{n_m}(x_{n_m}, \dots, \iota_{Q_{2_m}} x_{2_m} \varphi_{2_m}(x_{2_m}, \dots, \iota_{Q_{1_m}} x_{1_m} \varphi_{1_m}(x_{1_m}))) \dots)$$

(Likewise with $-\psi_j$.)

2. *Abbreviations for parallel nested negative qualified definiteness: mutatis mutandis.*

$$\begin{aligned}
 & \{QU_{1k}\}: \\
 & \underbrace{\forall u_{11} \forall v_{11} ((\varphi_{11}(u_{11}) \& \varphi_{11}(v_{11})) \supset u_{11} \stackrel{\pm}{=}_{Q_{11}} v_{11}), \dots, \forall u_{1_m} \forall v_{1_m} ((\varphi_{1_m}(u_{1_m}) \& \varphi_{1_m}(v_{1_m})) \supset u_{1_m} \stackrel{\pm}{=}_{Q_{1_m}} v_{1_m})}_{QU_{1,m}} \\
 & \{QU_{2k}\}: \\
 & \underbrace{\forall u_{21} \forall v_{21} ((\varphi_{21}(u_{21}, \dots, \iota_{Q_{11}} x_{11} \varphi_{11}(x_{11})) \& \varphi_{21}(v_{21}, \dots, \iota_{Q_{11}} x_{11} \varphi_{11}(x_{11}))) \supset u_{21} \stackrel{\pm}{=}_{Q_{21}} v_{21}), \dots, \forall u_{2_m} \forall v_{2_m} ((\varphi_{2_m}(u_{2_m}, \dots, \iota_{Q_{1_m}} x_{1_m} \varphi_{1_m}(x_{1_m})) \& \varphi_{2_m}(v_{2_m}, \dots, \iota_{Q_{1_m}} x_{1_m} \varphi_{1_m}(x_{1_m}))) \supset u_{2_m} \stackrel{\pm}{=}_{Q_{2_m}} v_{2_m})}_{QU_{2,m}} \\
 & \vdots \\
 & \{QU_{nk}\}: \\
 & \underbrace{\forall u_{n1} \forall v_{n1} ((\varphi_{n1}(u_{n1}, \dots, \iota_{Q_{21}} x_{21} \varphi_{21}(x_{21}, \dots, \iota_{Q_{11}} x_{11} \varphi_{11}(x_{11}))) \& \varphi_{n1}(v_{n1}, \dots, \iota_{Q_{21}} x_{21} \varphi_{21}(x_{21}, \dots, \iota_{Q_{11}} x_{11} \varphi_{11}(x_{11}))))}_{QU_{n1}} \dots \\
 & \underbrace{\iota_{Q_{11}} x_{11} \varphi_{11}(x_{11}), \dots, \iota_{Q_{n1}} x_{n1} \varphi_{n1}(x_{n1}, \dots, \iota_{Q_{2_m}} x_{2_m} \varphi_{2_m}(x_{2_m}, \dots, \iota_{Q_{1_m}} x_{1_m} \varphi_{1_m}(x_{1_m})))}_{QU_{n,m}} \dots \\
 & \underbrace{\iota_{Q_{1_m}} x_{1_m} \varphi_{1_m}(x_{1_m}), \dots, \iota_{Q_{2_m}} x_{2_m} \varphi_{2_m}(x_{2_m}, \dots, \iota_{Q_{1_m}} x_{1_m} \varphi_{1_m}(x_{1_m}))) \supset u_{nm} \stackrel{\pm}{=}_{Q_{nm}} v_{nm}}_{QU_{nm}} \dots
 \end{aligned}$$

Figure 1 : QU-Abbreviations (Generalization B)

DEFINITION 6.11 (Derivations ($\iota_{\mathcal{Q}}B$ -systems)). The following rules replace the rules for parallel qualified definiteness in Definition 6.4.

1. *Rules for parallel nested positive qualified definiteness:*

$$\frac{\begin{array}{ccc} \{\mathcal{D}_{1_1}\} & \{\mathcal{D}_{2_1}\} & \mathcal{D}_{3_1} \\ \{E_{1_k}\} & \{QU_{1_k}\} & P_1 \end{array}}{QD_1} (\iota_{\mathcal{Q}}I_{i,1}^B)$$

$$\vdots$$

$$\frac{\begin{array}{ccc} \{\mathcal{D}_{1_n}\} & & \{\mathcal{D}_{2_n}\} \quad \mathcal{D}_{3_n} \\ \{E_{n_k}\} & & \{QU_{n_k}\} \quad P_n \end{array}}{QD_n} (\iota_{\mathcal{Q}}I_{i,n}^B)$$

$$\frac{\mathcal{D}_1}{E_{jk}} (\iota_{\mathcal{Q}}E_{k,j}^B 1) \quad \frac{\mathcal{D}_1}{QU_{jk}} (\iota_{\mathcal{Q}}E_{k,j}^B 2) \quad \frac{\mathcal{D}_1}{P_j} (\iota_{\mathcal{Q}}E_{i,j}^B 3)$$

where $i \in \{1, \dots, m\}$ (arity of predicate in QD_j), $j \in \{1, \dots, n\}$ (level of nesting),
 $k \in \{1, \dots, m\}$

2. *Rules for parallel nested negative qualified definiteness:*

$$\frac{\begin{array}{ccc} \{\mathcal{D}_{1_1}\} & \{\mathcal{D}_{2_1}\} & \mathcal{D}_{3_1} \\ \{-E_{1_k}\} & \{-QU_{1_k}\} & -P_1 \end{array}}{-QD_1} (\iota_{\mathcal{Q}}-I_{i,1}^B)$$

$$\vdots$$

$$\frac{\begin{array}{ccc} \{\mathcal{D}_{1_n}\} & & \{\mathcal{D}_{2_n}\} \quad \mathcal{D}_{3_n} \\ \{-E_{n_k}\} & & \{-QU_{n_k}\} \quad -P_n \end{array}}{-QD_n} (\iota_{\mathcal{Q}}-I_{i,n}^B)$$

$$\frac{\mathcal{D}_1}{-E_{jk}} (\iota_{\mathcal{Q}}-E_{k,j}^B 1) \quad \frac{\mathcal{D}_1}{-QU_{jk}} (\iota_{\mathcal{Q}}-E_{k,j}^B 2) \quad \frac{\mathcal{D}_1}{-P_j} (\iota_{\mathcal{Q}}-E_{i,j}^B 3)$$

where $i \in \{1, \dots, m\}$ (arity of predicate in QD_j), $j \in \{1, \dots, n\}$ (level of nesting),
 $k \in \{1, \dots, m\}$

(Likewise with $-\psi_j$.)

DEFINITION 6.12 (Detour conversions ($\iota_{\mathcal{Q}}B$ -systems)).

1. *Detour conversions for parallel nested positive qualified definiteness:*

$$\begin{array}{c}
 \frac{\{\mathcal{D}_{1_1}\} \quad \{\mathcal{D}_{2_1}\} \quad \mathcal{D}_{3_1}}{\{\mathcal{E}_{1_k}\} \quad \{QU_{1_k}\} \quad P_1} (\iota_{\mathcal{Q}}I_{i,1}^B) \\
 \hline
 \begin{array}{c}
 \vdots \\
 \{\mathcal{D}_{1_n}\} \quad \quad \quad \{\mathcal{D}_{2_n}\} \quad \mathcal{D}_{3_n} \\
 \{\mathcal{E}_{n_k}\} \quad \quad \quad \{QU_{n_k}\} \quad P_n
 \end{array} \\
 \hline
 \frac{QD_n}{E_{n_k}} (\iota_{\mathcal{Q}}E_{k,n}^B \ 1) \\
 \text{conv} \\
 \frac{\{\mathcal{D}_{1_1}\} \quad \{\mathcal{D}_{2_1}\} \quad \mathcal{D}_{3_1}}{\{\mathcal{E}_{1_k}\} \quad \{QU_{1_k}\} \quad P_1} (\iota_{\mathcal{Q}}I_{i,1}^B) \\
 \hline
 \begin{array}{c}
 \vdots \\
 \mathcal{D}_{1_n} \\
 E_{n_k}
 \end{array} \\
 \\
 \frac{\{\mathcal{D}_{1_1}\} \quad \{\mathcal{D}_{2_1}\} \quad \mathcal{D}_{3_1}}{\{\mathcal{E}_{1_k}\} \quad \{QU_{1_k}\} \quad P_1} (\iota_{\mathcal{Q}}I_{i,1}^B) \\
 \hline
 \begin{array}{c}
 \vdots \\
 \{\mathcal{D}_{1_n}\} \quad \quad \quad \{\mathcal{D}_{2_n}\} \quad \mathcal{D}_{3_n} \\
 \{\mathcal{E}_{n_k}\} \quad \quad \quad \{QU_{n_k}\} \quad P_n
 \end{array} \\
 \hline
 \frac{QD_n}{QU_{n_k}} (\iota_{\mathcal{Q}}E_{k,n}^B \ 2)
 \end{array}
 \quad \text{conv} \quad \begin{array}{c} \mathcal{D}_{2_n} \\ QU_{n_k} \end{array}$$

$$\begin{array}{c}
 \frac{\{\mathcal{D}_{1_1}\} \quad \{\mathcal{D}_{2_1}\} \quad \mathcal{D}_{3_1}}{\{E_{1_k}\} \quad \{QU_{1_k}\} \quad P_1} (\iota_{\mathcal{Q}^{\text{I}^{\text{B}}}_{i,1}}) \\
 \vdots \\
 \frac{\{\mathcal{D}_{1_n}\} \quad \{\mathcal{D}_{2_n}\} \quad \mathcal{D}_{3_n}}{\{E_{n_k}\} \quad \{QU_{n_k}\} \quad P_n} (\iota_{\mathcal{Q}^{\text{I}^{\text{B}}}_{i,n}})
 \end{array}
 \text{conv}
 \begin{array}{c}
 \mathcal{D}_{3_n} \\
 P_n
 \end{array}$$

2. *Detour conversions for parallel nested negative qualified definiteness: mutatis mutandis.*

Remark 6.13. A remark analogous to Remark 6.6 applies to $\iota_{\mathcal{Q}}B$ -systems.

Example 6.14. We construct a canonical derivation for (9). We use the following symbolization:

$$L^2(\iota_{\mathcal{Q}_{2_1}}x(K^2(x, \iota_{\mathcal{Q}_{1_1}}y(J^1(y))))), \iota_{\mathcal{Q}_{2_2}}z(Q^2(z, \iota_{\mathcal{Q}_{1_2}}u(D^1(u))))$$

Let $\mathcal{Q}_{1_1}, \mathcal{Q}_{1_2}, \mathcal{Q}_{2_1}, \mathcal{Q}_{2_2} \subseteq \mathcal{P}$.

$$\begin{array}{ccc}
 \mathcal{D}_{1(E)} & \mathcal{D}_{2(QU)} & \mathcal{D}_{3(P)} \\
 \exists y J^1(y) \quad \forall u_1 \forall v_1 ((J^1(u_1) \& J^1(v_1)) \supset u_1 \stackrel{\pm}{=}_{\mathcal{Q}_{1_1}} v_1) & & \forall w_1 (J^1(w_1) \supset K^2(\alpha_1, w_1)) \\
 \\
 \mathcal{D}'_{1(E)} & \mathcal{D}'_{2(QU)} & \mathcal{D}'_{3(P)} \\
 \exists u D^1(u) \quad \forall u'_1 \forall v'_1 ((D^1(u'_1) \& D^1(v'_1)) \supset u'_1 \stackrel{\pm}{=}_{\mathcal{Q}_{1_2}} v'_1) & & \forall w'_1 (D^1(w'_1) \supset Q^2(\alpha_2, w'_1))
 \end{array}$$

The first level of parallel nesting:

$$\mathcal{D}_{4(E)} = \frac{\frac{\mathcal{D}_{1(E)} \quad \mathcal{D}_{2(QU)} \quad \mathcal{D}_{3(P)}}{K^2(\alpha_1, \iota_{\mathcal{Q}_{1_1}}y(J^1(y)))} (\iota_{\mathcal{Q}^{\text{I}^{\text{B}}}_{2,1}})}{\exists x (K^2(x, \iota_{\mathcal{Q}_{1_1}}y(J^1(y))))}$$

$$\begin{array}{c}
 \mathcal{D}_{5(QU)} \\
 \forall u_2 \forall v_2 ((K^2(u_2, \iota_{\mathcal{Q}_{1_1}}y(J^1(y))) \& K^2(v_2, \iota_{\mathcal{Q}_{1_1}}y(J^1(y)))) \supset u_2 \stackrel{\pm}{=}_{\mathcal{Q}_{2_1}} v_2)
 \end{array}$$

$$\begin{aligned}
 \mathcal{D}'_{4(E)} &= \frac{\frac{\mathcal{D}'_{1(E)} \quad \mathcal{D}'_{2(QU)} \quad \mathcal{D}'_{3(P)}}{Q^2(\alpha_2, \iota_{\mathcal{Q}_{1_2}} u(D^1(u)))} (\iota_{\mathcal{Q}} \mathbb{I}_{2,1}^B)}{\exists z(Q^2(z, \iota_{\mathcal{Q}_{1_2}} u(D^1(u))))} \\
 &\quad \mathcal{D}'_{5(QU)} \\
 &\quad \forall u'_2 \forall v'_2 ((Q^2(u'_2, \iota_{\mathcal{Q}_{1_2}} u(D^1(u))) \& Q^2(v'_2, \iota_{\mathcal{Q}_{1_2}} u(D^1(u)))) \supset u'_2 \stackrel{\pm}{\mathcal{Q}_{2_2}} v'_2)
 \end{aligned}$$

The second level of parallel nesting:

$$\begin{aligned}
 \{\mathcal{D}_{(E)}\} &= \{\mathcal{D}_{4(E)}, \mathcal{D}'_{4(E)}\}, \{\mathcal{D}_{(QU)}\} = \{\mathcal{D}_{5(QU)}, \mathcal{D}'_{5(QU)}\} \\
 &\quad \mathcal{D}_{6(P)} \\
 \forall w_2 \forall w_3 ((K^2(w_2, \iota_{\mathcal{Q}_{1_1}} y(J^1(y))) \& (Q^2(w_3, \iota_{\mathcal{Q}_{1_2}} u(D^1(u)))) \supset L^2(w_2, w_3)) \\
 &\quad \frac{\{\mathcal{D}_{(E)}\} \quad \{\mathcal{D}_{(QU)}\} \quad \mathcal{D}_{6(P)}}{L^2(\iota_{\mathcal{Q}_{2_1}} x(K^2(x, \iota_{\mathcal{Q}_{1_1}} y(J^1(y))))), \iota_{\mathcal{Q}_{2_2}} z(Q^2(z, \iota_{\mathcal{Q}_{1_2}} u(D^1(u))))))} (\iota_{\mathcal{Q}} \mathbb{I}_{2,2}^B) \tag{6.5}
 \end{aligned}$$

A canonical derivation for (10) can be constructed in a similar manner.

6.3. Generalization C: Parallel conjunctively nested qualified definiteness

Finally, we generalize the systems for parallel nested qualified definiteness so as to allow for conjunctions also in the scope of qualified definiteness operators. This will allow us to deal with constructions such as (11)–(14).

DEFINITION 6.15. *CN-formulae:*

1. A *positive* CN-formula is either
 - (a) an atomic formula,
 - (b) a parallel QD-formula, or
 - (c) a parallel QD-formula containing a conjunction of formulae of the form of (1a), (1b), (1c).

2. A *negative* CN-formula is either

- (a) an atomic negative predication,
- (b) a parallel negative QD-formula, or
- (c) a parallel negative QD-formula containing a conjunction of formulae of the form of (2a), (2b), (2c).

DEFINITION 6.16 (Contextual definitions (\mathcal{L}^c)). We generalize the definitions for parallel nested qualified definiteness in Definition 6.8 by replacing them with definitions for parallel conjunctively nested qualified definiteness. Let $\mathcal{Q}_{1_1}, \dots, \mathcal{Q}_{n_m} \subseteq \mathcal{P}$ and let $C(x_{j_k})$ (resp. $-C(x_{j_k})$) be a CN-formula (negative CN-formula) for $j \in \{1, \dots, n\}$ and $k \in \{1, \dots, m\}$.

- 1. *Parallel conjunctively nested positive qualified definiteness*: like Definition 6.8(1), but with the occurrences of ‘ φ ’ replaced by ‘ C ’.
- 2. *Parallel conjunctively nested negative qualified definiteness*: like Definition 6.8(2), but with the occurrences of ‘ $-\varphi$ ’ replaced by ‘ $-C$ ’.

(Likewise with $-\psi$.)

Example 6.17. The symbolizations of the following examples make use of conjunction in the scope of the $\iota_{\mathcal{Q}}$ -operators.

- (11) The rabbit in the box looks at the rabbit in the hat.
 $Looks-at^2(\iota_{\mathcal{Q}'}x(Rabbit(x) \ \& \ In^2(x, \iota_{\mathcal{Q}_1}y(Box(y))))),$
 $\iota_{\mathcal{Q}'}z(Rabbit(z) \ \& \ In^2(z, \iota_{\mathcal{Q}_2}u(Hat(u))))$
- (12) The man wearing the beret with the button is French.
 $French(\iota_{\mathcal{Q}_3}x(Man(x) \ \& \ Wears^2(x, \iota_{\mathcal{Q}_2}y(Beret(y) \ \& \ Has^2(y,$
 $\iota_{\mathcal{Q}_1}z(Button(z))))))$
- (13) The man wearing the beret and carrying the newspaper is French.
 $French(\iota_{\mathcal{Q}_2}x(Man(x) \ \& \ Wears^2(x, \iota_{\mathcal{Q}_1}y(Beret(y))) \ \& \ Carries^2(x,$
 $\iota_{\mathcal{Q}_1}z(Newspaper(z))))$
- (14) The man wearing the beret and carrying the newspaper walks his dog.

$$Walks^2(\iota_{Q_2}x(Man(x)\&Wears^2(x,\iota_{Q_1}y(Beret(y))) \& Carries^2(x, \iota_{Q_1}'z(Newspaper(z))))), \iota_{Q_3}u(Dog(u)\&Owns^2(\iota_{Q_2}x(Man(x)\& (Wears^2(x,\iota_{Q_1}y(Beret(y)))\&Carries^2(x,\iota_{Q_1}'z(Newspaper(z))))),u)))$$

No conjunction surfaces in (11) and (12). Moreover, (11) and (14) are parallel also on the outermost level. The latter contains the possessive construction ‘his dog’, where the possessive pronoun ‘his’ stands in for a definite description. Examples (1)–(10) are special cases of parallel conjunctively nested qualified definiteness.

Remark 6.18. Note that (11), like (8), involves multiple uses of an incomplete description (i.e., ‘the rabbit’). If our symbolization of (11) were to reflect the relevant part of the scenario depicted by Haddock (i.e., there are three rabbits, two hats, and one box, where one rabbit is in a hat and one is in the box; cf. [8, p. 661]), we might consider using the symbolization $Looks-at^2(\iota_{Q_2}x(Rabbit(x) \& In^2(x, \iota_{Q_1}y(Box(y))))), \iota_{Q_2}'z(Rabbit(z) \& In^2(z, \iota_{Q_3}u(Hat(u))))$ instead, thereby leaving room for the symbolization of a further use of ‘the rabbit’ and a further use of ‘the hat’³

DEFINITION 6.19. *Abbreviations ($\iota_Q C$ -systems).* The abbreviations are like those of $\iota_Q B$ -systems in Definition 6.10, except that the occurrences of φ (resp. $-\varphi$) are replaced by C ($-C$).

DEFINITION 6.20. *Derivations ($\iota_Q C$ -systems).* The following rules replace the rules for parallel nested qualified definiteness in Definition 6.11.

1. *Rules for parallel conjunctively nested positive qualified definiteness:*

$$\frac{\begin{array}{ccc} \{\mathcal{D}_{1_1}\} & \{\mathcal{D}_{2_1}\} & \mathcal{D}_{3_1} \\ \{E_{1_k}\} & \{QU_{1_k}\} & P_1 \end{array}}{QD_1} (\iota_Q I_{i,1}^C)$$

$$\vdots$$

$$\frac{\begin{array}{ccc} \{\mathcal{D}_{1_n}\} & \{\mathcal{D}_{2_n}\} & \mathcal{D}_{3_n} \\ \{E_{n_k}\} & \{QU_{n_k}\} & P_n \end{array}}{QD_n} (\iota_Q I_{i,n}^C)$$

³I would like to thank Jan Köpping for making me aware of Haddock’s Puzzle [8] by which (11) is inspired.

$$\frac{\mathcal{D}_1}{\frac{QD_j}{E_{jk}} (\iota_{\mathcal{Q}} \text{Ec}_{k,j} 1)} \quad \frac{\mathcal{D}_1}{\frac{QD_j}{QU_{jk}} (\iota_{\mathcal{Q}} \text{Ec}_{k,j} 2)} \quad \frac{\mathcal{D}_1}{\frac{QD_j}{P_j} (\iota_{\mathcal{Q}} \text{Ec}_{i,j} 3)}$$

where $i \in \{1, \dots, m\}$ (arity of predicate in QD_j), $j \in \{1, \dots, n\}$ (level of nesting),
 $k \in \{1, \dots, m\}$

2. *Rules for parallel conjunctively nested negative qualified definiteness:*

$$\frac{\frac{\{\mathcal{D}_{1_1}\} \quad \{\mathcal{D}_{2_1}\} \quad \mathcal{D}_{3_1}}{\{-E_{1_k}\} \quad \{-QU_{1_k}\} \quad -P_1} (\iota_{\mathcal{Q}} -\text{I}_{i,1}^{\text{C}})}{-QD_1}}{\vdots}$$

$$\frac{\frac{\{\mathcal{D}_{1_n}\} \quad \{\mathcal{D}_{2_n}\} \quad \mathcal{D}_{3_n}}{\{-E_{n_k}\} \quad \{-QU_{n_k}\} \quad -P_n} (\iota_{\mathcal{Q}} -\text{I}_{i,n}^{\text{C}})}{-QD_n}}$$

$$\frac{\mathcal{D}_1}{\frac{-QD_j}{-E_{jk}} (\iota_{\mathcal{Q}} -\text{Ec}_{k,j} 1)} \quad \frac{\mathcal{D}_1}{\frac{-QD_j}{-QU_{jk}} (\iota_{\mathcal{Q}} -\text{Ec}_{k,j} 2)} \quad \frac{\mathcal{D}_1}{\frac{-QD_j}{-P_j} (\iota_{\mathcal{Q}} -\text{Ec}_{i,j} 3)}$$

where $i \in \{1, \dots, m\}$ (arity of predicate in QD_j), $j \in \{1, \dots, n\}$ (level of nesting),
 $k \in \{1, \dots, m\}$

(Likewise with $-\psi_j$.)

DEFINITION 6.21 (Detour conversions ($\iota_{\mathcal{Q}}C$ -systems)). The detour conversions for parallel conjunctively nested qualified definiteness are, *mutatis mutandis*, like those of Definition 6.12.

Remark 6.22. A remark analogous to Remark 6.6 applies also to $\iota_{\mathcal{Q}}C$ -systems.

Example 6.23. We construct a canonical derivation for (12). Symbolization:

$$F^1(\iota_{\mathcal{Q}_3} x(M^1(x) \ \& \ W^2(x, \iota_{\mathcal{Q}_2} y(B_1^1(y) \ \& \ H^2(y, \iota_{\mathcal{Q}_1} z(B_2^1(z)))))))$$

Let $\mathcal{Q}_j, \mathcal{Q}'_j \subseteq \mathcal{P}$.

$$\begin{array}{c}
 \mathcal{D}_{1(E)} \qquad \qquad \qquad \mathcal{D}_{2(QU)} \qquad \qquad \qquad \mathcal{D}_{3(P)} \\
 \exists z B_2^1(z) \quad \forall u_1 \forall v_1 ((B_2^1(u_1) \& B_2^1(v_1)) \supset u_1 \stackrel{\pm}{=}_{\mathcal{Q}_1} v_1) \quad \forall w_1 (B_2^1(w_1) \supset H^2(\alpha_1, w_1)) \\
 \\
 \frac{\mathcal{D}_4 \quad \frac{\mathcal{D}_{1(E)} \quad \mathcal{D}_{2(QU)} \quad \mathcal{D}_{3(P)}}{H^2(\alpha_1, \iota_{\mathcal{Q}_1} z(B_2^1(z)))} (\iota_{\mathcal{Q}} I_{2,1}^c)}{B_1^1(\alpha_1)} \\
 \mathcal{D}_{5(E)} = \frac{B_1^1(\alpha_1) \& H^2(\alpha_1, \iota_{\mathcal{Q}_1} z(B_2^1(z)))}{\exists y (B_1^1(y) \& H^2(y, \iota_{\mathcal{Q}_1} z(B_2^1(z))))} \\
 \\
 \mathcal{D}_{6(QU)} \\
 \forall u_2 \forall v_2 (((B_1^1(u_2) \& H^2(u_2, \iota_{\mathcal{Q}} z(B_2^1(z)))) \& (B_1^1(v_2) \& H^2(v_2, \iota_{\mathcal{Q}} z(B_2^1(z)))))) \supset u_2 \stackrel{\pm}{=}_{\mathcal{Q}_2} v_2) \\
 \\
 \mathcal{D}_{7(P)} \\
 \forall w_2 ((B_1^1(w_2) \& H^2(w_2, \iota_{\mathcal{Q}} z(B_2^1(z)))) \supset W^2(\alpha_2, w_2)) \\
 \\
 \mathcal{D}_8 \quad \frac{\mathcal{D}_{5(E)} \quad \mathcal{D}_{6(QU)} \quad \mathcal{D}_{7(P)}}{W^2(\alpha_2, \iota_{\mathcal{Q}_2} y(B_1^1(y) \& H^2(y, \iota_{\mathcal{Q}_1} z(B_2^1(z))))))} (\iota_{\mathcal{Q}} I_{2,2}^c) \\
 \mathcal{D}_{9(E)} = \frac{M^1(\alpha_2) \quad \frac{M^1(\alpha_2) \& W^2(\alpha_2, \iota_{\mathcal{Q}_2} y(B_1^1(y) \& H^2(y, \iota_{\mathcal{Q}_1} z(B_2^1(z))))))}{\exists x (M^1(x) \& W^2(x, \iota_{\mathcal{Q}_2} y(B_1^1(y) \& H^2(y, \iota_{\mathcal{Q}_1} z(B_2^1(z))))))}} \\
 \\
 \mathcal{D}_{10(QU)} \\
 \forall u_3 \forall v_3 (((M^1(u_3) \& W^2(u_3, \iota_{\mathcal{Q}_2} y(B_1^1(y) \& H^2(y, \iota_{\mathcal{Q}_1} z(B_2^1(z)))))) \& (M^1(v_3) \& W^2(v_3, \iota_{\mathcal{Q}_2} y(B_1^1(y) \& H^2(y, \iota_{\mathcal{Q}_1} z(B_2^1(z))))))))) \supset u_3 \stackrel{\pm}{=}_{\mathcal{Q}_3} v_3) \\
 \\
 \mathcal{D}_{11(P)} \\
 \forall w_3 ((M^1(w_3) \& W^2(w_3, \iota_{\mathcal{Q}_2} y(B_1^1(y) \& H^2(y, \iota_{\mathcal{Q}_1} z(B_2^1(z)))))) \supset F^1(w_3)) \\
 \\
 \frac{\mathcal{D}_{9(E)} \quad \mathcal{D}_{10(QU)} \quad \mathcal{D}_{11(P)}}{F^1(\iota_{\mathcal{Q}_3} x(M^1(x) \& W^2(x, \iota_{\mathcal{Q}_2} y(B_1^1(y) \& H^2(y, \iota_{\mathcal{Q}_1} z(B_2^1(z))))))} (\iota_{\mathcal{Q}} I_{1,3}^c) \quad (6.6)
 \end{array}$$

Example 6.24. Consider (14). Symbolization:

$$\begin{array}{l}
 W_1^1(\iota_{\mathcal{Q}_2} x(M^1(x) \& W_2^2(x, \iota_{\mathcal{Q}_1} y(B^1(y))) \& C^2(x, \iota_{\mathcal{Q}_1'} z(N^1(z))))), \iota_{\mathcal{Q}_3} u(D^1(u) \& \\
 O^2(\iota_{\mathcal{Q}_2} x(M^1(x) \& (W_2^2(x, \iota_{\mathcal{Q}_1} y(B^1(y))) \& C^2(x, \iota_{\mathcal{Q}_1'} z(N^1(z))))), u))
 \end{array}$$

Note that the second occurrence of $\&$ in the first description is the one which corresponds to the conjunction figuring at the surface of (14). Let $\mathcal{Q}_j, \mathcal{Q}'_j \subseteq \mathcal{P}$.

$$\begin{array}{ccc}
\mathcal{D}_{1(E)} & \mathcal{D}_{2(QU)} & \mathcal{D}_{3(P)} \\
\exists y B^1(y) & \forall u_1 \forall v_1 ((B^1(u_1) \& B^1(v_1)) \supset u_1 \stackrel{\pm}{=}_{\mathcal{Q}_1} v_1) & \forall w_1 (B^1(w_1) \supset W_2^2(\alpha_1, w_1)) \\
\mathcal{D}'_{1(E)} & \mathcal{D}'_{2(QU)} & \mathcal{D}'_{3(P)} \\
\exists z N^1(z) & \forall u'_1 \forall v'_1 ((N^1(u'_1) \& N^1(v'_1)) \supset u'_1 \stackrel{\pm}{=}_{\mathcal{Q}'_1} v'_1) & \forall w'_1 (N^1(w'_1) \supset C^2(\alpha_1, w'_1)) \\
\mathcal{D}_4 & \frac{\mathcal{D}_{1(E)} \quad \mathcal{D}_{2(QU)} \quad \mathcal{D}_{3(P)}}{W_2^2(\alpha_1, \iota_{\mathcal{Q}_1} y(B^1(y)))} (\iota_{\mathcal{Q}} I_{2,1}^C) & \frac{\mathcal{D}'_{1(E)} \quad \mathcal{D}'_{2(QU)} \quad \mathcal{D}'_{3(P)}}{C^2(\alpha_1, \iota_{\mathcal{Q}'_1} z(N^1(z)))} (\iota_{\mathcal{Q}} I_{2,1}^C) \\
M^1(\alpha_1) & \frac{W_2^2(\alpha_1, \iota_{\mathcal{Q}_1} y(B^1(y))) \& C^2(\alpha_1, \iota_{\mathcal{Q}'_1} z(N^1(z)))}{W_2^2(\alpha_1, \iota_{\mathcal{Q}_1} y(B^1(y))) \& C^2(\alpha_1, \iota_{\mathcal{Q}'_1} z(N^1(z)))} (\& I) \\
\mathcal{D}_{5(E)} = & \frac{M^1(\alpha_1) \& W_2^2(\alpha_1, \iota_{\mathcal{Q}_1} y(B^1(y))) \& C^2(\alpha_1, \iota_{\mathcal{Q}'_1} z(N^1(z)))}{\exists x (M^1(x) \& W_2^2(x, \iota_{\mathcal{Q}_1} y(B^1(y))) \& C^2(x, \iota_{\mathcal{Q}'_1} z(N^1(z))))} (\& I) \\
& \underbrace{\hspace{10em}}_{=A_1(x)} \\
& \mathcal{D}_{6(QU)} \\
& \forall u_2 \forall v_2 ((A_1(u_2) \& A_1(v_2)) \supset u_2 \stackrel{\pm}{=}_{\mathcal{Q}_2} v_2) \\
& \mathcal{D}_{7(P)} \\
& \forall w_2 ((M^1(w_2) \& W_2^2(w_2, \iota_{\mathcal{Q}_1} y(B^1(y))) \& C^2(w_2, \iota_{\mathcal{Q}'_1} z(N^1(z)))) \supset O^2(w_2, \alpha_2)) \\
& \mathcal{D}_8 \quad \frac{\mathcal{D}_{5(E)} \quad \mathcal{D}_{6(QU)} \quad \mathcal{D}_{7(P)}}{O^2(\iota_{\mathcal{Q}_2} x(M^1(x) \& W_2^2(x, \iota_{\mathcal{Q}_1} y(B^1(y))) \& C^2(x, \iota_{\mathcal{Q}'_1} z(N^1(z))))), \alpha_2)} (\iota_{\mathcal{Q}} I_{2,2}^C) \\
& \frac{D^1(\alpha_2) \& O^2(\iota_{\mathcal{Q}_2} x(M^1(x) \& W_2^2(x, \iota_{\mathcal{Q}_1} y(B^1(y))) \& C^2(x, \iota_{\mathcal{Q}'_1} z(N^1(z))))), \alpha_2)}{D^1(\alpha_2) \& O^2(\iota_{\mathcal{Q}_2} x(M^1(x) \& W_2^2(x, \iota_{\mathcal{Q}_1} y(B^1(y))) \& C^2(x, \iota_{\mathcal{Q}'_1} z(N^1(z))))), \alpha_2)} (\& I) \\
\mathcal{D}_{9(E)} = & \frac{D^1(\alpha_2) \& O^2(\iota_{\mathcal{Q}_2} x(M^1(x) \& W_2^2(x, \iota_{\mathcal{Q}_1} y(B^1(y))) \& C^2(x, \iota_{\mathcal{Q}'_1} z(N^1(z))))), \alpha_2)}{\exists u (D^1(u) \& O^2(\iota_{\mathcal{Q}_2} x(M^1(x) \& W_2^2(x, \iota_{\mathcal{Q}_1} y(B^1(y))) \& C^2(x, \iota_{\mathcal{Q}'_1} z(N^1(z))))), u))} \\
& \underbrace{\hspace{10em}}_{A_2(u)} \\
& \mathcal{D}_{10(QU)} \\
& \forall u_3 \forall v_3 ((A_2(u_3) \& A_2(v_3)) \supset u_3 \stackrel{\pm}{=}_{\mathcal{Q}_3} v_3) \\
& \mathcal{D}_{11(P)} \\
& \forall w_3 \forall w_4 ((M^1(w_3) \& W_2^2(w_3, \iota_{\mathcal{Q}_1} y(B^1(y))) \& C^2(w_3, \iota_{\mathcal{Q}'_1} z(N^1(z)))) \& \\
& (D^1(w_4) \& O^2(\iota_{\mathcal{Q}_2} x(M^1(x) \& W_2^2(x, \iota_{\mathcal{Q}_1} y(B^1(y))) \& C^2(x, \iota_{\mathcal{Q}'_1} z(N^1(z))))), w_4))) \& \\
& \supset W_1^2(w_3, w_4)) \\
& \frac{\mathcal{D}_{5(E)} \quad \mathcal{D}_{9(E)} \quad \mathcal{D}_{6(QU)} \quad \mathcal{D}_{10(QU)} \quad \mathcal{D}_{11(P)}}{(\iota_{\mathcal{Q}} I_{2,3}^C)} \\
& W_1^2(\iota_{\mathcal{Q}_2} x(M^1(x) \& W_2^2(x, \iota_{\mathcal{Q}_1} y(B^1(y))) \& C^2(x, \iota_{\mathcal{Q}'_1} z(N^1(z))))), \\
& \iota_{\mathcal{Q}_3} u (D^1(u) \& O^2(\iota_{\mathcal{Q}_2} x(M^1(x) \& W_2^2(x, \iota_{\mathcal{Q}_1} y(B^1(y))) \& C^2(x, \iota_{\mathcal{Q}'_1} z(N^1(z))))), u))
\end{array}$$

7. Concluding remarks

We have shown that dropping the standard primitive notion of identity from Russell's analysis of definite descriptions in favour of the defined notion of qualified identity [24] can lead us to proof systems for definiteness which not only enjoy good proof-theoretic properties (normalization, subexpression property, subformula property, internal completeness), but which also admit a formulation of an intuitionistically acceptable proof-theoretic semantics for natural language constructions which involve various kinds of definite descriptions (complete, incomplete, generic, parallel, nested, Haddock, predicative). It is hoped that the framework developed above is versatile enough to be adapted so as to handle a wider range of definiteness constructions.

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