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DIALOGICAL ECUMENISM

Abstract

Ecumenical logics are systems where two logics can coexist, sharing vocabulary and avoiding collapses between them. The literature has focused mainly on ecumenism between classical and intuitionistic logic, and several calculi of Natural Deduction and Sequents have been proposed. In this paper I contribute to this project with a dialogical ecumenical system. This Game utilizes an extension of the intuitionistic structural rules that permits to handle classical disjunctions and conditionals. I show that this is indeed an ecumenical dialogical system, where classical formulas and intuitionistic formulas can be validated without collapses between them, and provide a philosophical defense of its design.

Keywords: dialogics, logical ecumenism, game-theoretic semantics, intuitionistic logic.

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1. Introduction

A few years ago Prawitz presented a logical framework in which classical and intuitionistic logic can coexist without one being subordinated to the other. This system is known as *ecumenical logic* [18].[1] Prawitz's proposal shows that a classicist and an intuitionist may agree about the validity of some logical laws (the ecumenical ones) without abandoning their own understanding of the connectives. As he says: "The classical logician is not asserting what the intuitionistic logician denies. For instance, the classical logician asserts $A \vee_c \neg A$ to which the intuitionist does not object; he objects to the universal validity of $A \vee_i \neg A$, which is not asserted by the classical logician" [19, 29]. But this sentence would be absolutely trivial if both formulas were not part of the same system. This is why he goes on to say:

> If they are sufficiently ecumenical and *can use the other's vocabulary in their own speech*, a classical logician and an intuitionist can both *adopt the present mixed system*, and the intuitionist must then agree that $A \vee_c \neg A$ is trivially provable for any sentence A, even when it contains intuitionistic constants, and the classical logician must admit that he has no ground for universally asserting $A \vee_i \neg A$ even when A contains only classical constants. [19, 30] (my emphasis)

Further research followed Prawitz's seminal paper. In [16], a normalization theorem for the propositional fragment of the calculus was presented, together with a proof of its soundness and completeness for a Kripke-style semantics. In [14], an extensive study of ecumenicality is given, together with a broad consideration of the known logics that can be obtained in the framework. And [17] introduced several sequent calculi that extends Prawitz's original idea.

This paper contributes to this literature by introducing the first dialogical account of ecumenism. I translate a strategy introduced in [15] into the dialogical framework of [13], and show how this can be exploited to handle

[1] Though antecedents of the idea can be found in [12, 9] and [6].

the structural rule that differentiates between classical and intuitionistic dialogues at the level of the language. This allows for a game-theoretic semantics of the ecumenical connectives.

The paper is structured as follows. Sections 2 and 3 introduce the ecumenical system $\mathcal{LE}_p$ and the standard dialogical framework respectively. Then in section 4 the Game $D_{\mathcal{E}}$ is presented. This is a dialogical variant of $\mathcal{LE}_p$, with a dialogical interpretation of (what I call) states, new particle rules to distinguish between intuitionistic and classical connectives and structural rules to handle them ecumenically. Section 5 addresses some philosophical concerns and deliver some semantic clarifications. A final section summarizes the conclusions.

2. Ecumenical logic with a stoup

A well-known argument favoring classical over intuitionistic logic is that an intuitionistic negation cannot survive alongside a classical one, for the former collapses into the latter. On the other hand, some inferentialists have defended intuitionistic logic because it has some properties, such as harmony, that classical logic (allegedly) does not. Prawitz's ecumenical logic (EL), introduced in [19], surmounts these two challenges with a system where the collapse result does not obtain and harmony is preserved in the classical fragment. The formalism is a Natural Deduction system with:

1. Logical connectives that are indistinguishable from the classical or intuitionistic point of view, and that we may call *common*: $\wedge, \neg, \bot$. Their rules are their usual intuitionistic ones;

2. Those that characterize the difference between the classical and the intuitionistic interpretation: $\vee, \rightarrow$. These receive a sub-index depending on their interpretation: "i" for *intuitionist*, "c" for *classical*. The former are governed by their usual intuitionistic rules, but the latter receive new ones.

These rules illustrate Prawitz's point, but are not very elegant from a proof-theoretic perspective. In subsequent works, others introduced variants and improvements of the original system. In what follows we will be concerned with one of these proposals, the system $\mathcal{LE}_p$ introduced in [15].

We work with a propositional language defined as usual. Following Prawitz, we shall distinguish a *neutral fragment* of the language, in which the logical constants are limited to the signature $\{\neg, \wedge\}$. The classical and intuitionistic fragments of the language are obtained from the neutral by adding the connectives $\{\vee, \rightarrow\}$, with a sub-index c or i for classical and intuitionistic respectively.

The calculus is defined over structures I call *states*. A state is an expression of the form

$$\Gamma \ ; \ \Delta$$

Where Γ and Δ are sets of formulas, and the latter has at most one formula. The place Γ is called *context* and the place Δ is called *stoup*. We represent an empty stoup or an empty context as $\emptyset$.[2]

The calculus $\mathcal{LE}_p$ is defined over states, not formulas. Its rules are the following:

1. Operational rules:

   (a) Neutral rules:

   i. Negation:

   A. Introduction:

$$\dfrac{\begin{matrix}[\emptyset \ ; \ A]^n \\ \vdots \\ \Gamma \ ; \ \emptyset\end{matrix}}{\Gamma \ ; \ \neg A} \neg \text{ I } (n)$$

   B. Elimination:

$$\dfrac{\Gamma \ ; \ A \qquad \Delta \ ; \ \neg A}{\Gamma, \Delta \ ; \ \emptyset} \neg \text{ E}$$

[2] In [15] and related works the overall expression $\Gamma \ ; \ \Delta$ is called *stoup*, being Γ its context. I see my divergence on this point as nothing more than a personal preference. I do not attribute any special, or different, meaning to the word *state* as used here.

ii. Conjunction:

A. Introduction:

$$\frac{\Gamma \ ; \ A \qquad \Delta \ ; \ B}{\Gamma, \Delta \ ; \ A \wedge B} \wedge \text{I}$$

B. Elimination: $(j \in \{1, 2\})$

$$\frac{\Gamma \ ; \ A_1 \wedge A_2}{\Gamma \ ; \ A_j} \wedge \text{E}$$

(b) Intuitionistic rules:

i. Conditional:

A. Introduction:

$$\frac{\begin{array}{c}[\varnothing \ ; \ A]^n \\ \vdots \\ \Gamma \ ; \ B\end{array}}{\Gamma \ ; \ A \to_i B} \to_i \text{I } (n)$$

B. Elimination:

$$\frac{\Gamma \ ; \ A \qquad \Delta \ ; \ A \to_i B}{\Gamma, \Delta \ ; \ B} \to_i \text{E}$$

ii. Disjunction:

A. Introduction: $(j \in \{1, 2\})$

$$\frac{\Gamma \ ; \ A_j}{\Gamma \ ; \ A_1 \vee_i A_2} \vee_i \text{I}$$

B. Elimination:

$$\frac{\Gamma \ ; \ A \vee_i B \qquad \begin{array}{c}[\varnothing \ ; \ A]^m \\ \vdots \\ \Delta \ ; \ C\end{array} \qquad \begin{array}{c}[\varnothing \ ; \ B]^n \\ \vdots \\ \Pi \ ; \ C\end{array}}{\Gamma, \Delta, \Pi \ ; \ C} \vee_i \text{E } (m, n)$$

(c) Classical rules:

i. Conditional:

A. Introduction:

$$\dfrac{\begin{array}{c}[\varnothing\ ;\ A]^n\\ \vdots\\ \Gamma, B\ ;\ \varnothing\end{array}}{\Gamma\ ;\ A \to_c B}\ \to_c \mathrm{I}\ (n)$$

B. Elimination:

$$\dfrac{\Gamma\ ;\ A \to_c B \qquad \Delta\ ;\ A \qquad \begin{array}{c}[\varnothing\ ;\ B]^n\\ \vdots\\ \Pi\ ;\ \varnothing\end{array}}{\Gamma, \Delta, \Pi\ ;\ \varnothing}\ \to_c \mathrm{E}\ (n)$$

ii. Disjunction:

A. Introduction:

$$\dfrac{\Gamma, A, B\ ;\ \varnothing}{\Gamma\ ;\ A_1 \vee_c A_2}\ \vee_c \mathrm{I}$$

B. Elimination:

$$\dfrac{\Gamma\ ;\ A \vee_c B \qquad \begin{array}{c}[\varnothing\ ;\ A]^m\\ \vdots\\ \Delta\ ;\ \varnothing\end{array} \qquad \begin{array}{c}[\varnothing\ ;\ B]^n\\ \vdots\\ \Pi\ ;\ \varnothing\end{array}}{\Gamma, \Delta, \Pi\ ;\ \varnothing}\ \vee_c \mathrm{E}\ (m, n)$$

2. Structural Rules:

(a) Dereliction:

$$\dfrac{\Gamma\ ;\ A}{\Gamma, A\ ;\ \varnothing}\ \mathrm{Der}$$

(b) Contraction:

$$\dfrac{\Gamma, A, A\ ;\ B}{\Gamma, A\ ;\ B}\ \mathrm{C}$$

(c) Intuitionistic Weakening:

$$\dfrac{\Gamma \ ; \ \emptyset}{\Gamma \ ; \ A}\ \mathrm{IW}$$

(d) Classical Weakening:

$$\dfrac{\Gamma \ ; \ B}{\Gamma, A \ ; \ B}\ \mathrm{CW}$$

Hypothesis have always the form $\emptyset \ ; \ A$.

There is one major result concerning this logic:

Fact 2.1. The calculus $\mathcal{LE}_p$ strongly normalizes; if a formula has a derivation, then it has a unique normal derivation.

Proof: [15, th. 5]. □

I use some examples to deliver further explanations. The standard approach to Natural Deduction for classical logic is to define an intuitionistic calculus and then add a rule (double negation elimination, proof by cases) or an axiom (excluded middle, *consequentia mirabilis*) to enlarge its expressive power. This introduces some inelegancies in the classical proofs. For instance, the 'canonical' proof of the law of excluded middle in Gentzen's NK looks like this:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[p]^2}{p \vee \neg p}\ \vee \mathrm{I} \qquad [\neg(p \vee \neg p)]^1}{\bot}\ \neg \mathrm{E}}{\neg p}\ \neg \mathrm{I}\ (2)}{p \vee \neg p}\ \vee \mathrm{I} \qquad [\neg(p \vee \neg p)]^1}{\bot}\ \neg \mathrm{E}}{\neg\neg(p \vee \neg p)}\ \neg \mathrm{I}\ (2)}{p \vee \neg p}\ \neg\neg \mathrm{E}$$

This is clearly an unnatural way to account for the validity of the excluded middle. Compare it with the normal proof of the law of non contradiction:

$$\dfrac{\dfrac{\dfrac{[p \wedge \neg p]^1}{p}\wedge \text{E} \qquad \dfrac{[p \wedge \neg p]^1}{\neg p}\wedge \text{E}}{\bot}\to \text{E}}{\neg(p \wedge \neg p)}\to \text{I (1)}$$

This proof is more perspicuous: the formulas that appear in the derivation are all but only subformulas of the final one (provided we take $\neg A$ as an abbreviation of $A \to \bot$) and the last rule applied is the introduction of the main connective in the final formula. These are the kind of properties that one may expect from a more 'natural' proof of a principle, and is one way of understanding harmony [7]. The calculus $\mathcal{LE}_p$ provides proofs for the classical principles that have these properties. For instance, the proof for the law of excluded middle in this calculus is:

$$\dfrac{\dfrac{\dfrac{\dfrac{[\varnothing \ ; \ p]^1}{p \ ; \ \varnothing}\text{Der}}{p \ ; \ \neg p}\neg \text{ I (1)}}{p, \neg p \ ; \ \varnothing}\text{Der}}{\varnothing \ ; \ p \vee_c \neg p}\vee_c \text{I}$$

The detour through $\neg\neg(p \vee_i \neg p)$ has been avoided, and the properties of perspicuity (the subformula and the final rule) are respected.

Another interesting aspect of ecumenical systems in general (and of $\mathcal{LE}_p$ in particular) is that it allows to show the interaction between classical and intuitionistic connectives. Consider $p\vee(p \to q)$, one of the non-intuitionistic classical laws. In an ecumenical logic $p \vee_c (p \to_c q)$ can be derived and $p \vee_i (p \to_i q)$ cannot, as expected, but an interesting fact is that you can also derive $p \vee_c (p \to_i q)$:

$$\begin{prooftree}\AxiomC{$[\varnothing\ ;\ p]^1$}\RightLabel{Der}\UnaryInfC{$p\ ;\ \varnothing$}\RightLabel{IW}\UnaryInfC{$p\ ;\ q$}\RightLabel{$\to_i$ I (1)}\UnaryInfC{$p\ ;\ p \to_i q$}\RightLabel{Der}\UnaryInfC{$p, p \to_i q\ ;\ \varnothing$}\RightLabel{$\vee_c$ I}\UnaryInfC{$\varnothing\ ;\ p \vee_c (p \to_i q)$}\end{prooftree}$$

But not $p \vee_i (p \to_c q)$. For assume that there is a proof of this formula. As the calculus strongly normalizes (fact 2.1), there is a normal proof of the formula. In this normal proof, the last rule applied is the introduction of $\vee_i$; but neither p nor $p \to_c q$ are validities. Therefore, the proof does not exist.

The last example shows that ecumenicality serves to highlight some relations between classical and intuitionistic logic that other approaches (for instance, double-negation translations) tend to miss, or at least make less evident.

3. Dialogics

We now turn to Dialogics. For the most part, my presentation of the framework follows [21], chapters 3 to 5. See [5] for an updated presentation of the topic.[3]

In the working metaphor of Dialogics, arguments are debates where, provided one accepts certain *concessions* (the premises), it is possible to defend a *thesis* (the conclusion). These debates are turn-based games that consist in a series of assertions and requests made by two players: the Proponent, who has to defend the thesis (and challenge the concessions) and the Opponent, who seeks to challenge the thesis (and defend the concessions). By convention, we call the Proponent a ‘he’ and the Opponent a ‘she’.

[3]There is a brand in philosophy of Logic, whose main promoter is Catarina Dutilh Novaes, that is usually associated with Dialogics. That literature, interesting and appealing as it is, is nonetheless intentionally deviant to the dialogical tradition presented here, and so it is not considered as part of the bibliography. See [5, section 3.3] for a description of the difference between these two traditions.

These games are governed by two kind of rules: *particle* and *structural.* The first regulate *what* movements they can perform, while the second regulate *how* they are supposed to perform them.

Let X and Y be two different players. The standard particle rules are the following:

1. If X asserts $A \wedge B$, Y may challenge it by *requesting* A or by *requesting* B; in each case, the proper defense is that X asserts the requested conjunct.

2. If X asserts $A \vee B$, Y may challenge it by *requesting* one of the disjuncts; the proper defense is that X asserts A or B.

3. If X asserts $A \rightarrow B$, Y may challenge it by *asserting* A; the proper defense is that X asserts B.

4. If X asserts $A \rightarrow B$, Y may challenge it by *asserting* A; the proper defense is that X asserts B.

5. If X asserts $\neg A$, Y may challenge it by *asserting* A; and there is no corresponding defense for this.

These rules capture the game-theoretic meaning of the connectives. To understand a logical connective is to understand how an assertion with that connective can be challenged and what is to be expected as a defense. So, for example, the duality between disjunction and conjunction is captured as a difference in the prerogative of choice. If someone asserts a conjunction:

Love is wise and hatred is foolish.

it is to be expected that this someone is able to deliver a satisfactory defense of any of the two conjuncts: $\wedge_L$ ('Why (you say that) love is wise?') or $\wedge_R$ ('Why (you say that) hatred is foolish?'). So, the choice is up to the challenger. On the other hand, if someone asserts a disjunction:

There is a magic ring or ten coins in my pocket.

it is to be expected that this someone is able to defend one of the two disjuncts, but the choice is theirs to make. So the challenge is an open-ended request: $\vee_?$ ('What is in your pocket?').

The justification for the particle rule of negation may be less evident. To challenge a negation, one has to assert the negated proposition. And the rule states that this move has not a defense per se. But then the only option left is to challenge the assertion of the negated proposition (a move sometimes called counter-challenge). And this makes sense: to defend a negation amounts to be able to successfully challenge what is negated.

Conditionals are justified in a similar vein. When someone states a conditional:

If $2+2=3$ then I am the Pope in Rome.

the natural way to challenge this assertion is to concede the antecedent ('Assume $2+2=3$...') and wait for the other to assert (and defend) the consequent.

These are the *standard* particle rules, and the reason for this label is that, at least in the case of the classical and the intuitionistic Games, these rules are invariant.[4] So the difference between the two (dia)logics will be captured by the second set of rules—the structural ones.

The rules that define the *Classical Dialogic Game* $D_{\mathcal{C}}$ are the following (adapted from [21]):

SR0: Starting rule A dialogue starts with the Opponent stating initial concessions (if any) and then the Proponent stating the thesis. Then each of them will choose a positive integer. This number is their *repetition rank*.

SR1c: (Classical) Development rule Players move alternatively. Each move is either a challenge or a defense over the moves of the other, in accordance to the particle rules. A player can repeat a move up to as many times as the number (s)he chose as her/his repetition rank.

[4] Although it is not standard usage, I use 'Game' to mean all the games played under certain rules. In Dialogics inferences are games, not plays. A *play* in this context is one possible outcome of a game. Validity is defined in terms of winning strategies (see below), and these are captured by core strategies: sets of plays that show the strategy for one of the players, depending on all the relevant choices of the contender.

SR2: Copy-Cat rule The Proponent can only assert an atomic formula if the Opponent asserts it too. We make this requirement precise in the following way: if the Proponent asserts an atomic formula, the Opponent may challenge this move by requesting: *ut?* ('how?'); the Proponent then can defend his move with $sic(k)$ ('Thus: k'), where k refers to a move in which the Opponent makes the same assertion. The Proponent cannot challenge the assertion of an atomic formula.

SR3: Winning rule After all possible moves have been made, whoever has made the last move wins the game.

The repetition ranks mentioned in the rule SR0 are introduced to avoid redundant and infinite plays. They set the number of times each player can repeat a previous move (if no other constraint forbids to do so; see below). It is customary to call m the repetition rank of the Opponent and n the repetition rank of the Proponent. One important result concerning repetition ranks is the following:

FACT 3.1. If the Proponent has a winning strategy over a certain thesis, then he has a winning strategy for that thesis with repetition rank $n := 2$ regardless of the choice of his contender. If the Opponent has a winning strategy over a certain thesis, then she has a winning strategy for that thesis with repetition rank $m := 1$ regardless of the choice of her contender.

PROOF: [3, th. 1.2.2]. See also [4, section 4]. □

Based on this fact we will consider only games where $m := 1$ and $n := 2$. So the game with concessions Γ and thesis A, where $m = 1$ and $n = 2$ will be noted: $D\langle\Gamma, A\rangle$ instead of the usual $D^{1,2}\langle\Gamma, A\rangle$. As an *abus de language*, we will sometimes say 'the game A' to mean the game $D\langle\emptyset, A\rangle$.

Given a game $D\langle\Gamma, A\rangle$, the Proponent can play wisely or poorly. But in certain games there will be a recipe for victory in all possible choices of the Opponent. When this is the case, we say that there is a *winning strategy* for the Proponent. And validity in Dialogics is captured by this notion. So we are entitled to call $D_{\mathcal{C}}$ a dialogical presentation of classical logic because the following fact holds:

FACT 3.2. The Proponent has a *winning strategy* (that is, a way to win no matter what the Opponent does) in the game $D_{\mathcal{C}}\langle\Gamma, A\rangle$ if and only if A is a classical consequence of Γ.

PROOF: [1, th. 2]. □

The *Intuitionistic Dialogic Game* $D_{\mathcal{I}}$ obtains from the classical one by making only one change to the structural rules. If we add to SR1c the clause called *Last Duty First*, we obtain SR1i, the characteristic rule for intuitionistic games:

SR1i: (Intuitionistic) Development rule Players move alternatively. Each move is either a challenge or a defense over the moves of the other, in accordance to the particle rules. A player can repeat a move up to as many times as the number (s)he chose as her/his repetition rank.

Last Duty First: A player can only defend the last unanswered challenge made by the contender.

FACT 3.3. The Proponent has a *winning strategy* in the game $D_I\langle\Gamma, A\rangle$ if and only if A is an intuitionistic consequence of Γ.

PROOF: The two main theorems in [8]. □

Let's see some examples. Consider the game $p \vee \neg p$. The game begins with the Proponent stating the thesis, and each player choosing their repetition rank:

	O			**P**	
				$p \vee \neg p$	0
1	$m := 1$			$n := 2$	2

Then the Opponent can challenge the thesis:

	O			**P**	
				$p \vee \neg p$	0
1	$m := 1$			$n := 2$	2
3	$\vee_?$	0			

Note how we keep track of the moves in the outer columns, and use the inner column to match a challenge with the move being challenged.

The Proponent may now choose to answer with either p or $\neg p$. We place a defense alongside the corresponding challenge:

	O			**P**	
				$p \vee \neg p$	0
1	$m := 1$			$n := 2$	2
3	$\vee_?$	0		p	4

The Proponent has asserted an atomic proposition, so the Opponent is in a position to request a Copy-Cat move, according to our understanding of rule SR2:

	O			**P**	
				$p \vee \neg p$	0
1	$m := 1$			$n := 2$	2
3	$\vee_?$	0		p	4
5	*ut? p*	4			

The Opponent has not asserted p yet, so the Proponent cannot defend move 4. But he is not out of moves; in particular, he can return and defend the thesis *again*:

	O			**P**	
				$p \vee \neg p$	0
1	$m := 1$			$n := 2$	2
3	$\vee_?$	0		p	4
5	*ut? p*	4			
(3)	$\vee_?$	0		$\neg p$	6

As his repetition rank is 2, he is authorized to repeat a move. Now the Proponent has the last word, so the Opponent is forced to challenge move 6:

	O			**P**	
				$p \vee \neg p$	0
1	$m := 1$			$n := 2$	2
3	$\vee_?$	0		p	4
5	*ut? p*	4			
(3)	$\vee_?$	0		$\neg p$	6
7	p	6		-	

As negations cannot be defended, we place a hyphen in the corresponding spot to indicate that the Proponent's next move cannot be a defense of move 6. But note that now the Opponent has allowed a Copy-Cat move that enables the Proponent to come back to his previous defense of move 4 and win the game. So the final depiction of the play looks like this:

P_1:

	O			**P**	
				$p \vee \neg p$	0
1	$m := 1$			$n := 2$	2
3	$\vee_?$	0		p	4
5	*ut? p*	4		*sic*(7)	8++
(3)	$\vee_?$	0		$\neg p$	6
7	p	6		-	

As explained before, this table is read as follows:

1. We label the moves of the Proponent as '**P**' and the moves of the Opponent as '**O**'.
2. The outer columns keep track of the order of the moves.
3. The inner columns keep track of challenges.
4. Every defense is placed alongside the challenge it responds.
5. If one challenge is defended twice (as in this example), the move is repeated but its number appears in parenthesis.
6. The sign '++' marks the last move.

In this play the Proponent wins, because he retains the last word. It is also the best way for him to play: with this order of moves, he is able to win no matter what the Opponent does. This shows that $p \vee \neg p$ is a classical validity.

Consider now the intuitionistic restraint, *Last Duty First.* A player can only defend the last challenge of the contender. In the previous example, after move 5 the Proponent can only reply to the previous challenge, *ut?* p, but the Opponent has not conceded p yet, so he cannot defend this move. He had only one other alternative: to defend move 3 with $\neg p$ instead.

P_2:

	O			**P**	
				$p \vee \neg p$	0
1	$m := 1$			$n := 2$	2
3	$\vee_?$	0		$\neg p$	4
5++	p	4		-	

Now the Opponent concedes p, but the last unanswered challenge is move 5 and therefore the Proponent cannot 'come back' to defend again move 3. So he is out of moves. The Opponent has a way to win in every scenario, and therefore $p \vee \neg p$ is invalid, as expected.

Here's another example. The thesis is $(p \to q) \vee (q \to r)$.

P_3:

	O			**P**	
				$(p \to q) \vee (q \to r)$	0
1	$m := 1$			$n := 2$	2
3	$\vee_?$	0		$p \to q$	4
5	p	0		q	6
7	*ut?* q	6			

If this is an intuitionistic game, then the Proponent is out of moves: he cannot challenge any other move by the Opponent, and the Opponent has not conceded q, so he cannot defend move 7. But if the rules are classical, then he can continue the play as follows:

P'_3:

	O				P	
					$(p \to q) \vee (q \to r)$	0
1	$m := 1$				$n := 2$	2
3	$\vee_?$	0			$p \to q$	4
5	p	4			q	6
7	*ut? q*	6			*sic*(9)	10++
(3)	$\vee_?$	0			$q \to r$	8
9	q	8				

Note that move 9 is left without a defense, but this is not forbidden by the rules.

Allow me a brief digression here. In the two examples above the crucial step is the defense of a previous, already defended, challenge. But some may have noticed that these moves also exploit the fact that the Proponent can make this second defense because of his repetition rank. This fact is not really relevant for the distinction between classical and intuitionistic validities. For instance, the following is a game that the Proponent can win with the classical rules, where the repetition rank plays no role:

P_4:

	O				P	
					$\neg\neg p \to p$	0
1	$m := 1$				$n := 2$	2
3	$\neg\neg p$	0			p	4
5	*ut? p*	4			*sic*(7)	8++
	-			3	$\neg p$	6
7	p	6			-	

On the other hand, here is a game of an intuitionistic (and classical) validity, where the Proponent exploits the fact that he has a repetition rank of 2:

P_5:

	O			**P**	
0.1	$p \wedge (q \vee r)$			$(p \wedge q) \vee (p \wedge r)$	0
1	$m := 1$			$n := 2$	2
3	$\vee_?$	0		$p \wedge q$	8
5	$q \vee r$		0.1	$\wedge_R$	4
7	q		5	$\vee_?$	6
9	$\wedge_L$	8		p	12
11	p		0.1	$\wedge_L$	10
13	*ut? p*	12		*sic*(11)	14++

In this play the Proponent needs to challenge the concession twice. He has to request the second conjunct of the concession (move 4) *before* he defends the thesis, because this defense depends on whether the Opponent concedes q or r. Under the intuitionistic rules this movement is legitimate, since *Last Duty First* obliges only to *defend* in a certain way, but he is always free to challenge previous moves of his contender. By the time he finally defends move 3 (at move 8) this is still the last *unanswered challenge* of the Opponent.

I hope that these examples and explanations are illustrative enough of the dialogical idiosyncrasies. In the next section, these will be exploited to produce an ecumenical Game, in the vein of $\mathcal{LE}_p$.

4. Dialogical Ecumenism

The main difference between the classical and the intuitionistic Games is the clause in SR1i called *Last Duty First*. We can paraphrase the effect of this clause as saying that in the intuitionistic games the defenses have to be performed *one at a time*, whereas in the classical game a defense can be performed in the context of another defense. As we saw in the case of $(p \to q) \vee (q \to r)$ (P_3), to defend successfully one of the disjuncts, the Proponent needs to force the Opponent to challenge the other. This results in a heuristic of the following kind:

1. Defend the thesis $((p \to q) \vee (q \to r))$ with the first disjunct $(p \to q)$
2. The Opponent challenges it (p)

3. Defend it (q)
4. The Opponent counter-challenges this defense (*ut? q*)
   (a) Hold on and come back to defend the thesis again with the second disjunct ($q \to r$)
   (b) The Opponent challenges it (q)
5. Defend the counter-challenge (*sic*(4*b*))

The nested list in this recipe corresponds to a *sub-defense*. The main defense is left in stand-by, and it is reprised after the second defense has been performed.

As we saw in section 2, Prawitz's insight in his approach to ecumenicality was that the distinction between classical and intuitionistic logic can be expressed solely on the $\{\vee, \to\}$ fragment of the language. From the point of view of dialogics, these connectives are the ones in which the Defender has a choice:

1. When a conjunction is challenged, the Defender can only defend with the conjunct requested.
2. When a negation is challenged, the Defender cannot defend it.
3. When a disjunction is challenged, the Defender can choose the disjunct to assert.
4. When a conditional is challenged, the Defender can choose to assert the consequent or counter-challenge the antecedent.

And as we saw in the examples of the previous section, this choice plays an important role in the use of the classical liberality of rule SR1c. In all cases considered, the Proponent chooses one of his possibilities and uses (part of) this defense to prepare the other. The intuitionistic rules do not forbid this strategy, but they force the Proponent to finish the first defense before coming back and perform the other. So in the following example:

P_6:

	O			**P**	
				$(p \wedge q) \to (p \vee r)$	0
1	$m := 1$			$n := 2$	2
3	$p \wedge q$	0		$p \vee r$	6
5	p		3	$\wedge_L$	4
7	$\vee_?$	6		p	8
9	*ut? p*	8		*sic*(5)	10++

The Proponent chooses to counter-challenge the antecedent first (move 4) and then defend the consequent (move 6), but this is in agreement with *Last Duty First*, since a counter-challenge is not a defense, and therefore before move 6 the last unanswered challenge is indeed 3. So what differentiates the intuitionistic from the classical disjunction and conditional is that only in the latter the defense can be reprised at any moment of the play. This is the insight that we will exploit to define an ecumenical version of these games.

The *Ecumenical Dialogic Game* $D_{\mathcal{E}}$ are games in which the players use ecumenical formulas. Moves are not depicted as formulas but as states. When talking about dialogues we call the context of a state *duties* and the stoup *current move*.

The particle rules in this new Game are the following:

1. If X asserts $A \wedge B$, Y may challenge it by *requesting* A or B; in each case, the proper defense is that X asserts the requested conjunct.

2. If X asserts $\neg A$, Y may challenge it by *asserting* A; and there is no corresponding defense for this.

3. If X asserts $A \vee_i B$, Y may challenge it by *requesting* one of the disjuncts; the proper defense is that X asserts A or B.

4. If X asserts $A \to_i B$, Y may challenge it by *asserting* A; the proper defense is that X asserts B.

5. If X asserts $A \vee_c B$, Y may challenge it by *requesting* one of the disjuncts; the proper defense is that X asserts A or B and stores the other as a duty, or that (s)he stores both as a duty and makes another move.

6. If X asserts $A \rightarrow_c B$, Y may challenge it by *asserting* A; the proper defense is that X asserts B or store it as a duty and makes another move.

The structural rules are as follows:

SR0e: (Ecumenical) Starting rule A dialogue starts with the Opponent stating initial concessions as states $\emptyset$; A (if any) and then the Proponent stating the thesis Γ ; B. Then each of them will choose a positive integer. This number is its *repetition rank*.

SR1e: (Ecumenical) Development rule Players move alternatively. Each move is:

1. either a challenge or a defense over the moves of the other, in accordance to the particle rules, or
2. the assertion of one of the duties of the player. When a duty is asserted, it is removed from the list of duties of the player. This move is called *settling*.

A player can repeat a move up to as many times as the number (s)he chose as her/his repetition rank.

Last Duty First: A player can only defend the last unanswered challenge made by the contender.

SR2: Copy-Cat rule The Proponent can only assert an atomic formula if the Opponent asserts it too. We make this requirement precise in the following way: if the Proponent asserts an atomic formula, the Opponent may challenge this move by requesting: *ut?* ('how?'); the Proponent then can defend his move with *sic*(k) ('Thus: k'), where k

refers to a move in which the Opponent makes the same assertion. The Proponent cannot challenge the assertion of an atomic formula.

SR3: Winning rule After all possible moves have been made, whoever has made the last move wins the game.

As before, I deliver some explanations through examples. We begin with a game that corresponds to one of the ecumenical formulas we commented early on.

P_7:

	O			**P**	
				$\emptyset\ ;\ p \vee_c (p \rightarrow_i q)$	0
1	$m := 1$			$n := 2$	2
3	$\vee_?$	0		$p\ ;\ p \rightarrow_i q$	4
5	$\emptyset\ ;\ p$	4		$q\ ;\ p$	6
7	*ut? p*	6		*sic*(5)	8++

As per *Last Duty First*, after move 5 this is the last unanswered challenge, so the Proponent cannot come back to defend again move 3. But his move 6 is an appropriate defense of the previous move: he stores the consequent as a duty, and then settles a previous duty. Therefore, this move is legitimate and the Proponent is able to win the game.

A play may end with one or both players having duties to fulfill, as this example shows. But the winning condition is the same as before, and there are no moves to force a player to settle. So this fact has no impact on the result of the play. Here is another example. The Proponent does not settle r; but the Opponent has no reasons to request for it, so he wins anyway.

P_8:

	O			**P**	
				$\emptyset\ ;\ (p \rightarrow_c q) \vee_c (q \rightarrow_c r)$	0
1	$m := 1$			$n := 2$	2
3	$\vee_?$	0		$q \rightarrow_c r\ ;\ p \rightarrow_c q$	4
5	$\emptyset\ ;\ p$	4		$q\ ;\ q \rightarrow_c r$	6
7	$\emptyset\ ;\ q$	6		$r\ ;\ q$	8
9	*ut? q*	8		*sic*(7)	10++

Moves 6 and 8 are defenses where the consequent is stored as a duty and a settling is performed instead.

A play can also start with duties from the Proponent. In the following game the thesis is the state $p\ ;\ p \rightarrow_c q$.

P_9:

	O			**P**	
				$p\ ;\ p \rightarrow_c q$	0
1	$m := 1$			$n := 2$	2
3	$\emptyset\ ;\ p$	0		$q\ ;\ p$	4
5	*ut? p*	4		*sic*(3)	6++

The Proponent has been challenged with $\emptyset\ ;\ p$; but the thesis is a classical conditional, so he can defend it by storing the consequent as a duty and then performing a different move. In this case, as he had a previous duty (p), he chooses to settle it, thus asserting the state:

$$q\ ;\ p$$

And he can defend p, since the Opponent has conceded it.

Note that, although the Proponent can assume duties at the very start of the play, the Opponent can only concede duty-free propositions. We will come back to this feature and try to justify it in the next section.

The examples above show plays where a settling occurs as a defense. This may not be always the case though. There are plays in which settlings will occur neither as defenses nor as challenges of other moves. See the following example:

P_{10}:

	O			**P**	
				$p\ ;\ \neg p$	0
1	$m := 1$			$n := 2$	2
3	p	0		-	
				p	4
5	*ut? p*	4		*sic*(3)	6++

In this play the Proponent cannot defend the thesis, since it is a negation. But he has a duty, p, and so he asserts it as one of his available moves, without being a challenge or a defense (move 4).

We end this section with some important metalogical results about this Game.[5]

THEOREM 4.1. *Let $\{A_1, \dots, A_n, B\}$ be a set of formulas from the intuitionistic fragment of the language, and $\{A'_1, \dots, A'_n, B'\}$ the same set of formulas without the indexes $(_i)$. The Proponent has a winning strategy in the game*

$$D_{\mathcal{E}}\langle\{\varnothing\ ;\ A_1, \dots, \varnothing\ ;\ A_n\}, \varnothing\ ;\ B\rangle$$

If and only if he has a winning strategy in the game $D_{\mathcal{I}}\langle\{A'_1, \dots A'_n\}, B'\rangle$.

PROOF: Immediate. □

THEOREM 4.2. *Let $\{A_1, \dots, A_n, B\}$ be a set of formulas from the classical fragment of the language, and $\{A'_1, \dots, A'_n, B'\}$ the same set of formulas without the indexes $(_c)$. If the Proponent has a winning strategy in the game*

$$D_{\mathcal{E}}\langle\{\varnothing\ ;\ A_1, \dots, \varnothing\ ;\ A_n\}, \varnothing\ ;\ B\rangle$$

He has a winning strategy in the game $D_{\mathcal{C}}\langle\{A'_1, \dots A'_n\}, B'\rangle$.

PROOF: Assume the antecedent. The Proponent has a winning strategy in the game $D_{\mathcal{E}}\langle\{\varnothing\ ;\ A_1, \dots, \varnothing\ ;\ A_n\}, \varnothing\ ;\ B\rangle$. As the thesis in this game has no duties, every formula stored and (eventually) settled during the course of a play must come from the defense of a $C \vee_c D$ or $C \rightarrow_c D$ formula.

As games are finite, there is a finite set of finite series of moves that completely describe the winning strategy for the Proponent. These can be displayed in a single tree-like diagram, in which every node represents a move and a branching occurs every time the Opponent has a choice.[6] We assume that the tree is oriented from the root (the beginning of the play) to the leaves (the last move in every possible development of the strategy). We sketch an algorithm to transform this diagram into the diagram of a winning strategy for the Proponent in the game $D_{\mathcal{C}}\langle\{A'_1, \dots A'_n\}, B'\rangle$.

[5]Concerning theorem 4.2: a previous version of this article included a (faulty) result about classical recapture. I am most grateful with an anonymous referee who raised suspicions on the result and lead me to correct it.

[6]See [21], chapter 5 for a detailed explanation on how to obtain such a diagram.

1. Eliminate all indexes ($_c$).
2. Eliminate every empty set of duties ($\varnothing$; ...) and check for the overall status of the diagram:
   (a) If all moves have been replaced by standard classical moves, go to step 4.
   (b) If there are moves with duties in the tree, look for the first leftmost appeareance of such a move in the tree. Go to step 3.
3. As this move is the first in which a duty appears in the tree, this must correspond to a storage after a defense of a formula $C \to_c D$ or $C \vee_c D$. Let i be the move in which the formula being defended was asserted.
   (a) If it is the defense of $C \to_c D$, then D is the duty being stored. Justify this move as whatever is the justification of the current move, and look forward for the move in which the duty D is settled. If there is none, eliminate D from all subsequent duties. If the settling occurs at move j, justify this move as a defense of move i and then eliminate D from all previous duties. The classical rules allow this move.
   (b) If it is the defense of $C \vee_c D$, we distinguish which player makes the defense:
      i. *If the Proponent makes the move:* C, D or both are the duty being stored. Justify this move as whatever is the justification of the current move, and look forward for the move in which the duty (or duties) is (are) settled. If there is none, eliminate C and D from all subsequent duties. If the settling of one of them occurs at move j, justify this move as a defense of move i and then eliminate the duty settled from all the previous moves. The classical rules allow this move. Repeat with the other duty if applies.
      ii. *If the Opponent makes the move:* C, D or both are the duties being stored. Justify this move as whatever is the justification of the current move, and look forward for the moves in

which each duty (or duties) is (are) settled. If there is none, eliminate C and D from all subsequent duties. If the settling of one of them occurs at move j and the settling of the other occurs at move k, transform the tree into a branching: in one branch, deploy the strategy after the assertion of C, and in the other, after the assertion of D. In each branch, eliminate all moves that relate to the other disjunct. Justify these moves as a defense of move i and then eliminate the duty settled from all previous moves. The classical rules allow this move.

Then go back to step 2.

4. The transformation is complete.

Concerning step 3(b)(ii), the algorithm assumes that the defenses of C and D by the Opponent are independent from each other, and so the moves can be evenly divided between the two branches. This is not *a priori* necessary. Yet the algorithm still works if the two defenses are intertwined: just treat this case as in 3(b)(ii)) and give the Opponent a repetition rank of $m := 2$. As a consequence of fact 3.1, this change does not affect the validity status of the game. □

THEOREM 4.3. *Let A be a formula of the intuitionistic fragment of the language. A is an intuitionistic validity if and only if $D_{\mathcal{E}}\langle \emptyset, \varnothing \; ; \; A\rangle$.*

PROOF: Immediate consequence of theorem 4.1. □

THEOREM 4.4. *Let A be a formula of the classical fragment of the language. A is a classical validity if and only if $D_{\mathcal{E}}\langle \emptyset, \varnothing \; ; \; A\rangle$.*

PROOF: ($\Leftarrow$) Immediate consequence of theorem 4.2.

($\Rightarrow$) We will show that if A is a theorem of an axiomatic calculus for classical logic, then $D_{\mathcal{E}}\langle \emptyset, \varnothing \; ; \; A\rangle$.[7] The calculus is introduced in [11] (see sections 19 and 29 therein).

[7] I am pretty confident that this theorem may be proved constructively, but for the time being this is the best I have. As an anonymous referee pointed out, the translation from classical to ecumenical strategies is not straightforward, because Last Duty First is

We begin by showing that all the axiom schemes of this calculus induce games in which the rules for $\to_c$ and $\vee_c$ are enough for preserving all the relevant duties in a winning strategy for the Proponent. In what follows, every play starts with:

	O			**P**	
				$\emptyset$; T	0
1	$m := 1$			$n := 2$	2
3	...				

Where T is the schema under consideration. The tabular format is avoided for economy of space.

1. $A \to_c (B \to_c A)$: The Opponent challenges move 0. The Proponent may defend or store $B \to_c A$ as a duty and challenge A. No information about pending duties gets lost. If A is atomic, the Proponent has the resources to Copy-Cat. In all cases, he wins.

2. $(A \to_c B) \to_c ((A \to_c (B \to_c C)) \to_c (A \to_c C))$: The Proponent defends all the challenges up to this point:

   O: $\emptyset$; $A \to_c B$ (challenge: 0) (3)

   P: $\emptyset$; $(A \to_c (B \to_c C)) \to_c (A \to_c C)$ (defense) (4)

   O: $\emptyset$; $A \to_c (B \to_c C)$ (challenge: 4) (5)

   P: $\emptyset$; $A \to_c C$ (defense) (6)

   O: $\emptyset$; A (challenge: 6) (7)

   If A cannot be defended, the Proponent should challenge it. He can store C and no information about pending duties gets lost.

   If A can be defended, then the Proponent may use it to challenge $A \to_c B$ (move 3) and $A \to_c (B \to_c C)$ (move 5). As a counter-challenge of A is not strategically relevant for the Opponent, her

a global restriction in the latter games. The current proof establishes that this restriction does not affect the proof of classical validities; but it does not provide a means to find the corresponding strategy in the ecumenical Game.

only option is to defend these moves. Now the Proponent is in a position to either challenge B or defend C. A similar situation obtains: if B cannot be defended, he must challenge it (he can store C and no information about pending duties gets lost). If it can be defended, he must challenge $B \rightarrow_c C$. The Opponent then grants C, and the Proponent can Copy-Cat her moves to properly defend move 6 and win.

3. $A \rightarrow_c (B \rightarrow_c (A \wedge B))$: If A or B can be challenged successfully, the strategic move for the Proponent is to do so (he may store $A \wedge B$, but this is strategically innocuous). If not, he will be able to successfully defend $A \wedge B$.

4. $(A \wedge B) \rightarrow_c A$: The Proponent stores A and challenges $A \wedge B$ by requesting A. After the Opponent defends, he is in a position to settle A and Copy-Cat the Opponent's moves and win.

5. $(A \wedge B) \rightarrow_c B$: Analogous to the previous one.

6. $A \rightarrow_c (A \vee_c B)$: The Proponent may choose to store B in the defense of $A \vee_c B$, but this is innocuous. The counter-challenge of A will force the Opponent to defend it: if it can be defended, the Proponent can Copy-Cat the moves and win; if not, the Proponent wins anyway.

7. $B \rightarrow (A \vee_c B)$: Analog to the previous one.

8. $(A \rightarrow_c C) \rightarrow_c ((B \rightarrow_c C) \rightarrow_c ((A \vee_c B) \rightarrow_c C))$: The Proponent defends all the challenges up to this point:

   O: $\varnothing$; $A \rightarrow_c C$ (challenge: 0) (3)
   P: $\varnothing$; $(B \rightarrow_c C) \rightarrow_c ((A \vee_c B) \rightarrow_c C)$ (defense) (4)
   O: $\varnothing$; $B \rightarrow_c C$ (challenge: 4) (5)
   P: $\varnothing$; $(A \vee_c B) \rightarrow_c C$ (defense) (6)
   O: $\varnothing$; $A \vee_c B$ (challenge: 6) (7)
   P: C ; $\vee_?$ (stores the defense of 6; challenge: 7) (8)

The Opponent may store one of the disjuncts and play the other; no information about pending duties gets lost. Anyway, this is strategically innocuous for her. Eventually she will have to assert C. Then the Proponent can settle C and Copy-Cat her moves to win.

9. $(A \to_c B) \to_c ((A \to_c \neg B) \to_c \neg A)$: The Proponent defends all the challenges up to this point:

   O: $\emptyset$; $(A \to_c B)$ (challenge: 0) (3)

   P: $\emptyset$; $(A \to_c \neg B) \to_c \neg A$ (defense) (4)

   O: $\emptyset$; $(A \to_c \neg B)$ (challenge: 4) (5)

   P: $\emptyset$; $\neg A$ (defense) (6)

   O: $\emptyset$; A (challenge: 6) (7)

   Move 7 is the last duty, so at this point of the play no defenses of previous moves can be performed. If A can be defended, then the Proponent has to challenge 3 and 5. Out of B and $\neg B$, the Proponent then must challenge the one that cannot be defended. If A cannot be defended, then the Proponent has to counter-challenge 7. In either case, the Proponent wins, and *Last Duty First* is innocuous.

10. $\neg\neg A \to_c A$: After move 3, where the Opponent asserts $\neg\neg A$, the Proponent stores A and counter-challenges $\neg\neg A$. After the defense of $\neg\neg A$, defenses of previous challenges are forbidden, because of Last Duty First. But, as the Proponent stored A, he is able to settle and the play continues as a defense of A. If it can be defended, the Proponent can Copy-Cat the Opponent's moves and win; if not, the Proponent wins anyway.

Finally, we prove that Modus Ponens is an external consequence of $D_{\mathcal{E}}$-games. Assume that the Proponent cannot defend $\emptyset$; B. If the Proponent has a winning strategy for $\emptyset$; $A \to_c B$, then he is able to defend $\emptyset$; B or to challenge $\emptyset$; A successfully. But we already know that he is not able to defend B. Therefore, he is able to challenge A successfully. In case he needs part of the defense of B to properly challenge A, he may store B as

a duty. No information about duties gets lost, and by contraposition, if $D_{\mathcal{E}}\langle\emptyset, \varnothing \; ; \; A\rangle$ and $D_{\mathcal{E}}\langle\emptyset, \varnothing \; ; \; A \to_c B\rangle$ then $D_{\mathcal{E}}\langle\emptyset, \varnothing \; ; \; B\rangle$.
The theorem then holds by induction on the length of $H_{\mathcal{C}}$-derivations. □

This confirms the ecumenical nature of the Game. Also, the fact that there are also validities with mixed connectives (such as P_7 above) suggests that this Game is equivalent with $\mathcal{LE}_p$. I am not interested in the proof of this result though, so I state it only as a conjecture and give a sketch of the proof.

Conjecture 4.5. Let $\{A_1, \ldots, A_n, B\}$ be a set of ecumenical formulas. The Proponent has a winning strategy in the game

$$D_{\mathcal{E}}\langle\{\varnothing \; ; \; A_1, \ldots, \varnothing \; ; \; A_n\}, \varnothing \; ; \; B\rangle$$

If and only if $\varnothing \; ; \; A_1, \ldots, \varnothing \; ; \; A_n \vdash_{\mathcal{LE}_p} \varnothing \; ; \; B$.

Proof (Sketch): In [20] there is an algorithm to translate intuitionistic core strategies into Natural Deduction derivations. This algorithm should be adaptable without problems to our systems. □

What really interests me is the philosophical reading of the Game, to which I turn in the next section.

5. The dialogical meaning of states

In this section I focus on the semantic understanding of ecumenicality in the dialogical framework.

At the beginning of the last section we stressed that, to recover the meaning of classical disjunction and conditional in a game with intuitionistic rules, we need to allow exceptions to the *Last Duty First* condition. But we also commented that a duty means a 'challenge without defense'. This means that in both games a player can always (counter-)challenge a move made by the other. So one way to explicate the particle rules of the classical connectives is to take them as rules that transform defenses into counter-challenges.

Consider the intuitionistic rules and the following play:

P_{11}:

	O			**P**	
				$\neg(\neg A \wedge \neg B)$	0
1	$m := 1$			$n := 2$	2
3	$\neg A \wedge \neg B$	0		-	
5	$\neg A$		3	$\wedge_L$	4
	-		5	A	6

In this game the thesis cannot be defended, because it is a negation. But the Proponent can choose to defend either A or B by means of requesting from the Opponent either $\wedge_L$ or $\wedge_R$ (move 4). And most importantly, he can do so *at any stage of the game*, since this is a counter-challenge and not a defense. This shows that there is a certain symmetry between defending $A \vee_c B$ and $\neg(\neg A \wedge \neg B)$. On the other hand, consider again the intuitionistic rules and the following play:

P_{12}:

	O			**P**	
				$\neg(A \wedge \neg B)$	0
1	$m := 1$			$n := 2$	2
3	$A \wedge \neg B$	0		-	
5	$\neg B$		3	$\wedge_R$	4

For the remaining of this game, the Proponent is free to either challenge $A \wedge \neg B$ again (move 3) and ask for A or assert B as a challenge to 5 and then defend it. Both moves are compatible with *Last Duty First*, and therefore can be performed at any stage of the game. With these observations in sight, let's consider P_8 again. The thesis in that game was: $\emptyset\ ;\ (p \rightarrow_c q) \vee_c (q \rightarrow_c r)$. Compare that play with the following one:

P_{13}:

	O			**P**	
				$\neg(\neg\neg(p \wedge \neg q) \wedge \neg\neg(q \wedge \neg r))$	0
1	$m := 1$			$n := 2$	2
3	$\neg\neg(p \wedge \neg q) \wedge \neg\neg(q \wedge \neg r)$	0	-		
5	$\neg\neg(p \wedge \neg q)$		3	$\wedge_L$	4
	-		5	$\neg(p \wedge \neg q)$	6
7	$p \wedge \neg q$	6		-	
9	$\neg\neg(q \wedge \neg r)$		3	$\wedge_R$	8
	-		5	$\neg(q \wedge \neg r)$	10
11	$q \wedge \neg r$	10		-	
13	q		11	$\wedge_L$	12
15	$\neg q$		7	$\wedge_R$	14
	-		15	q	16
17	*ut? q*	16		*sic*(13)	18++

The two crucial moves here are 8 and 14, where the Proponent comes back to request what he needs to win. But these returns are in accordance with *Last Duty First*, since they are challenges, not defenses.[8]

In the same line, there is a suggestive relation between states

$$A_1, \ldots, A_n \ ; \ B$$

and the proposition

$$(\neg A_1 \wedge \cdots \wedge \neg A_n) \rightarrow_i B$$

[8] As a matter of fact, this definition:

$$\begin{aligned} A \vee_c B &= \neg(\neg A \wedge \neg B) \\ A \rightarrow_c B &= \neg(A \wedge \neg B) \end{aligned}$$

Corresponds to part of the Glivenko-Gödel negative translation [10]. To have a fully working translation, you need to apply double negations in the atomic fragment of the language. This translation (and its suboptimal handling of double negations) was the starting point for the design of LC, the logical system that inspired $\mathcal{LE}_p$ [9].

Compare P_9 with the following one:

P_{14}:

	O			**P**	
				$\neg p \to \neg(p \wedge \neg q)$	0
1	$m := 1$			$n := 2$	2
3	$\neg p$	0		$\neg(p \wedge \neg q)$	4
5	$p \wedge \neg q$	4		-	
7	p		5	$\wedge_L$	6
	-		3	p	8
9	*ut? p*	8		*sic*(7)	10++

In both plays the strategy is the same: wait for the Opponent to challenge $p \to_c q$ ($\neg(p \wedge \neg q)$) (P_9, move 3; P_{14}, move 7) and then settle the previous duty (P_9, move 4; P_{14}, move 8).

These observations suggest the correct interpretation of states as assertions. When a player asserts

$$A_1, \ldots, A_n \ ; \ B$$

We can read this as: "B, or else A_1 or... or A_n". Note how we invert the order in the reading (B first, the A's after).

Two tempting confusions have to be expelled at this point. The first is the suggestion to read the state as a sequent: "B provided you grant me A_1 and... and A_n". It is clear from all the examples that this reading is incorrect. Consider P_{10}. The thesis is the state $p \ ; \ \neg p$, and we saw that this is a valid state. Yet it is obviously mistaken to read it as: "$\neg p$, provided you grant me p". The reading that I offer is more accurate: "$\neg p$, or else p".

Yet here comes the second confusion. It may seem that the "or else... or..." used to read the states is a classical disjunction in the metalanguage. This is heavily suggested by the fact that something like the excluded middle, ("$\neg p$, or else p") is valid in this logic. But this is also incorrect, because this disjunction is not commutative, as the following example shows:

P_{15}:

	O			**P**	
				$\neg p$; p	0
1	$m := 1$			$n := 2$	2
3	*ut?* p	0			
				$\neg p$	4
5++	p	4		-	

The Proponent cannot defend move 3, because of *Last Duty First.* This also confirms that, when we showed that $A_1, \ldots, A_n$; B can be approximatively translated into

$$(\neg A_1 \wedge \cdots \wedge \neg A_n) \to B$$

The main conditional in this formula is intuitionistic. The reason is that in classical logic we have that

$$\neg A \to B \vdash \neg B \to A$$

But not in intuitionistic logic.

Another important consequence of the non commutative nature of this reading of states is that once a player chooses a way to defend a classical disjunction (or conditional), this decision cannot be reversed during the play. In fact, suppose that the game develops in this way:

1. **X** asserts: $A \vee_c B$
2. **Y** requests: $\vee_?$ (challenge)
3. **X** asserts: A ; B (defense)
4. **Y** *challenges* B
5. **X** asserts: B ; A (?)

Move 5 in this example is not justified by any of the rules. If X makes this move as a settling of A, then B is lost, not stored; and he cannot make this move as a second defense of move 3, since his last duty is the challenge in move 4. So, in general, during a play a player cannot change their last current move with one of their duties.

Despite this evident departure from classical logic, the 'or else' reading is still a reasonable disjunction. In particular, because both $D_{\mathcal{E}}\langle\{\emptyset\ ;\ A\}, B\ ;\ A\rangle$ and $D_{\mathcal{E}}\langle\{\emptyset\ ;\ A\}, A\ ;\ B\rangle$ are valid games:

P_{16} :

	O			**P**	
0.1	$\emptyset$; p			q ; p	0
1	$m := 1$			$n := 2$	2
3	*ut? p*	0		*sic*(0.1)	4++

P_{17} :

	O			**P**	
0.1	$\emptyset$; p			p ; q	0
1	$m := 1$			$n := 2$	2
3	*ut? q*	0			
				$\emptyset$; p	4
5	*ut? p*	0		*sic*(0.1)	6++

States in $D_{\mathcal{E}}$ represent primarily the assertion of the current state, and duties represent alternative assertions offered secondarily by the player. This is why, in principle, we do not want the Opponent to be able to deliver duties in the concessions, because the point of making concessions is to deliver an opening basis for the reasoning. Consider the following dialogue:

P_{18} :

	O			**P**	
0.1	p, q ; r			$\emptyset$; r	0
1	$m := 1$			$n := 2$	2
3	*ut? r*	0		*sic*(0.1)	4
5	p ; q				
(3)	*ut? r*	0		*sic*(0.1)	6
7++	$\emptyset$; p				

The Proponent loses this game, but because of a ‘dishonest’ move of the Opponent: she begins by conceding r, and the Proponent then asserts r too. But from move 5 onwards the game is no longer a ‘dialogue’: the Opponent asserts q and then p, but in this case these are not concessions, because there is no thesis. (Move 6 in the previous example is just a repetition of the previous move.) An instance of a dialogue with this form is as equally odd:

Socrates:	What is good is beautiful. Or else, Spartans are greedy.
Citizen:	What is good is beautiful, indeed.
Socrates:	Why do you say that?
Citizen:	You said so.
Socrates:	OK, then Spartans are greedy.
Citizen:	...

If the Citizen replies with "Spartans are greedy, indeed", we would say that a new debate has started. Socrates, as the Opponent, has the role of giving concessions for the Citizen (as Proponent) to defend a thesis, and not for the sake of asserting things.

Before the end of this section I want to stress some relevant similarities of this work with a previous one. In [2], Blass provides a dialogical interpretation of Linear Logic, where multiplicative and additive connectives are differentiated by the possibility of re-challenging and re-defending an assertion. In particular, additive disjunction corresponds to the disjunction that cannot be re-defended by the Proponent, and multiplicative disjunction the one that can. So, in our proposal, the first would correspond to $\vee_i$ and the second to $\vee_c$.

This insight makes it tempting to reintroduce $D_{\mathcal{E}}$ as a dialogue between formulas (not states), with the standard particle rules and the following alternative version of SR1e:

SR1e': (Alternative Ecumenical) Development rule Players move alternatively. Each move is a challenge or a defense over the moves of the other, in accordance to the particle rules. A player can repeat a move up to as many times as the number (s)he chose as her/his repetition rank.

Last Duty First: A player can only defend an intuitionistic formula if it is the last unanswered intuitionistic formula challenged by the contender.

(Recall that a formula is intuitionistic if its main connective is neutral or intuitionistic.) As in this Game we lost the explicit track of duties, this dialogue may be seen as the dialogical counterpart of EL, Prawitz's original ecumenical calculus.

6. Conclusions

Logical ecumenism is the search for systems where two logics can coexist, sharing vocabulary and avoiding collapses between them. The literature has focused mainly on ecumenism between classical and intuitionistic logic, and several calculi of Natural Deduction and Sequents have been proposed. In this paper I contributed to this project with a dialogical variant of the ecumenical system of [15]. This Game, $D_{\mathcal{E}}$, utilizes an extension of the intuitionistic structural rules (with the *Last Duty First* clause) that permits to handle classical disjunctions and conditionals. I showed that this is indeed an ecumenical dialogical system, where classical formulas and intuitionistic formulas can be validated without collapses between them. It is also speculated that it can be proved to be equivalent to $\mathcal{LE}_p$.

This proposal internalizes the difference between the two logics at the level of the connectives (the *particle* rules), while this is usually presented at the level of plays (the *structural* rules). In fact, if we take the fragment of games $D_{\mathcal{E}}$ without intuitionistic connectives ($\to_i, \vee_i$) one obtains a Game for classical validities with the *Last Duty First* clause, the distinctive feature of intuitionistic games (theorem 4.4). This shows that Dialogics is well suited to account for the distinction between classical and intuitionistic logic both as a variance in meaning and as a variance in structure.

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