Logarithmic convexifying of polynomials

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SUMMARY OF THE DOCTORAL DISSERTATION

One of the fundamental problems of analysis, technology, economics and other branches of science is the search for minima and critical points of functions. One of the methods leading to this goal is the deformation of a given function to a convex function, searching for critical points of this deformation and iterating this process. Reducing a function to a convex or strongly convex function leads to easy determination of critical points and minima of this deformation. These are the exact points where the gradient is zero.The classic approach to convexifying of a function $f : \mathbb{R}^n \to \mathbb{R}$ on bounded and convex sets is to add a strongly convex function $b : \mathbb{R}^n \to \mathbb{R}$ such that $f + b$ is a strongly convex function on this set (see for instance: A.N.Tikhonov, W.B.Liu, C.A.Flouudas and S.Zlobec for quadratic function $b(x) = \gamma|x|^2$, $\gamma > 0$). We describe this more precisely.

Let $b : \mathbb{R}^n \to \mathbb{R}$ be a $\mathscr{C}^k$ class μ-strongly convex function, $k \geq 2$, $\mu > 0$. Let $X \subset \mathbb{R}^n$ be a compact and convex set, let $f : \mathbb{R}^n \to \mathbb{R}$ be a function of class $\mathscr{C}^k$ and let $D \in \mathbb{R}$ be a positive number such that

$$|\partial_\beta^2 f(x)| \leq D \quad \text{for } \ x \in X \text{ and } \beta \in S^{n-1},$$

where S^{n-1} the unit sphere in $\mathbb{R}^n$, i.e., $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$, and $\partial_\beta^2 f(x)$ is the second order derivative of f in the direction β at x. One can directly check that

For any $\xi \in \mathbb{R}^n$ and $N > D/\mu$, the function $\phi_{N,\xi} : \mathbb{R}^n \to \mathbb{R}$ defined by

$$\phi_{N,\xi}(x) = Nb(x - \xi) + f(x), \quad x \in \mathbb{R}^n,$$

is strongly convex on X (more precisely $N\mu - D$-strongly convex).

We will compare the above approach to convexifying of a function that takes only positive values with another approach of multiplying it by a power of a strongly convex function . The latter approach was proposed in 2015 by K. Kurdyka, S. Spodzieja and continued by K. Kurdyka, K. Rudnicka, S. Spodzieja. More precisely, by K. Kurdyka, S. Spodzieja a positive function f of class $\mathscr{C}^2$ is convex on a compact and convex set $X \subset \mathbb{R}^n$ by multiplying the function f by $(1 + |x|^2)^N$ for some N, and by K. Kurdyka, K. Rudnicka, S. Spodzieja – by multiplying the function f by $\exp(N|x|^2)$.

So, we generalize these results and show that: If $X \subset \mathbb{R}^n$ is a compact and convex set and $f : X \to \mathbb{R}$ is a function of the class $\mathscr{C}^2$ that takes only positive values, then for any strongly convex function $b : \mathbb{R}^n \to \mathbb{R}$ there is $N_0 > 0$ such that for each $N \geq N_0$ and $\xi \in X$, the function

$$\varphi_{N,\xi}(x) = b^N(x - \xi)f(x), \quad x \in \mathbb{R}^n, \tag{1}$$

is strongly convex on X.

In the case where the function f is a polynomial, the exponent N can be estimated efficiently in terms of the radius of the set X (i.e., $\sup\{|x| : x \in X\}$) of the modules of the polynomial coefficients and $m = \inf\{f(x) : x \in X\}$ 2.2.2). Therefore, in the case of positive functions on compact and convex sets, both adding a multiple of a strongly convex function to the function and multiplying it by the power of such a function have a similar effect, but the first method uses a smaller coefficient N. If we additionally assume that b is a logarithmically strongly convex function (i,e., $\ln b$ is a strongly convex function), then $\varphi_{N,\xi}$ is also a logarithmically strongly convex function. In the case when X is a semialgebraic, compact and convex set, the coefficients of the polynomials describing X and the coefficients of the polynomial f are integers (or rational numbers), the exponent N can be determined fully efficiently. These theorems are obtained using the result of G. Jeronimo, D. Perrucci, E. Tsigaridas.

We will compare the classical approach to problem convexifying of a function with the above for any strongly convex function $b : \mathbb{R}^n \to \mathbb{R}$ and a positive function on a closed and convex (not necessarily bounded) set. We convexiying of a function f by multiplying it by $b(N(x-\xi))$ instead of $b^N(x-\xi)$. This approach simplifies some calculations.

Under the above assumptions, let

$$N(|\xi|) = \frac{D}{\mu}\left(|\xi| + 1 + \sqrt{\frac{\alpha}{\mu}}\right)^{\alpha} + 1$$

Then, for any $\xi \in \mathbb{R}^n$ a function $\phi_\xi : \mathbb{R}^n \to \mathbb{R}$ defined by

$$\phi_\xi(x) = N(|\xi|)b(x-\xi) + f(x), \quad x \in \mathbb{R}^n,$$

is strongly convex on X (more precisely μ-strongly convex).

In particular, the assertion of the above can be obtained for the function $\psi_\xi(x) = Nb(\xi)b(x-\xi) + f(x)$, for a sufficiently large constant N.

In the case when we obtain the convexifying of a function by multiplying it by $x \mapsto b(N(x-\xi))$, where b is a strongly convex function or logarithmically strongly convex, we must of course assume that the function only takes positive values on X.

The main difficulty in applying the above fact is the estimation of the constant C. This difficulty can be overcome when we convexifying of the polynomial. More specifically, let $f \in \mathbb{R}[x]$, where $x = (x_1, \ldots, x_n)$ is a system of a variable, be a polynomial of degree d, and let $f = f_0 + \cdots + f_d$, where f_j is a homogeneous polynomial of degree j or zero. Let $f_{d*} = \min_{|x|=1} f_d(x)$. Obviously, $f_{d*} > 0$ if and only if the leading form f_d of the polynomial f takes only positive values $\mathbb{R}^n \setminus \{0\}$.

Then we can get a convexifying of polynomial f by multiplying it by the function $b(N(x-\xi))$, $\xi \in \mathbb{R}^n$.

We deal with iterations of a mapping that assigns to each point the only critical point of the convexifying of function f.

We assume that $X \subset \mathbb{R}^n$ is a convex and compact set, and that the function $b : \mathbb{R}^n \to \mathbb{R}$ is strongly convex of class $\mathscr{C}^k$, $k \geq 2$ such that $0 = \operatorname{argmin}_{\mathbb{R}^n}$. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function of the class $\mathscr{C}^k$. Then there is a number $N \geq 1$ such that for any $\xi \in \mathbb{R}^n$, the function

$$\phi_{N,\xi}(x) = Nb(x - \xi) + f(x)$$

is strongly convex on X. We define a mapping

$$\kappa_N : X \ni \xi \mapsto \operatorname{argmin}_X \phi_{N,\xi} \in \mathbb{R}^n.$$

If

$$X_{f \leq r} := \{x \in \mathbb{R}^n : f(x) \leq r\} \subset X,$$

we show that: If $f : \mathbb{R}^n \to \mathbb{R}$ is a semialgebraic function of class $\mathscr{C}^2$ then for any $\xi \in X_{f \leq r}$, the limit point $\lim_{\nu \to \infty} \kappa_N^\nu(\xi)$ exists and belongs to Σ_f of critical values of f.

Assuming that the function f is semialgebraic, the above theorem allows us to define the mapping

$$\kappa_{N,*} : X_{f \leq r} \to \Sigma_f \cap X_{f \leq r} \, ,$$

given by $\kappa_{N,*}(\xi) = \lim_{\nu \to \infty} \kappa_N^\nu(\xi)$.

Assuming that the function f has only one critical value on $X_{f \leq r}$, we show that the mapping $\kappa_{N,*}$ is continuous.

We transfer some properties of the mapping κ_N to the case of unbounded sets. Among other things, in the case where the convexification of the function f is of the form

$$\psi_\xi(x) = N(\xi)b(x - \xi) + f(x), \qquad (\xi, x) \in \mathbb{R}^n \times \mathbb{R}^n$$

and for a convex and closed set X,

$$\kappa(\xi) = \operatorname{argmin}_X \psi_\xi \in X,$$

At the end, we deal with the convergence problem of the sequence $\frac{\xi_\nu}{|\xi_\nu|}$, where $\xi_\nu = \kappa_N^\nu(\xi)$, $\nu \in \mathbb{N}$, and $\xi \in \mathbb{R}^n$, the sequence $|\xi_\nu|$ is strictly decreasing from a certain point.

Part of the results of this work has already been published in the work of A.N. Abdullah, K. Rosiak, S. Spodzieja. This applies to point 2.2 and Chapter 6.