Sergiy Yakovlev, Oksana Pichugina, Liudmyla Koliechkina

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Abstract

The monograph is dedicated to exploring combinatorial point configurations derived from mapping a set of combinatorial configurations into Euclidean space. Various methods for this mapping, along with the typology and properties of the resultant configurations, are presented. In addition, the study revolves around combinatorial polytopes defined as convex hulls of combinatorial point configurations. The primary focus lies in examining multipermutation and partial multipermutation point configurations alongside their associated combinatorial polytopes known as multipermutohedra and partial multipermutohedra. Our theoretical contributions are substantiated through the proof of theorems and supporting auxiliary statements. Examples and illustrations are included to enhance the comprehension of the material.

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Introduction

The term configuration derives from the late Latin word "configuratio", meaning shaping or arrangement. Thus, configuration refers to the outward appearance, shape, and relative position of items or their constituent parts.

In mathematics, particularly in the research domain of projective geometry, configuration typically denotes a specific arrangement of a set of points and lines on a plane or surfaces in space [11, 21, 63]. In [17, 88], point configurations are treated as collections of points in Euclidean space. The focus is on studying the arrangement and properties of these points in the given space.

The modern classification of such configurations was provided by B. Grunbaum [21], distinguishing their three main classes: topological, geometric, and combinatorial. Grunbaum typology framework comprehensively provides typology configurations concerning their underlying mathematical structures.

A configuration is topological if it represents an arrangement of pseudo lines within the projective plane and their corresponding intersection points.

Geometric configurations are those in which the lines are considered in either Euclidean or projective plane, and the points arise from the intersections of these lines. The primary focus of research in geometric configurations theory revolves around two challenging problems. The first is the selection and establishment of the existence of different classes of configurations. The second is determining the count of non-isomorphic configurations within the respective class.

A complete review of publications on geometric configurations can be found in [19].

Initially, the study of geometric configurations involved geometric methods, where the configurations were built on a plane. However, V. Martinetti introduced a combinatorial approach to investigating geometric configurations [35]. Concurrently, the problem of determining the existence and enumeration of configurations was addressed through a recursive approach, building upon solutions for configurations with fewer points. This recursive methodology proved valuable in tackling the challenges posed by varying-size configurations.

H. Schroter made an important observation that not all of V. Martinetti's configurations could be constructed geometrically on a plane. Consequently, H. Schroter shifted focus away from the geometric aspect of configurations, emphasizing the combinatorial perspective instead [69]. He introduced the concept of a configuration in terms of "elements" and "columns", as opposed to "points" and "lines". In this new formulation, a configuration was represented by a binary matrix, where the columns corresponded to straight lines, and the rows represented their intersection points. The matrix determines the assignment of points to lines while its elements indicate whether a given point lies on a specific line. An arbitrary binary matrix, not necessarily square, is considered a configuration representing an equivalence class of matrices obtained by permuting rows and columns [67].

Combinatorial configurations predominantly focus on geometric aspects while treating points and lines as abstract sets within a combinatorial context. The study of combinatorial configurations became a separate research domain in discrete mathematics with C. Berge as its founder.

According to C. Berge, a combinatorial configuration is understood as mapping an initial set of elements, which can be of any nature, onto a finite abstract resulting set with a certain structure [6]. This mapping is subject to constraints that govern the relationships and arrangement of elements. By studying these configurations, researchers explore the interactions between constraints, structure, and mappings in combinatorics.

The combinatorial configurations are studied in publications [7,14,24–27,33,34,71,81,82], where there is an assumption that the initial and resulting sets are finite. In [24, 27, 71], the concept of a combinatorial configuration was further developed by relaxing the requirement that the resulting set is finite, allowing it to be countable. As a result, the definition of

combinatorial objects and combinatorial objects of a certain order was introduced. This made it possible to significantly expand the range of real problems that can be formulated and analyzed using such a concept.

Any combinatorial configuration can be associated with a certain point in Euclidean space. Then, the set of combinatorial configurations corresponds to a finite point configuration, which we call a combinatorial point configuration. It is obvious that there are an infinite number of such mappings.

We will, therefore, require that the combinatorial point configuration have a special structure. In this regard, we will use the concept of Euclidean combinatorial sets introduced by Yu. Stoyan in the preprints [74, 75] and further studied in [53, 59, 60, 76–81]. In accordance with Yu. Stoyan, elements of Euclidean combinatorial sets differ in constituent items or their order. A mapping is proposed for such sets, called immersion into Euclidean space. As a result of immersion, finite point configurations with specific properties are formed. This enables the combinatorial polytope theory to study various classes of finite point configurations.

In combinatorics, the basic combinatorial configurations are permutations and partial permutations. Evidently, permutation and partial permutation sets are Euclidean combinatorial. Therefore, the monograph focuses on the finite point configurations corresponding to these sets. We singled out the classes permutation point configurations, multipermutation point configurations, partial permutation point configurations, and partial multipermutation point configurations then studied their properties.

The monograph is organized as follows.

The first chapter presents the basic definitions of sets, multisets, logical operations over them and mappings. The concept of a finite point configuration, as a collection of singleton sets in Euclidean space, is introduced. Methods are proposed for decomposing finite point configurations into planes and surfaces. A serface-located set is defined, and special classes of spherically-located and superspherically-located sets are singled out. The decomposition of finite point configurations on various surfaces are carried out. The properties of combinatorial polytopes as convex hulls of finite point configurations are described. Classes of multilevel and vertex-located sets are identified. Methods for functional-analytical description of some

classes of finite point configurations are offered.

In the second chapter, an in-depth exploration of the properties of different types of finite point configurations is undertaken, focusing on vertex-located and surface-located sets. An important class of polyhedral-surface sets is introduced, and its subclasses are singled out. Moreover, we establish a relationship between finite point configurations and vertex-located, surface-located, and polyhedral-surface sets. The chapter also delves into the properties of multilevel sets and the corresponding multilevel polytopes, enabling the exploration of their specific characteristics and geometrical properties. Furthermore, the properties of a finite point configurations formed as a result of logical operations over them are investigated. This chapter comprehensively studies the properties and relationships between different types of finite point configurations, including vertex-located, surface-located, and polyhedral-surface sets. It also sheds light on the behavior of configurations obtained under logical operations of other suvh configurations.

The third chapter examines the concept of combinatorial configurations, as defined by C.Berge. Combinatorial configurations are characterized as mappings that transform an initial set of items of any nature onto a finite resulting set with a specific structure, subject to a given set of constraints. Next, we explore sets of combinatorial configurations and introduce their mapping into Euclidean space. In this way, finite point configurations are formed called combinatorial point configurations. We have defined a mapping for a set of combinatorial configurations such that the corresponding combinatorial point configuration has the properties described in Chapter 2, particularly vertex-located or polyhedral-surface-located. It is justified that so-called Euclidean combinatorial sets under specific mapping into Euclidean space have this property.

A typology of combinatorial point configurations is proposed. In particular, permutation point configurations, multipermutation point configurations, partial permutation point configurations, and partial multipermutation point configurations are introduced. Examples are given to illustrate the combinatorial structure of such configurations. Combinatorial point configurations generated by sets of all permutations and multipermutations, partial permutations, partial multipermutations, and unbounded partial permutations are called

entire ones.

Chapters 4 and 5 focus on an in-depth exploration of two main classes of combinatorial point configurationsets: entire multipermutation and partial multipermutation point configurations. The cardinality, symmetry and multilevelness of these sets are explored. We delve into the concept of combinatorial polytopes as convex hulls of combinatorial point configurations. The dimension of polytopes and criteria for identifying their vertices are analyzed, methods of analytical description are proposed, and their equivalence in a certain sense is investigated. The properties of vertex- and surface-located combinatorial point configurations are studied by considering the spatial relationships between the vertices of polytopes and the circumscribed surfaces around them. Relevant illustrations and explanations accompany the theoretical material.

In conclusion, the main contributions of this monographic study are summarized. Additionally, potential promising avenues for future developments are outlined.

When describing the main material, our approach involved using conventional and specific terminology in accordance with the provided list of references. Furthermore, at the outset of each chapter, we list the main sources that formed the basis of its content.

Chapter 1

Background of finite point configurations

In this chapter, we study finite sets of isolated points in n-dimensional Euclidean space called finite point configurations (FPC) [17, 88]. When presenting the main material of the monograph, definitions of set and multiset are used in accordance with the generally accepted terminology. The classification of finite point configurations is carried out using the theory of convex surfaces. In this case, the concepts of surface-located and vertex-located sets introduced in [39, 40, 91] were used. The decomposition of finite point configurations is based on the papers [1, 2, 17]. Approaches to the analytic description of finite point configurations are proposed based on the so-called f-representations [39–42, 48, 50]. When considering convex hulls of finite point configurations, the theoretical principles of polytopes [4, 20, 23, 65, 68, 106] were applied.

1.1 Sets and multisets, order relations and mappings

A concept of a set is initial and has no definition. A set is believed to be a collection of elements united by a common feature and considered a whole.

As a rule, capital letters of the Latin alphabet are used to designate sets, and the corresponding small letters are used to denote their elements. For example, if a collection of elements of a set X is characterized by a feature Ω, then the notation $X = \{x \mid \Omega\}$ is used, and the feature Ω itself is called a characteristic property of X.

Depending on the number of elements in sets, finite, countable, and continuum cardi-

nality sets exist. For finite sets, the notation is used:

$$X = \{x_1, ..., x_\eta\}, \tag{1.1}$$

and for counting ones:

$$X = \{x_1, ..., x_\eta, ...\}.$$

To specify the number of elements of a finite set X, we will use the notation $|X|$, i.e. in the representation (1.1), $\eta = |X|$.

We introduce the notation $J_n = \{1, ..., n\}$ for a set of the first n natural numbers, and for finite sets, we will further use the notation $X = \{x_i\}_{i \in J_n}$.

Using the concept of a set is not convenient enough if a set is considered which elements are different in themselves while coinciding in the chosen characteristic property. For example, the sets $A = \{a, a, b, c, c\}$ and $A = \{a, b, c\}$ are equivalent, and $|A| = 3$. If the collection contains the same elements and is required to consider their multiplicity, then we will use the concept of a multiset.

By multiset, we mean a collection of elements

$$G = \{g_1, ..., g_\eta\}, \tag{1.2}$$

not necessarily different with a common feature.

Various elements of a multiset G form a set called an underlying set

$$S(G) = \{e_1, ..., e_k\}. \tag{1.3}$$

To specify a multiset, multiplicities of its elements have to be specified. Let η_i be a multiplicity of e_i ($i \in J_k$). A vector of multiplicities

$$[G] = (\eta_1, \eta_2, ..., \eta_k), \ \sum_{i=1}^{k} \eta_i = \eta$$

is called the primary specification of the multiset G.

On the other hand, the multiset G can be represented as

$$G = \{e_1^{\eta_1}, ..., e_k^{\eta_k}\}\,. \tag{1.4}$$

Thus, there are several ways to represent a multiset.

Further, we will use the notation $\{.\}$ for unordered sequences of elements and $[.]$ (or $\langle . \rangle$) for ordered sequences of elements; $(.)$ will be utilized for vectors, i.e. ordered sequences of real numbers.

Note that a multiset G coinciding with its underlying set $S(G)$ is a set, i.e.

$$G = S\,(G)\,,\ \ [G] = (1^{\eta})\,.$$

Moreover, in order to emphasize that among the elements of G, our interest in distinct ones, we will move to consideration of $S(G)$.

In these notations: if G is a multiset of the form (1.4), that $\langle G \rangle$ is an ordered sequence of its elements of the form $\langle g_1, ..., g_\eta \rangle$; (G) is a vector $(g_1, ..., g_\eta) \in \mathbb{R}^\eta$.

Note that the transition operation from G to $\langle G \rangle$ is always defined, while the transition from G to (G) is defined only for a numerical multiset G.

In this case, for numerical multisets, we will distinguish between the representations $\langle G \rangle$ and (G), assuming that they consist of elements of different natures allowing a one-to-one mapping between them.

The following basic operations are defined on multisets: union, intersection, complement, symmetric difference, arithmetic sum, direct product, etc.

A union of multisets A and B is the multiset $C = A \cup B$ consisting of all elements present in at least one of these multisets, where the multiplicity of each element in C is equal to the maximum multiplicity of elements in A and B. In other words, a pairwise comparison of each element of the multisets A and B is performed, and an element with the highest multiplicity is selected in each pair.

An intersection of multisets A and B is the multiset $C = A \cap B$ consisting of all elements present in each of the multisets, and the multiplicity of each element in C is equal to

the minimum multiplicity of elements of A and B. In this case, elements of the multisets A and B are pairwise compared, and the element with the smallest multiplicity is selected in each pair. In this case, elements of multisets A and B is compared in pairs, and in each pair the element with the smallest multiplicity is selected.

An arithmetic sum of multisets A and B is the multiset $C = A \bigoplus B$ consisting of all elements present in at least one of these multisets, and the multiplicity of each element in C is equal to the sum of its multiplicities in A and B.

Further, we will use the inverse operation of transition from ordered sequences and vectors to a multiset of their coordinates. For this, we will use the notation $\{x\}$. For example, if x is a vector of the form $(x_1, ..., x_n)$, then a set of its coordinates is $\{x_1, ..., x_n\}$.

Similarly to sets, one can introduce various types of relations on multisets.

A binary relation R on a multiset G is called a <u>nonstrict partial order relation</u> if it satisfies the following conditions:

- reflectivity: $\forall x \in G : xRx$;
- antisymmetry: $\forall x, y \in G : xRy \wedge yRx \Rightarrow x = y$;
- transitivity: $\forall x, y, z \in G : xRy \wedge yRz \Rightarrow xRz$.

A nonstrict partial order relation R, denoted by $\preccurlyeq$, is called a <u>linear order</u> if the condition $\forall x, y \in G$ $(xRy \vee yRx)$, where a multiset G where the linear order relation is introduced, is called <u>linearly ordered</u>.

An antireflective, antisymmetric, and transitive order relation is called <u>strict</u> and is denoted by the symbol $\prec$. A set G on which a strict order is introduced (i.e. $\forall x, y \in G$ $(x \prec y \vee y \prec x)$) is called <u>strictly ordered</u>.

Suppose that G is linearly ordered, i.e.

$$g_i \preceq g_{i+1},\ i \in J_{\eta-1}, \tag{1.5}$$

then its underlying set $S(G)$ will be strictly ordered, i.e.

$$e_i \prec e_{i+1},\ i \in J_{k-1}. \tag{1.6}$$

If $S(G)$ is a numerical set, i.e.

$$e_i \in \mathbb{R}^1,\ i \in J_k,$$

then, since any two numbers $x, y \in \mathbb{R}^1$ are comparable and are in one of three relations $x > y$, $x < y$, or $x = y$, the conditions (1.5), (1.6) are always met, and (1.5) becomes

$$g_i \leq g_{i+1},\ i \in J_{\eta-1}, \tag{1.7}$$

and (1.6) becomes

$$e_i < e_{i+1},\ i \in J_{k-1}.$$

The defining principle of the concept of combinatorial configurations and their classification is the use of different types of set mapping. Therefore, we give the following definitions.

It is assumed that the mapping χ of the set B into the set A is given if each element $b \in B$ is matched by a unique element $a \in A$. The correspondence between a and b is written as equality $a = \chi(b)$, and the mapping χ is denoted by $\chi : B \to A$.

In this case, the element a is an image of the element b under the mapping χ, and the element b is a preimage of the element a denoted as $b = \chi^{-1}(a)$.

Similarly, a set A is an image of a set B and is denoted by $A = \chi(B)$, and a set B is a preimage of a set A and is denoted by $B = \chi^{-1}(A)$.

A mapping $\chi : B \to A$ is surjective if any element $a \in A$ has at least one preimage $b \in B$ under this mapping. In other words, for each $a \in A$ there exists $b \in B$ such that $a = \chi(b)$. If χ it is a surjective mapping, then $\chi(b) = a$. For finite sets A and B, surjective mapping $\chi : B \to A$ means that $|B| \geq |A|$.

A mapping $\chi : B \to A$ is injective if for any $b', b'' \in B$, such that $b' \neq b''$, holds $\chi(b') \neq \chi(b'')$. If B and A are finite, then an injective mapping $\chi : B \to A$ means that $|B| \leq |A|$.

A mapping $\chi : B \to A$ is bijective if it is both surjective and injective. Therefore,

a bijective mapping χ means that under the condition $a = \chi(b)$, for each $a \in A$, $b \in B$ is uniquely determined. In this case, a bijective mapping $\chi : B \to A$ establishes a one-to-one correspondence between the sets B and A. We have $|B| = |A|$ if B and A are finite,

1.2 Finite point configurations and their decompositions

Let E be a finite set of isolated points in $\mathbb{R}^n$. Denote:

$$E = \{x_1, ..., x_{n_E}\} \subset \mathbb{R}^n. \tag{1.8}$$

where n_E is the cardinality of E.

Clearly, the set E is a finite point configuration.

Let $x_i = (x_{ij})^\top_{j \in J_n}$, $i \in J_{n_E}$. We introduce a set of values of coordinates of E-points:

$$X^j = \{e_{j1}, ..., e_{jk_j}\} = S(\{x_{ij}\}_{i \in J_{n_E}}),\ j \in J_n, \tag{1.9}$$

forming from them the n-dimensional finite lattice:

$$X = \bigotimes_{j=1}^{n} X^j.$$

It is easy to see that the points of E are the grid nodes:

$$\forall x \in E \quad x \in X.$$

Remark 1.1. Without loss of generality, we assume that $k_j \geq 2$, $j \in J_n$. Otherwise, some coordinates of E-points can be fixed, and the dimension can be reduced.

Let us form a set of different coordinates of E-points by combining the sets (1.9) and single outing the underlying set of the resulting multiset:

$$\mathcal{A} = \{e_1, ..., e_k\} = S(\bigcup_{j=1}^{n} X^j). \tag{1.10}$$

The set $\mathcal{A}$ will be called the generating set of E.

We note the following features of the discrete lattice $\mathcal{A}^n$ constructed on its basis:

- $k \geq \max\limits_{j \in J_n} k_j$, which implies that $k \geq 2$, $|\mathcal{A}^n| \geq 2^n$;
- $X \subseteq \mathcal{A}^n$, respectively, $\forall x \in E \quad x \in \mathcal{A}^n$.

Consider an issue of decomposition of a set E by surfaces and its decomposition into pairwise disjoint subsets.

Let a function $h(x)$ be defined on the set E. The problem of expanding E into the family of surfaces given by the function $h(x)$ is to find level surfaces of this function:

$$\mathcal{S}^i = \{x \in \mathbb{R}^n : \, h(x) = h_i\}, \; i \in J_{m_{h(x)}}, \tag{1.11}$$

such that

$$h_i < h_{i+1}, \; i \in J_{m_{h(x)}-1}, \tag{1.12}$$

$$\mathcal{E}^i = E \cap \mathcal{S}^i \neq \emptyset, \; i \in J_{m_{h(x)}}, \tag{1.13}$$

$$E = \bigcup_{i=1}^{m_{h(x)}} \mathcal{E}^i. \tag{1.14}$$

This means that

$$\forall \, i \neq i' \; \mathcal{E}^i \cap \mathcal{E}^{i'} = \emptyset, \tag{1.15}$$

wherefrom it is seen that the decomposition (1.14) of E into pairwise disjoint subsets (1.13) is constructed with the number of components that depend essentially on the type of $h(x)$. Below we will call the set E $m_{h(x)}$-level with respect to the function $h(x)$.

This raises the following questions to be considered:

- what properties the resulting do the discrete sets $\{\mathcal{E}^i\}_{i \in J_{m_{h(x)}}}$ have depending on the form of a set E and a function $h(x)$;
- how to find feasible points of these sets;
- how to specify the function $h(x)$ to decrease or increase the number of components in the decomposition.

Depending on which function $h(x)$ will be taken as the basis of the decomposition, the formulas (1.13) and (1.14) will define the decomposition of E into parallel hyperplanes, nested hyperspheres, ellipsoids, piecewise linear surfaces, etc.

Suppose $h(x)$ is convex[1] , and the convex set $C^1 = \{x \in \mathbb{R}^n : h(x) \leq h_1\}$ contains interior points, i.e. C^1 is a convex body. Then the surface $S^1 = \{x \in \mathbb{R}^n : \ h(x) = h_1\}$ is the boundary of the convex body C^1 and is called a complete convex surface, and its arbitrary subset, including itself, is a convex surface. Then, taking into account (1.12), we have

$$C^i = \{x \in \mathbb{R}^n : h(x) \leq h_i\}\,,\ i \in J_{m_{h(x)}} \tag{1.16}$$

is a set of convex bodies. Respectively, (1.11) is a family of complete convex surfaces, and the formulas (1.13) and (1.14) define a decomposition of E into these convex surfaces.

Introducing the notation $f_i(x) = h(x) - h_i$ we can rewrite (1.11) as

$$\mathcal{S}^i = \{x \in \mathbb{R}^n : \ f_i(x) = 0\},\ i \in J_{m_{h(x)}}. \tag{1.17}$$

Convex surfaces have the peculiarity that the Gaussian curvature and all the principal curvatures of all their points are non-negative. If the function $h(x)$ is strictly convex, then the set of its non-differentiability points is at most countable. Accordingly, a set of points of surfaces (1.17) with zero Gaussian curvature will also be at most countable, i.e. strictly convex surfaces. Therefore, the formulas (1.13) and (1.14) define a decomposition of E into a family of strictly convex surfaces. Finally, if $h(x)$ is a strongly convex function, then the Gaussian curvature of all points of surfaces (1.17) is positive, i.e. these surfaces are $n-1$-convex.

[1]Let $D \subseteq \mathbb{R}^n$ be convex.

A function $f(x)$ defined on D is called convex on D if for any $x, y \in D$ and, for an arbitrary $\lambda \in [0,1]$, the following inequality holds:

$$f(\lambda x + (1-\lambda)\, y) \leq \lambda f(x) + (1-\lambda)\, f(y)\,.$$

If on D this inequality holds as strict, i.e. $\forall \lambda \in (0,1)$,

$$f(\lambda x + (1-\lambda)\, y) < \lambda f(x) + (1-\lambda)\, f(y)\,,$$

function $f(x)$ is called strictly convex.

If, in addition, there exists $\rho > 0$ such that $\forall \lambda \in (0,1)$, $\forall\, x, y \in D$:

$$f(\lambda x + (1-\lambda)\, y) \leq \lambda f(x) + (1-\lambda)\, f(y) - \lambda(1-\lambda)\rho||x-y||_2,$$

the function $f(x)$ is strongly convex on D with a strict convexity parameter ρ.

Respectively, (1.13), (1.14) will be expanded in a family of $n-1$-convex surfaces.

We will mainly consider two types of decompositions of finite sets in families of convex surfaces: a) in parallel hyperplanes; b) in strictly convex surfaces.

Let us consider a one-level set with respect to a function $h(x)$. Let $m_{h(x)} = 1$. Then, omitting the index i in the formulas (1.16) and (1.17), we get

$$C = C^1 = \{x \in \mathbb{R}^n : f_1(x) \leq 0\} = conv\, S, \tag{1.18}$$

$$S = S^1 = \{x \in \mathbb{R}^n : f_1(x) = 0\}. \tag{1.19}$$

If $f_1(x)$ is a strictly convex function, the body (1.18) will be called a strictly convex body. Its boundary is a strictly convex surface S coinciding with a set of extreme points of C. Recall that a point x is called the extreme point of a set C if it cannot be represented as a convex combination of any other two points of this set.

We introduce the following definitions.

Definition 1.1. *A finite point configuration E is called a surface-located set (SLS), if there exists a strictly convex surface S such that*

$$E \subseteq S. \tag{1.20}$$

In other words, E is a SLS if there exists $f_1(x)$ such that

$$\forall \lambda \in (0,1)\ \forall x, y \in \mathbb{R}^n\ f_1(\lambda x + (1-\lambda) y) < \lambda f_1(x) + (1-\lambda) f_1(y), \tag{1.21}$$

the formula (1.19) defines a strictly convex surface, and the condition (1.20) is satisfied. SLS can be classified depending on the shape of S.

Definition 1.2. *A finite point configuration E is called a spherically-located set if there exists a hypersphere of radius r centered at the point $a = (a_i)_{i \in J_n} \in \mathbb{R}^n$:*

$$S_r(a) = \left\{ x \in \mathbb{R}^n : \sum_{i=1}^{n} (x_i - a_i)^2 = r^2 \right\} \tag{1.22}$$

such that

$$E \subseteq S_r(a).$$

Thus, if the surface (1.19) has the shape

$$S = S_r(a),$$

the set (1.20) is spherically-located.

Similarly, we introduce two more classes of finite point configurations:

- a finite point configuration E of the shape (1.20) is called an ellipsoidally-located set if (1.19) is an ellipsoid centered at $a \in \mathbb{R}^n$:

$$S = \left\{ x \in \mathbb{R}^n : (x-a)^T A (x-a) = 1 \right\}, \tag{1.23}$$

  where $A \succ 0$ is a positive definite matrix of order n;

- a finite point configuration E is called a superspherically-located set if the equation defines the surface (1.19):

$$S = \left\{ x \in \mathbb{R}^n : \sum_{i=1}^{n} (x_i - a_i)^{2\alpha} = r^{2\alpha} \right\}, \tag{1.24}$$

  where $\alpha \in (0.5, \infty)$, i.e. S is a supersphere [36] of radius r centered on a point $a \in \mathbb{R}^n$ and deformation coefficient α.

The supersphere (1.24) is conveniently defined using the l_p-norm[2]:

$$\|x - a\|_{2\alpha} = r.$$

The hypersphere (1.22) is a special case of the supersphere for $\alpha = 1$ and can be defined

[2] $\|x\|_p = (\sum_{i=1}^{n} x_i^p)^{1/p}$ is a ℓ_p-norm $p \geq 1$.

using the Euclidean norm[3]:

$$\|x-a\|_2 = r.$$

The equation

$$x^\top Ax + b^\top x + c = 0 \tag{1.25}$$

defines the ellipsoid (1.23), where $A \succ 0$, $x, b \in \mathbb{R}^n$, $c \in \mathbb{R}^1$.

If only (1.21) is violated from the conditions (1.19)-(1.21), then E is located on the surface S (surface-located, S-surface-located). At the same time, the convex body (1.18) can be either bounded and unbounded (further referred to as Case 1.2.1), a subset of points of S where the function $f_1(x)$ is non-differentiable, may be uncountable (further referred to as Case 1.2.2).

Let us illustrate this with examples.

Case 1.2.1: if $f_1(x)$ is linear, then (1.18) defines a half-space, i.e. unbounded domain. In this case, we say that E is located on the hyperplane (1.19) or is plane-located.

Case 1.2.2: As an example of a bounded domain (1.18), we consider $f_1(x) = \|x-a\|_1 - 1$[4] or $f_1^*(x) = \|x-a\|_\infty - 1$ [5]. Because the $f_1(x)$, $f_1^*(x)$ - convex piece-wise linear functions, i.e. are not strictly convex. In this case, we say that E is located on a polyhedral surface (a piece-wise linear surface):

- in the first case, the surface is a surface of a hyperoctahedron:

$$S = \{x \in \mathbb{R}^n : \|x-a\|_1 = 1\}; \tag{1.26}$$

- in the second case, the surface is a surface on a hypercube (see Sec. 5.9):

$$S = \{x \in \mathbb{R}^n : \|x-a\|_\infty = 1\}. \tag{1.27}$$

[3] $\|x\|_2 = \sqrt{\sum\limits_{i=1}^n x_i^2}$ - ℓ_2-norm (the Euclidean norm)

[4] $\|x\|_1 = \sum\limits_{i=1}^n |x_i|$ - ℓ_1-norm (Manhattan distance, the Taxicab norm)

[5] $\|x\|_\infty = \max\limits_i |x_i|$ - ℓ_∞-norm (maximum norm)

Accordingly, the convex body (1.18) itself will be a polytope, which is a hyperoctahedron in the case of (1.26), and it is a hypercube in the case of (1.27).

Suppose that $h(x)$ is a strictly convex function such that the formulas (1.11) and (1.12) yield the decomposition of E in a family of strictly convex surfaces, particularly the decomposition (1.13)-(1.15) of E into surface-located sets of the form (1.13).

$$C^i = \{x \in \mathbb{R}^n : \ h(x) \leq h_i\}, \ i \in J_{m_{h(x)}} \tag{1.28}$$

defines a sequence of strictly convex bodies nested inside each other:

$$C^i \subset C^{i+1}, \ i \in J_{m_{h(x)}-1}.$$

Particularly, if $h(x) = (x - x_0)^2$, the formulas (1.11) and (1.12) define the decomposition of E into a family of hyperspheres, resulting in the decomposition of E into spherically-located sets. The set (1.28) is a finite sequence of nested balls.

Now we move to the case when the function $h(x)$ is linear:

$$h(x) = \overline{n}^\top x, \ \text{where } \overline{n}^\top \neq \mathbf{0}, \tag{1.29}$$

where $\mathbf{0} \in \mathbb{R}^n$ is zero vector.

The function $h(x)$ is determined by a vector $\overline{n}$ of a normal to the hyperplane (normal vector):

$$H(\overline{n}) = \{x \in \mathbb{R}^n : \overline{n}^\top x = 0\}. \tag{1.30}$$

Thus, the formulas (1.11)-(1.15) can be rewritten in terms of $\overline{n}$, using the notation $m_{(\overline{n})}$ for the number of different values the function (1.29) taken on E, which implies that $m_{(\overline{n})} = m_{h(x)}$ satisfies the constraints:

$$\begin{aligned} &E = \bigcup_{i=1}^{m_{(\overline{n})}} E^i(\overline{n}), \\ &\forall\, i \neq i' \ E^i(\overline{n}) \cap E^{i'}(\overline{n}) = \varnothing; \\ &E^i(\overline{n}) = H^i(\overline{n}) \cap E \neq \varnothing, \end{aligned} \tag{1.31}$$

where

$$H^i(\overline{n}) = \{x \in \mathbb{R}^n : \overline{n}^\top x = h_i\},\ i \in J_{m_{(\overline{n})}}. \tag{1.32}$$

The formula (1.32) defines the decomposition of E into a family of hyperplanes parallel to the hyperplane (1.30) (further referred to as a decomposition of E into parallel hyperplanes toward the vector $\overline{n}$. The finite point configuration E itself will be called $m_{(n)}$-level toward the vector $\overline{n}$.

Further, to emphasize that we deal with a decomposition of E toward the normal vector of the hyperplane H, we will use the notation $\overline{n}_H$. Respectively, $m_{(\overline{n}_H)}$ will be referred to as the levelness of E towards the direction $\overline{n}_H$. Suppose (1.30) is a coordinate hyperplane $x_j = 0$. In that case, we will use the following notation:

$$H(\overline{n}_j) = \{x \in \mathbb{R}^n : x_j = 0\}, \tag{1.33}$$

where

$$\overline{n}_j = (n_{ij})_{i \in J_n} :\ n_{ij} = \begin{cases} 1,\ i = j, \\ 0,\ i \neq j \end{cases} \quad (i \in J_n)$$

The decomposition of E into hyperplanes parallel to the coordinate hyperplane (1.33) is referred to as the decomposition of E along the coordinate x_j, and the value $m_{(\overline{n}_j)}$ is called the levelness of E along the coordinate x_j ($j \in J_n$).

The maximum of $m_{(\overline{n}_j)}$ ($j \in J_n$) is denoted by

$$m'(E) = \max_{j \in J_n} m_{(\overline{n}_j)} \tag{1.34}$$

and is called the levelness of E along coordinates, and this set itself is called $m'(E)$-level along coordinates.

Let us establish a connection between $m'(E)$ and the cardinality k (see (1.10)) of the generating set of E.

First, we form the decompositions of E along coordinates based on the formula (1.9)

and the following obvious representations of this set:

$$E = \bigcup_{i=1}^{k_j} E^{ij},\ j \in J_n. \tag{1.35}$$

where

$$H^{ij} = H^i(\overline{n}_j) = \{x \in \mathbb{R}^n : x_j = e_{ij}\}; \tag{1.36}$$

$$E^{ij} = H^{ij} \cap E \neq \emptyset,\ i \in J_{k_j},\ j \in J_n. \tag{1.37}$$

This leads to $m_{(\overline{n}_j)} = k_j$ $(j \in J_n)$ and

$$m'(E) = \max_{j \in J_n} k_j.$$

Note that $m'(E)$ is bounded from above by the cardinality of a set $\mathcal{A}$ generating E, and from below, according to Remark 1.1, by the value 2. Hence, $2 \leq m'(E) \leq k$.

If a finite point configuration E is known to be generated by a set $\mathcal{A} = \{e_1, ..e_k\}$, while the representation (1.9), (1.10) is unknown, E can be represented as follows:

$$E = \bigcup_{i=1}^{k} E^{ij},\ j \in J_n; \tag{1.38}$$

$$H^{ij} = H^i(\overline{n}_j) = \{x \in \mathbb{R}^n : x_j = e_i\},\ i \in J_k,\ j \in J_n; \tag{1.39}$$

$$E^{ij} = H^{ij} \cap E. \tag{1.40}$$

Moreover, some of the sets (1.40) may be empty, respectively, the representation (1.38)-(1.40) will not be a decomposition of E into parallel hyperplanes in the case if the last of the conditions (1.31) is violated.

To move to the decomposition of E, we introduce sets

$$I^j \subseteq J_k :\ E^{ij} \neq \emptyset,\ k_j = |I^j|,\ j \in J_n. \tag{1.41}$$

Now, we can derive an E-decomposition along coordinates:

$$E = \underset{i \in I^j}{\cup} E^{ij},\ j \in J_n, \tag{1.42}$$

where I^j is given by (1.41).

Notice, that $X^j = \{e_{j1}, ..., e_{jk_j}\} = \{e_i\}_{j \in I^j}$, $j \in J_n$, i.e. the decompositions (1.39)-(1.42) and (1.35)-(1.37) are identical.

Also, we introduce the notation for projections of the sets (1.40) onto hyperplanes:

$$E'^{ij} = Pr_{H^{ij}} E^{ij},\ i \in J_k,\ j \in J_n. \tag{1.43}$$

Note that the sets (1.40) are usually representable as a projection of the initial finite point configuration E onto H^{ij}:

$$E'^{ij} = Pr_{H^{ij}} E,\ i \in J_k,\ j \in J_n. \tag{1.44}$$

In terms of projections (1.43), the condition (1.41) can be rewritten as

$$I^j \subseteq J_k :\ E'^{ij} \neq \emptyset,\ k_j = |I^j|,\ j \in J_n.$$

Accordingly, an indicator that (1.40)-(1.42) yields the decomposition of E along coordinates is the condition:

$$E'^{ij} \neq \emptyset,\ i \in J_k,\ j \in J_n, \tag{1.45}$$

which, in turn, is equivalent to $E^{ij} \neq \emptyset$, $i \in J_k$, $j \in J_n$.

1.3 Convex hulls of finite point configurations

Introduce a set $D \subseteq \mathbb{R}^n$. Its <u>convex hull</u> is a set of all convex combinations of points of D, and its <u>affine hull</u> is a set of all affine linear combinations of its points. The convex and affine hulls of D are denoted by $conv\ D$ and $aff\ D$, respectively.

Definition 1.3. *A convex hull of a finite set D is called a* <u>*polytope*</u> *induced by D.*

Based on this definition, the convex hull

$$P = conv\ E \tag{1.46}$$

of a finite point configuration E is a polytope.

<u>The dimension</u> of a polytope P, dim P, is the dimension of its affine hull:

$$d = \dim\ P = \dim\ aff\,(P)\,, \tag{1.47}$$

i.e. it is the maximum number of linearly independent vectors in $aff\,(P)$.

A polytope satisfying the condition (1.47) is called a <u>d- dimensional polytope</u> (d-polytope).

A polytope is called <u>full-dimensional</u> if its dimension coincides with the dimension of Euclidean space where it is given. For example, the polytope $P \subset \mathbb{R}^{\ltimes}$ is full-dimensional if and only if dim $P = n$.

Since by assumption, E is non-empty, P is also a non-empty set, so it has a <u>support hyperplane</u> H, i.e. a hyperplane that has common points with P and such that the entire polytope lies in one of the two half-spaces defined by the hyperplane H. The resulting non-empty set $F = P \cap H$ is called the <u>face</u> of the polytope P generated by the supporting hyperplane H, and $i =$ dim F is called the <u>face dimension</u> of the polytope. Faces of dimension i are called <u>i-faces</u>. Depending on the value of $i \in J^0_{d-1} = J_{d-1} \cup \{0\}$, the following terminology is used for denoting i-faces of P: 0-faces are <u>vertices</u> of P, 1-faces are its <u>edges</u>, ..., $d-1$-faces are <u>hyperfaces</u> (facets).

Let $\mathbf{F}_i$ be a set of i-faces of P, and f_i be their number ($f_i = |\mathbf{F}_i|$), $i \in J^0_{d-1}$, $\mathbf{H}$ is a set of supporting hyperplanes to P-facets. The values f_i, $i \in J^0_{d-1}$ are coordinates of a polytope P's <u>f-vector</u>, $f = (f_0, f_1, ..., f_{d-1})$.

A set of vertices of a polytope P is denoted by $vert\ P$ or $V(P)$, a set of edges by

edges P or $E(P)$, and a set of facets by *facets* P or $F(P)$, i.e.

$$vert\ P = \mathbf{F}_0,\ edges\ P = \mathbf{F}_1,\ facets\ P = \mathbf{F} = \mathbf{F}_{d-1}.$$

Respectively,

$$|vert\ P| = f_0,\ |edges\ P| = f_1,\ |\mathbf{F}| = |\mathbf{H}| = f_{d-1}.$$

For a set of vertices of a polytope P we will use the notation V, We will use the notation V for a set of vertices of a polytope P and the notation $v_{[.]}$ for its elements, i.e.

$$V = vert\ P = \{v_i\}_{i \in J_{n_V}},$$

where $n_V = |V| = f_0$. Similarly, we will use the notation $F_{[.]}$ for P-facets and $H_{[.]}$ for their corresponding support hyperplanes:

$$\mathbf{F} = \{F_i\}_{i \in J_{f_{d-1}}},\ \mathbf{H} = \{H_i\}_{i \in J_{f_{d-1}}}.$$

Let us introduce the so-called V- and H-representations of a polytope P.

The <u>V-representation</u> (defining in a parametric form) of a polytope P is defining it as a convex hull of its vertex set:

$$P = conv\ V. \tag{1.48}$$

Note that the representations (1.46) and the V-representation (1.48) are special cases of P-representations of the form:

$$\begin{aligned} &P = conv\ E', \\ &\text{where } E:\ V \subseteq E' \subseteq E, \end{aligned} \tag{1.49}$$

for $E' = V$ and $E' = E$, respectively.

A distinctive feature of the V-representation in the family (1.49) is that it is minimal in the sense that the elimination of any element from V and the subsequent formation of a

convex hull leads to a formation of a polytope, different from P:

$$\forall\, v \in V \quad P \supset P' = conv(P \backslash \{v\}).$$

Accordingly, it is minimal in terms of the number of points in a set generating P, i.e. in the number of elements in the V-representation of the polytope. The case when the family (1.49) includes a single representation of P that is only possible if

$$E = V = vert\ P, \tag{1.50}$$

in other words, $|n_E| = |n_V|$, will be considered specifically.

Definition 1.4. *A finite point configuration E is called a* <u>*vertex-located set*</u> *(*<u>*VLS*</u>*) if it coincides with a set of vertices of its convex hull:*

$$E = vert\ conv\ E. \tag{1.51}$$

Given (1.46), the condition (1.51) can be rewritten as an expression (1.50), which will be used further as a vertex locality condition for E.

If E is not a VLS, then the condition $\overline{E} = E \backslash V \neq \emptyset$ is satisfied, i.e. it contains interior points of the polytope P or its faces.

Checking the condition (1.50), and then in case of its failure, deriving conditions (necessary and sufficient) of whether an arbitrary $x \in E$ is a vertex P is <u>the problem of constructing V-representations of a polytope</u> P given in the form (1.46). A solution to this problem for a VLS is trivial. Indeed, membership of E is a necessary and sufficient condition for a point x to be a vertex of P. However, for an arbitrary E, checking the fulfillment of the condition (1.50), extracting the set $\overline{E}$ from E, and, accordingly, the problem of constructing a V-representation can be challenging.

This problem for a VLS is solved if its elements give themselves. In addition, these sets possess many specific features that will be studied below. Any non-vertex-located set E allows the decomposition of (1.14), (1.15) into vertex-located subsets.

Now we move to the consideration of the H-representation (H-presentation, the analytic form) of a polytope P, which is its representation by a linear system of constraints:

$$\begin{aligned} &A'x \le a'_0,\ A''x = a''_0, \\ &x \in \mathbb{R}^n,\ A' = (a'_{ij}) \in \mathbb{R}^{n' \times n},\ A'' = (a''_{ij}) \in \mathbb{R}^{n'' \times n}, \\ &a'_0 = (a'_{i0}) \in \mathbb{R}^{n'},\ a''_0 = (a''_{i0}) \in \mathbb{R}^{n''}. \end{aligned} \tag{1.52}$$

The problem of constructing an H-representation of a polytope P consists in finding the numbers n', n'', matrices A', A'' and vectors a'_0, a''_0 such that (1.52) defines P.

Let $A' = (a'_{ij})_{i \in J_{n'}, j \in J_n}$, $a'_0 = (a'_{i0})^\top_{i \in J_{n'}}$; $A'' = (a''_{ij})^\top_{i \in J_{n''},\, j \in J_n}$, $a''_0 = (a''_{i0})_{i \in J_{n''}}$. (1.52) in a vector form:

$$\overline{a}'_i x \le a'_{i0},\ i \in J_{n'},\ \overline{a}''_i x = a''_{i0},\ i \in J_{n''}, \tag{1.53}$$

$$\text{where } \overline{a}'_i = (a'_{ij})_{j \in J_{n'}},\ \overline{a}''_i = (a'_{ij})_{j \in J_{n''}}, \tag{1.54}$$

where

$$\overline{a}'_i = (a'_{ij})_{j \in J_{n'}},\ \overline{a}''_i = (a'_{ij})_{j \in J_{n''}}.$$

Remark 1.2. The rank of the matrix A'' is

$$\rho = rank\ A''. \tag{1.55}$$

The value (1.55) allows estimating the dimension of P from below:

$$\dim\ P \ge n - \rho. \tag{1.56}$$

If, in addition, an interior point P of the affine subspace $\overline{a}''_i x = a''_{i0}$, $i \in J_{n''}$, this inequality turns into equality, i.e. if

$$\exists x^0 \in E:\ A'x^0 = a'_0,\ A''x^0 < a''_0 \Rightarrow$$

$$\dim\ P = n - \rho. \tag{1.57}$$

Each equation in an H-representation of a polytope P defines a hyperplane, and the inequality defines a half-space and its bounding hyperplane, which may or may not be the support hyperplane of its facet. Moreover, each facet of P is represented by a certain constraint from the H-representation. Accordingly, $n' + n'' \geq |\mathbf{F}| = f_{d-1}$ is valid.

Consider the case when

$$n' + n'' = |\mathbf{F}|, \tag{1.58}$$

i.e. this inequality is satisfied as equality.

An H-representation of a polytope P minimal in terms of the number of constraints is called an <u>irredundant H-representation</u> of P.

The system of constraints (1.52) is an irredundant H-representation of P if eliminating any constraint from it leads to a relaxation of P. Thus, the H-representation (1.53), (1.54) of a polytope P is irredundant if

$$\begin{aligned} &\forall\, i' \in J_{n'} \quad P \subset P'_{i'} = \{x \in \mathbb{R}^n : \overline{a}'_i x \leq a'_{i0}\}_{i \in J_{n'} \setminus \{i'\}}; \\ &\forall\, i'' \in J_{n''} \quad P \subset P''_{i''} = \{x \in \mathbb{R}^n : \overline{a}''_i x \leq a''_{i0}\}_{i \in J_{n''} \setminus \{i''\}}. \end{aligned} \tag{1.59}$$

If the condition (1.59) is violated, the H-representation (1.52) is said to be <u>redundant</u>.

Note that the sets $P'_{i'}$, $P''_{i''}$ ($i' \in J_{n'}$, $i'' \in J_{n''}$) in (1.59) can be either bounded or unbounded. In the first case, we will apply the term "relaxation polytope" to them.

Definition 1.5. *A polytope P' is called a <u>relaxation of a polytope</u> P if $P' \supset P$ and every H-representation of P' is a proper subsystem of an irredundant H-representation of P'.*

<u>The problem of constructing an irredundant H-representation of a polytope</u> is to find a linear system of constraints(1.52), (1.59). When this problem has been solved, then, due to each equation (1.52) defines a support hyperplane associated with a hyperface of P. At the same time, each inequality defines a certain such hyperplane, then the system of equations of these support hyperplanes (further referred to as a system of hyperfaces' equations) of P is extracted directly from the H representation:

$$\mathbf{H} : A'x = a'_0, A''x = a''_0.$$

Accordingly, we have the relation (1.58) for the number of facets P.

If the H-representation (1.53) and (1.54) is irredundant, then, by Remark 1.2, the value ρ will satisfy the equality:

$$\rho = n''. \tag{1.60}$$

Accordingly, the bound (1.56) and condition (1.57) become

$$\dim\ P \geq n - \rho,$$

$$\dim\ P = n',$$

i.e. the dimension of P coincides with the number of inequalities n' in its irredundant H-representation.

Remark 1.3. Constructing a V-representation from an H-representation is a vertex enumeration problem. Conversely, constructing an H-representation from a V-representation of a polytope is a problem of listing hyperfaces. Based on the relationship between vertices and facets of a polytope, these two problems are equivalent [12].

Each vertex x of P can be associated with a set of incident/adjacent facets and edges of the polytope (adjacent edges). The first set includes all facets at the intersection of which x is formed. The second set contains all P-edges with x as one endpoint and adjacent to x point as another endpoint. Each vertex $x \in V$ of the polytope P is associated with a set of vertices adjacent to it called a neighbourhood of x:

$$N(x) = N_P(x) = \{y \in V : \ y \leftrightarrow x\}, \tag{1.61}$$

They are connected with x by an edge $(x, y) \in edges\ P$. Further, the notation $N(x)$ will be used when considering a neighborhood of one polytope and $N_{[.]}(x)$ when considering several polytopes, where [.] is a polytope notation.

The number of vertices P adjacent to $x \in V$ is denoted by $\mathcal{R}(x)$:

$$\mathcal{R}(x) = |N(x)|.$$

For $x \in vertP$, $\mathcal{R}(x)$ is a vertex degree.

If all vertices of a polytope have the same number of adjacent vertices

$$\forall x \in V : \mathcal{R} = \mathcal{R}(x),$$

they are called regular, and the value $\mathcal{R}$ is the regularity degree of P-vertices.

Single-outing classes of polytopes with regular vertices, the formulation of conditions for selecting a neighborhood $N(x)$ from V for $x \in V$ for a polytope P, and the adjacency criterion for P-vertices are studied in polyhedral combinatorics and graph theory where graphs of polytopes are explored.

Here we note a single feature of polytopes with regular vertices. The formula $|edges\, P| = \frac{|V|\mathcal{R}}{2}$ or $f_1 = \frac{f_0\mathcal{R}}{2}$ can find the number of its edges.

One of the well-known types of polytopes is simple polytopes, characterized by the regularity degree of their vertices coinciding with their dimension, i.e.

$$\mathcal{R} = \dim\ P.$$

Simple polytopes include simplices, hypercubes, etc. An interesting peculiarity is that their faces of arbitrary dimension are also simple polytopes.

Another class of polytopes is simplicial polytopes, which faces are simplices. Among them are hyperoctahedrons, simplices, etc.

An important problem in the combinatorial theory of polytopes is the problem of classifying and enumerating polytopes with a given face structure [4, 65, 68, 106], in particular, polytopes with a given f-vector.

One can single out subclasses of so-called combinatorically equivalent polytopes in this class. We introduce the following definitions.

A graph $H(P)$ of a polytope P with a set of vertices $V(P)$ and edges $E(P)$ is a graph formed by vertices and edges P, i.e. $H(P) = (V(P), E(P))$.

Graphs $H(P)$ and $H(P')$ are isomorphic if there exists a bijection $V(P) \xrightarrow{\phi} V(P')$ between sets of vertices of polytopes P, P' such that any two vertices of the graph $H(P)$ are

adjacent if and only if the corresponding two vertices of the graph $H(P')$ are adjacent:

$$\begin{aligned} &\forall v_1, v_2 \in V(P): \ v_1 \leftrightarrow v_2 \ \Leftrightarrow \\ &v_1^{'} = \phi(v_1), v_2^{'} = \phi(v_2) : v_1^{'}, v_2^{'} \in V(P'), \ v_1^{'} \leftrightarrow v_2^{'}. \end{aligned} \tag{1.62}$$

Polytopes are combinatorically equivalent if the corresponding graphs of polytopes are isomorphic.

To each polytope P for which the origin is an interior point, a <u>dual polytope</u> P^{Δ} (a polar polytope) can be associated whose f-vector coincides with the f-vector of the polytope P up to reverse reordering of the coordinates. If $f = (f_i)_{i \in J^0_{d-1}}$ is the f-vector of P, then $f^{\Delta} = (f_{d-i+1})_{i \in J^0_{d-1}}$ will be the f-vector of its dual polytope P^{Δ}. In particular, $|\mathbf{F}_P| = |V_{P^{\Delta}}|$ and $|V_P| = |\mathbf{F}_{P^{\Delta}}|$.

For example, a hypercube and a hyperoctahedron of the same dimension are dual, while a simplex is dual to itself. A distinctive feature of simple polytopes is that their duals are simplicial, and vice versa. The hypercube and hyperoctahedron illustrate this property since a hypercube is a simple polytope, while a hyperoctahedron is simplicial.

Suppose V-, H-representations are found for the polytope P. In that case, we can discuss constructing a decomposition of the set of its vertices into parallel hyperplanes toward normal vectors to its facets.

Definition 1.6. *[1] A polytope P is called a <u>$m''(P)$-level polytope</u> if for any facet $F_1 \in \mathbf{F}$ and the corresponding supporting hyperplane $H_1 \in \mathbf{H}$, there exists a family of $(m''(P) - 1)$ hyperplanes $\{H_i\}_{i \in J_{m''(P)-1} \setminus \{1\}}$ parallel to H_1, such that all vertices of V lie on the hyperplanes $\{H_i\}_{i \in J_{m''(P)}}$.*

In terms of an $m_{(\overline{n})}$-level set toward the vector $\overline{n}$, this definition says that, in the decomposition of a set V toward normal vectors to P-facets, levelness does not exceed $m''(P)$. It reaches the upper bound toward a normal vector of a certain facet. That is, if the number $m(\overline{n}_F)$ of V decomposition levels toward normal vectors of its facets F needs to be found for $\forall F \in \mathbf{F}$, then $m''(P)$ is the maximum of all these values:

$$m''(P) = \max_{H \in \mathbf{H}} m(\overline{n}_F), \tag{1.63}$$

where, for each F, the vector $\overline{n}_F$ is found from the condition:

$$H_F = \{x \in \mathbb{R}^n : \overline{n}_F^\top x = b_F\} \in \mathbf{H}, H_F = \{x \in \mathbb{R}^n : \overline{n}_F^\top x = b_F\} \in \mathbf{H},\ F \in \mathbf{F},\ b_F \in \mathbb{R}^1. \quad (1.64)$$

The value $m''(P)$ is bounded from below:

$$m''(P) \geq 2, \quad (1.65)$$

because $\forall F \in \mathbf{F}\ E \cap H_F, E \setminus E \cap H_F \neq \emptyset$.

Polytopes for which the inequality (1.65) turns into equality $m''(P) = 2$ form a wide class of two-level polytopes (2-level polytopes), whose applications are known in polyhedral combinatorics, combinatorial optimization, communication complexity, statistics, etc. [1, 2, 8, 16, 17, 17].

Among two-level polytopes are hypercubes and hyperoctahedrons, polytopes of independent sets of perfect graphs (stable/independent set polytopes of perfect graphs), Hansen polytopes, Hanner polytopes, Birkhoff polytopes, order polytopes of finite posets, etc. A new subfamily of two-level polytopes has been discovered recently in matroid theory among the base polytopes of two-level matroids [17, 17]. Each class of two-level polytopes and their corresponding vertices sets are associated with a certain optimization problem. For example, the problem of finding the maximum independent set in a graph can be formulated as an optimization problem on a vertex set of a two-level polytope of an independent set of this graph. In contrast, the same problem on the polytope will be its relaxation.

Multilevel polytopes possess various outstanding properties. One of them is that each two-level polytope P is combinatorically equivalent to a certain $(0-1)$-polytope P', i.e. polytope whose vertices are $(0-1)$-vectors [111]:

$$\exists P' \subset \mathbb{R}^n :\ P \cong P^0,\ vertP' \subseteq B_n.$$

For example, an n-dimensional cube is combinatorically equivalent to a hypercube $PB_n = [0,1]^n$, and an octahedron is combinatorically equivalent to the following $(0-1)$-

polytope:

$$P = conv\{(1,0,0),(0,1,0),(0,0,1),(1,1,0),(1,0,1),(0,1,1)\}.$$

1.4 Functional representations of finite point configurations

Let E be a finite point configuration in $\mathbb{R}^n$, which is not singleton and

$$\mathcal{F} = \{f_j(x)\}_{j \in J_m} \tag{1.66}$$

is a family of functions defined on E and continuous on $convE \subseteq D$. As a rule, we will assume $D = R_n$.

Definition 1.7. *A representation of a set E by functional dependencies*

$$f_j(x) = 0,\ j \in J_{m'}, \tag{1.67}$$

$$f_j(x) \leq 0,\ j \in J_m \backslash J_{m'}. \tag{1.68}$$

are called an <u>*analytic representation*</u> *(*<u>*f-representation*</u>*) of E.*

The equations (1.67) is called a <u>strict part</u> of the f-representation, the inequalities (1.68) is an <u>unstrict part</u>, and the number of constraints is the representation <u>order</u>. Particularly, m is the order of the f-representation (1.67), (1.68), while m', $m'' = m - m'$ are the orders of its strict and unstrict parts, respectively.

Individual constraints of the f-representation (1.67), (1.68) are its components, and it itself is called an m-component functional representation of the set E.

Geometrically, an f-representation (1.67), (1.68) represents a finite point configuration E as an intersection of m'-surfaces:

$$S_j = \{x \in \mathbb{R}^n : f_j(x) = 0\},\ j \in J_{m'}, \tag{1.69}$$

subject to the inequality constraints (1.68) that form a discrete set E:

$$C_j = \{x \in \mathbb{R}^n : f_{j+m'}(x) \leq 0\},\ j \in J_{m''}. \tag{1.70}$$

In terms of (1.69), (1.70), the fact that (1.67), (1.68) is a functional representation of E can be represented as follows:

$$E = \left(\bigcap_{j \in J_{m'}} S_j \right) \bigcap \left(\bigcap_{i \in J_{m''}} C_j \right).$$

Let us classify functional representations depending on the type of functions involved, the order of its strict and unstrict parts, and the functional representation as a whole.

By the form of functions (1.66) and (1.68), functional representations of a set E can be linear and non-linear, continuous, differentiable, smooth, convex, polynomial, trigonometric, etc. Further classification of the representations can be introduced for these classes. For example, (1.67) and (1.68) is a polynomial f-representation of E if all functions in the family $\mathcal{F}$ are polynomials. Introducing the notion of degrees of an f-representation as the highest degree of these polynomials, one can single out linear, quadratic, cubic, biquadratic, and polynomial representations of higher degrees f-representations.

The general problem of constructing a functional representation of a finite point configuration E is to find a constraint system (1.67) and (1.68) analytically describing E. It is easy to see that E has unlimited number of functional representations, so this problem is always solvable. Indeed, to construct one of these representations, it suffices to form n different interpolation polynomials over points of E, imposing additional constraints on the linear independence of their gradients at these points. As a result of such a construction, the surfaces defined by these interpolation polynomials may also also have other common points, except for E. They can be cut off by constraints of type (1.68). For example, each "extra" point x^0 can be cut off by a constraint of the form $(x - x^0)^2 \geq r_0^2$, where $r_0 > 0$ is the radius of the x^0-neighborhood with no E-points. It results in a formation of a polynomial f-representation of E of the degree $|E| - 1$. Thus, the problem of the existence of an f-representation of an arbitrary FPC E turns out to be theoretically solvable along with the problem of deriving its

polynomial functional representation. Moreover, the issue of the existence and subsequent construction of polynomial representations of lower degrees can be addressed. Accordingly, the original problem of constructing a functional representation of E can be concretized with the problem of constructing a cubic, quadratic, linear, and other f-representations.

Note that the linear functional representation (1.67), (1.68) precisely coincides with the linear equality system (1.52) and defines a polytope. Since a set E is not singleton, its functional representations are always nonlinear. Therefore, the question is raised on the existence of quadratic representations as polynomial representations of E of minimal degree.

Let us classify functional representations depending on the ratio of parameters m, m', m''. We introduce into consideration several types of functional representations.

Definition 1.8. *The constraint system* (1.67) *and* (1.68) *is called:*

- *a* <u>*strict f-representation*</u> *of E if it contains only the strict part:*

$$m' = m,\ m'' = 0; \tag{1.71}$$

- *an* <u>*unstrict f-representation*</u> *of E if it has only an unstrict part:*

$$m' = 0,\ m'' = m;$$

- *the* <u>*general f-representation*</u> *if there are strict and unstrict parts in it:*

$$m'\,(m - m') > 0.$$

For example, a functional representation

$$f_i(x) = x_i^2 - x_i = 0,\ i \in J_n \tag{1.72}$$

is strict with and $m = m' = n$. It defines the set $B_n = \{0, 1\}^n$ of n-dimensional binary vectors. Geometrically, (1.72) represents B_n as an intersection of pairs of n parallel hyperplanes.

This set can be defined in another way:

$$\begin{aligned} &f_1(x) = (x - \mathbf{a})^2 - \frac{n}{4} = 0,\ \mathbf{a} = \frac{\mathbf{1}}{\mathbf{2}} \in \mathbb{R}^n; \\ &f_{i+1}(x) = x_i - 1 \le 0,\ f_{i+n+1}(x) = -x_i \le 0,\ i \in J_n \end{aligned} \tag{1.73}$$

(here and below $\mathbf{a} = (a, ..., a)^\top$ is a vector of the corresponding dimension, $a \in \mathbb{R}^1$). This representation has parameters $m' = 1$, $m'' = 2n$, $m = 2n + 1$, i.e. this is the general f-representation. Geometrically, it defines B_n as an intersection of a unit hypercube and a hypersphere circumscribed about it.

Note that both of the above functional representations are quadratic.

Let us generalize the concept of an irredundant H-representation of a polytope to a functional representation of a set. A system of constraints (1.67) and (1.68) is called an <u>irredundant f-representation</u> of the set E if the exclusion of any of its constraints results in a superset of E:

$$\forall\ j \in J_{m'}\ E \backslash S_j \supset E;$$

$$\forall\ i \in J_{m''}\ E \backslash C_i \supset E.$$

For irredundant strict representations, we introduce the following terminologies:

- a two-component strict functional representation of E is called a <u>tangent f-representation</u> if it coincides with a set of tangent points of the surfaces S_1 and S_2;
- a n-component irredundant strict functional representation is called an <u>intersecting f-representation</u>.

Example 1.1. Let us consider functional representations (1.72) and (1.73) of the set B_n. They are irredundant f-representations since eliminating any component from them leads to the specification of its superset.

Following the above classification of strict irredundant representations, (1.72) is an intersecting f-representation.

An example of a tangent representation of the set B_n is

$$S_1 : \sum_{i=1}^{n} \left(x_i - \frac{1}{2}\right)^2 = \frac{n}{4}; \quad S_2 : \sum_{i=1}^{n} \left(x_i - \frac{1}{2}\right)^4 = \frac{n}{16}, \tag{1.74}$$

representing the binary set as an intersection of the hypersphere S_1 and supersphere S_2 with a deformation coefficient of 2.

Let us demonstrate that one-component functional representations can define finite point configurations. For illustration, we take an example of a functional representation (1.72) by convolving all its components:

$$f_1(x) = \sum_{i=1}^{n} (x_i^2 - x_i)^2 = 0. \tag{1.75}$$

The equation (1.75) also defines the set B_n. It is non-convex, unlike the f-representation (1.72)-(1.74). This demonstrates that the minimum order m of a convex f-representation of a discrete set is two, and convex tangent representations are minimal f-representations by the number of components. Similarly, for an arbitrary finite set for which an m-component irredundant strict f-representation is known, various irredundant strict functional representations of order $m \in [1, m-1]$ can be found similarly by convolutions of its various components.

If a set $E \subset \mathbb{R}^n$ is described analytically in some lifted space $\mathbb{R}^{n'}$, $n' > n$ [3], we say about the existence of an <u>extended functional representation</u> of a set E. Lifting into a higher dimension space can be convenient when forming f-representations of a certain set, such as quadratic ones. If the extended functional representation is built, then projection into the original space allows finding an f-representation of this set.

Chapter 2

Finite and combinatorial point configurations

This chapter discusses the properties of various classes of finite point configurations. The class of polyhedral-surface sets and their special subclasses are defined in accordance with the publications [39, 40, 42].

For arbitrary finite point configurations, the relationship with vertex-located sets is studied in accordance with [3].

The concept of a multilevel set is introduced as a generalization of a multilevel polytope [1, 17]. The properties of multilevel sets and corresponding polytopes are investigated. The existence of a bijection between two-level sets, two-level polytopes [2, 8, 16, 17], and two-level $(0-1)$-sets [111] is established.

When studying finite point configurations obtained by logical operations on other such configurations, the terminologies and theoretical background from [20, 23, 112] are used.

2.1 Properties of finite point configurations

Let us establish a connection between vertex-located and surface-located sets. The following theorem holds.

Theorem 2.1. *An arbitrary finite surface-located set is vertex-located.*

The proof is based on the fact that all points of a strictly convex surface S and, accordingly, of a set $E \subset S$ are extreme, while the vertex set of a polytope $P = convE$ coincides with the set of its extreme points.

Corollary 2.1. *Finite spherically-, ellipsoidally-, and superspherically-located sets are vertex-located.*

Remark 2.1. Establishing vertex locality of discrete sets is, normally, rather complicated as it requires proof of inclusions $vert\ P \subseteq E$ and $vert\ P \supseteq E$. Theorem 2.1 offers a simpler way for the proof based on utilizing the concept of a strictly convex surface S circumscribed about E. Moreover, the vertex locality of the set E follows directly from the existence of a strictly convex function that defines the circumscribed surface S.

Let a function $f(x)$ be given on a finite point configuration E.

Definition 2.1. *An extension of the function $f(x)$ from E onto $E' \supset E$ is a function $F(x)$ defined on E' and coincided with $f(x)$ on E, i.e.*

$$\forall\, x \in E \quad F(x) = f(x)\,. \tag{2.1}$$

The condition (2.1) will be further written as

$$F(x) \underset{E}{=} f(x)\,. \tag{2.2}$$

The extension of a function from E to the entire space $\mathbb{R}^n$ will be further referred to as an extension of the function from the set E.

A function $F(x)$ is called a convex (strictly/strongly convex) extension of $f(x)$ from E onto E' if the set E' is convex and $F(x)$ is convex (strictly convex/strongly convex) on E'.

Theorem 2.2. *If E is a vertex-located set and $E' \supseteq E$ is a compact convex set, then for any function $f : E \to \mathbb{R}^1$ and for any $\rho > 0$, there exists a strongly convex function $F : E' \to \mathbb{R}^1$ with parameter at least ρ such that the condition* (2.2) *is satisfied.*

Theorem 2.2 establishes the existence of a strongly convex extension of an arbitrary function from an arbitrary vertex-located set to a convex compact set containing E.

Theorem 2.3. *An arbitrary vertex-located set is surface-located.*

The proof of Theorem 2.3 is based on applying Theorem 2.2 and constructing a strictly convex extension of a piecewise linear function defining the surface of the polytope P.

Thus, Theorems 2.1 and 2.3 establish a one-to-one correspondence between vertex-located and finite surface-located sets.

Remark 2.2. Without loss of generality, the function $f_1(x)$ in (1.21) can be seen as differentiable because otherwise, it can always be replaced by its strictly convex extension.

Theorem 2.4. *An arbitrary finite surface-located set E can be represented as the intersection of a strictly convex surface S' and a polytope P', i.e. there exists such surface S' and polytope P':*

$$E = P' \cap S'. \tag{2.3}$$

The proof of this theorem is based on the fact that if the set E is S-surface-located while P is its convex hull, then the strictly convex body C bounded by the surface S contains the polytope P. Moreover, the common points of S and P exactly form the set E. Therefore, there exists the following representation of E:

$$E = P \cap S, \tag{2.4}$$

where P is the polytope (1.46) and the surface S satisfies the conditions (1.19) and (1.21). Respectively, by choosing $S' = S$, $P' = P$, we come to (2.3).

For sets (2.3), we introduce a special terminology.

Definition 2.2. *A finite point configuration E is called a* <u>*polyhedral-surface set*</u> *if it is representable as the intersection of a polytope and a strictly convex surface.*

In this case, the functional representation of E of the form (2.3), which includes an H-representation of the polytope P' and the equation of a circumscribed strictly convex surface S', is called <u>a polyhedral-surface f-representation</u> of E.

Remark 2.3. Further, polyhedral-surface representations of sets will be constructed in the form (2.4), where P is given by the H-representation (1.52) and the strictly convex surface S is given by the equation $f_1(x) = 0$. Note that the representation (2.3) can also be valid

for an arbitrary surface-located set if P' is taken instead of P or its relaxation polytope. S' is also not required to be necessarily strictly convex. For example, it can be a polyhedral or even a non-convex surface. However, a functional representation of E constructed on the basis of (2.3) and consisting of the H-representation of P' and the equation of S' will be a polyhedral-surfaced representation of E only if the surface S' is strictly convex. Thus, we can assume that $S' = S$ and consider the representation of E in the form (2.3), i.e.:

$$E = P' \cap S. \tag{2.5}$$

We single out a class of surface-polyhedral sets consisting of finite point configurations representable in the form (2.4). Depending on the type of a strictly convex surface, its various subclasses can be introduced.

In particular, a finite point configuration E represented in the form (2.5) is a polyhedral-spherical set, if S is a hypersphere. Likewise, if S is an ellipsoid, E is called a polyhedral-ellipsoidal set; if S is a supersphere, E is a polyhedral-superspherical set. Thus, we have singled out the classes of polyhedral-ellipsoidal and polyhedral-superspherical sets along with the subclass of polyhedral-spherical sets formed in their intersection. Functional representations of these sets are called polyhedral-superspherical, polyhedral-ellipsoidal, and polyhedral-spherical f-representations, respectively.

For instance, the constraints (1.73) form a polyhedral-spherical representation of the set B_n. It can be generalized to polyhedral-superspherical case. Namely, a polyhedral-superspherical representation of B_n can be formed as follows:

$$f_1(x) = \sum_{i=1}^{n} \left| x_i - \frac{1}{2} \right|^k - \frac{n}{2^k} = 0;$$

$$f_{i+1}(x) = x_i - 1 \leq 0,\ f_{i+n+1}(x) = -x_i \leq 0,\ i \in J_n,$$

where $k > 1$ is a constant.

In terms of polyhedral-surface sets, Theorem 2.4 can be reformulated in the following way.

Theorem 2.5. *An arbitrary finite surface-located set is a polyhedral-surface set.*

Thus, all finite spherically-located sets are polyhedral-spherical.

Further, the following terminology will be used for the characterization of the vertex-located set E:

- "polyhedral-surface set", if an analytic description of P and S is known;
- "surface-located set", if we only know the equation of a strictly convex surface S.

For each polyhedral-surface set E, one can write a functional representation of the order $m = m' + m'' + 1$ having the form of (1.52) and

$$f_1(x) = 0 \tag{2.6}$$

being a polyhedral-surface representation of E. Its type is determined by the class of $f_1(x)$. For instance, if $f_1(x)$ is polynomial, then the entire polyhedral-surface representation (1.52), (2.6) is a polynomial strictly convex functional representation of E of the same degree as the polynomial $f_1(x)$. Polyhedral-spherical and polyhedral-ellipsoidal representations are examples of strictly convex quadratic functional representations. In contrast, a polyhedral-superspherical representation with $\alpha = 2$ is an example of such a biquadratic representation.

As mentioned above, f-representations of sets can be redundant. Let us examine the polyhedral-surface representation (1.52) and (2.6) on redundancy. Since, by assumption, E is not a singleton, the constraint (2.6) is essential since its elimination leads to consideration of the polytope P. Using an irredundant H-representation of P as a linear part of the polyhedral-surface representation is a necessary but insufficient condition for its non-redundancy. Indeed, if (1.52) contains at least one constraint whose elimination leads to the formation of a relaxation polytope $P' \supset P$ such that (2.5) holds, then the representation (1.52), (2.6) is a redundant f-representation of E.

However, some finite point configurations allow establishing that the constraint system (1.52) and (2.6) define their irredundant polyhedral-surface functional representation.

Theorem 2.6. *If E is a polyhedral-surface set and not a singleton, while its convex hull P is a simple polytope, then every irredundant H-representation of polytope P and the equation* (2.6) *form an irredundant functional representation of E.*

The proof of this theorem is based on the impossibility of constructing a relaxation polytope P' for a simple polytope P such that (2.5) holds.

Further typology of polyhedral-surface and vertex-located sets will be done by types of strictly convex surfaces and polytopes.

Finite point configurations can be formed as the intersection of polytopes with surfaces that are not necessarily strictly convex.

Let us formulate a generalization of the concept of a polyhedral-surface set. First, we introduce the required terminology.

The surface S given by (1.19) is called circumscribed about a set $M \subset \mathbb{R}^n$, and the set M is called inscribed in the surface S if the intersection of M and S forms a finite set

$$E = M \cap S \tag{2.7}$$

and for the points of M, one of the following conditions is satisfied:

$$\forall\, x \in M \;\; f_1(x) \leq 0 \tag{2.8}$$

or

$$\forall\, x \in M \;\; f_1(x) \geq 0.$$

Further, we suppose that $f_1(x)$ satisfies the condition (2.8).

Let M be a polytope. If it is inscribed in the surface S, this means the set (2.7) is S-surface-located if the condition (1.21) holds. If M is inscribed into S, this surface can be non-smooth convex like (1.26) or (1.27), non-convex, such as

$$S = \left\{ x \in \mathbb{R}^n : \prod_{i=1}^{n} x_i = 1 \right\}$$

etc. The main requirement remains that, likewise polyhedral-surface sets, a finite point configuration has to be formed as the intersection of S with M.

If in (2.7), the set M is a surface, then there exists a function $f_2\,(x)$ defining this

surface:

$$M = \{x \in \mathbb{R}^n : f_2(x) = 0\}. \tag{2.9}$$

Then, by analogy with (2.8), we can always assume that E is formed at the intersection of the surfaces S, M and at the intersection of spatial bodies:

$$C_1 = \{x\colon\ f_1(x) \leq 0\},$$

$$C_2 = \{x\colon\ f_2(x) \leq 0\}$$

providing that $f_2(x)$ satisfies the following condition:

$$\forall x \in S\ f_2(x) \geq 0.$$

Finally, two more E-representations are valid, namely, as the intersection of a surface with a spatial body:

$$E = C_1 \cap M,\ E = S \cap C_2.$$

It is easy to see that an FPC E inscribed in a surface is vertex-located. Then, there exists a strictly convex surface S' such that E is polyhedral-surface set with a functional representation involving the equation of the surface S'.

2.2 Multilevel finite point configurations and multilevel polytopes

There are convex polyhedral-surface representations for arbitrary vertex-located sets. On the other hand, according to Theorem 2.2, for any function defined on a vertex-located set, one can construct a convex (strictly/strongly convex) extension from the set. These peculiarities make it possible to use properties of the sets, their convex hulls, circumscribed strictly convex surfaces, and convex functions in constructing convex relaxations and solving other problems on these sets.

This is not always possible for an arbitrary finite point configuration. For example,

for a non-vertex-located FPC E that has points inside the polytope $convE$, there are no convex functional representations and convex extensions from them for functions given on E. Therefore, let us establish the connection of an arbitrary finite point configuration with vertex-located sets. We offer two ways to accomplish this.

The first one consists of selecting a strictly convex function $h(x)$ and constructing the decomposition of E into a family of strictly convex surfaces (1.11), (1.12). The sets (1.13) formed due to the decomposition are surface-located and, consequently, vertex-located according to Theorem 2.3. Thus, the transition from the set E to its vertex-located subsets is done.

The second way is based on lifting into Euclidean space $\mathbb{R}^{n'}$ $(n' > n)$, which dimension depends on the number of coordinates' values taken by E points, i.e. on the levelness $m'(E)$ of E along coordinates and on the number k_i of unique values of the coordinate x_i in the formula (1.9) $(i \in J_n)$.

Note that if E is not vertex-located, then the number $m'(E)$ is greater than two because

$$\exists\, i \in J_n :\ k_i > 2. \tag{2.10}$$

Moreover, for an arbitrary non-vertex-located set, there exists an extended polyhedral-surface representation.

Indeed, by the construction of sets $\{X_j\}_{j \in J_n}$ in (1.9), the coordinates of every point $x \in E$ take a finite number of different values from these sets, namely,

$$\forall\, x = (x_1, ..., x_n) \in E\ x_i \in X_i = \{e_{ij}\}_{j \in J_{k_i}},\ i \in J_n,$$

and as discrete variables, they are representable by binary variables:

$$x_i = \sum_{j=1}^{k_i} e_{ij} \cdot y_{ij},\ i \in J_n, \tag{2.11}$$

where $y_{ij} \in \{0,1\},\ j \in J_{k_i},\ i \in J_n,$

$$\sum_{j=1}^{k_i} y_{ij} = 1,\ i \in J_n. \tag{2.12}$$

The resulting discrete set

$$Y = \{y_{ij} \in \{0,1\} :\ y_{ij} \text{ satisfies the conditions (2.11), (2.12)}\}_{x \in E}$$

is vertex-located as a subset of the vertex-located binary set. Respectively, it allows the polyhedral-spherical representation (1.73).

Substituting (2.11) in the linear constraint system (1.52), we obtain

$$\begin{aligned} &\sum_{j=1}^{n} a'_{ij} x_j = \sum_{j=1}^{n} a'_{ij} \sum_{j'=1}^{k_j} e_{j'j} \cdot y_{j'j} = \sum_{j=1}^{n} \sum_{j'=1}^{k_j} (a'_{ij} e_{j'j}) y_{j'j} \le a'_{i0},\ i \in J_{n'}, \\ &\sum_{j=1}^{n} a''_{ij} x_j = \sum_{j=1}^{n} a''_{ij} \sum_{j'=1}^{k_j} e_{j'j} \cdot y_{j'j} = \sum_{j=1}^{n} \sum_{j'=1}^{k_j} (a''_{ij} e_{j'j}) y_{j'j},\ i \in J_{n''}. \end{aligned} \tag{2.13}$$

Complementing (2.13) with the equations (2.12) and two-sided constraints:

$$0 \le y_{ij} \le 1,\ j \in J_{k_i},\ i \in J_n, \tag{2.14}$$

we get an H-representation of the polytope $P' = conv\, Y$.

Now, complementing it by the equation:

$$\sum_{j=1}^{n} \sum_{i=1}^{k_j} \left(y_{ij} - \frac{1}{2} \right)^2 = \frac{n'}{4} \tag{2.15}$$

of the hypersphere circumscribed about Y, we come to a polyhedral-spherical representation (2.12)-(2.15) of the set Y, which is an extended functional representation of E in $\mathbb{R}^{n'}$, where $n' = \sum_{j=1}^{n} k_j \ge 2n$ according to Remark 1.1 and condition (2.10).

Let us connect the decomposition of a finite point configuration E into parallel hyperplanes with the H-representation of its convex hull, choosing normal vectors to the polytope P-facets as the direction vectors for the decomposition.

Since an arbitrary facet $F \in \mathbf{F}$ is associated with a support hyperplane $H_F \in \mathbf{H}$:

$$\exists \overline{n}_{H_F}, b_{H_F} : \ H_F = \{x \in \mathbb{R}^n : \overline{n}_{H_F}^{\top} x = b_{H_F}\},$$

then, when a certain $F \subseteq \mathbf{F}$ is chosen, we will consider the decomposition by (1.31), (1.32) toward the vector $\overline{n}_{H_F}$ into hyperplanes parallel to the facet F. At the same time, E is called $m_{(\overline{n}_{H_F})}$-level toward $\overline{n}_{H_F}$.

So, for example, if polytope P has facets parallel to the coordinate hyperplanes, then the formulas (1.35)-(1.39) define decompositions of E toward normal vectors of these facets. Respectively, k_j is the levelness of E along coordinates or towards normal vectors to coordinate hyperplanes ($j \in J_n$).

From the number of E levels toward normal vectors to all P-facets, we can determine the number

$$m(E) = \max_{F \in \mathbf{F}} m_{(\overline{n}_{H_F})}, \tag{2.16}$$

which is called the <u>levelness of E</u>, while E is called a <u>$m(E)$-level set</u>.

If E is an FPC and P is its convex hull, then comparison of the levelness $m(E)$ of E with the levelness (1.63) of the polytope P yields the estimate:

$$m(E) \geq m''(P). \tag{2.17}$$

If E is vertex-located, then since $V = E$ and the value $m''(P)$ specifies the levelness of the vertex set toward normal vectors to all P-facets. Respectively, (2.17) holds as equality:

$$m(E) = m''(P). \tag{2.18}$$

Now, we consider in detail the class of two-level sets and the corresponding polytopes.

For a two-level E,

$$m(E) = 2. \tag{2.19}$$

In this case, the formulas (1.65), (2.17), and (2.19) together yield:

$$m(E) = m''(P) = 2,$$

thus E and the polytope P are both two-level, and the condition (2.18) is also satisfied.

By analogy with a vertex set of two-level polytopes, a peculiarity of two-level sets is that they are two-level toward a normal vector of each facet of P, i.e.

$$\forall\, F \in \mathbf{F} \quad m_{(\overline{n}_{H_F})} = 2.$$

Therefore for two-level sets, the decomposition (1.31) becomes: for an arbitrary $F \in \mathbf{F}$,

$$\begin{aligned}
&E = E^1(\overline{n}_F) \cup E^2(\overline{n}_F),\\
&E^1(\overline{n}_F) \cap E^2(\overline{n}_F) = \varnothing,\\
&E^j(\overline{n}_F) = H^j(\overline{n}_F) \cap E \neq \varnothing,\ j = 1, 2.
\end{aligned}$$

For any facet F, we can always assume that $H^1(\overline{n}_F) = H_F$. Respectively,

$$E^1(\overline{n}_F) = E \cap H_F, E^2(\overline{n}_F) = E \backslash E^1(\overline{n}_F).$$

As it turns out, the two-levelness of a discrete set is sufficient for it to be surface-located, hence, vertex-located. This fact is established in the theorem below.

Theorem 2.7. *Every two-level finite point configuration is a surface-located set.*

Proof. Let E be a two-level FPC. Selecting its facet $F \in \mathbf{F}$, by (1.64), we have

$$\exists\, b_F \in \mathbb{R}^1 :\ H_F = \{x \in \mathbb{R}^n : \overline{n}_F^\top x = b_F\} \in \mathbf{F}.$$

In addition to the value b_F the function $h(x) = \overline{n}_H^\top x$ takes one more value on E. Let us denote it as b'_F and the corresponding parallel hyperplane as H'_F. Now, we have:

$\forall F \in \mathbf{F}\ \exists b_F, b'_F,\ b_F \neq b'_F,$

$$H_F = \{x \in \mathbb{R}^n : \overline{n}_F^\top x = b_F\},\ H'_F = \{x \in \mathbb{R}^n : \overline{n}_F^\top x = b'_F\},$$
$$E^1(\overline{n}_F) \subset H_F; E^2(\overline{n}_F) \subset H'_F,$$

in particular,

$$\forall\, x \in E\ \overline{n}_F^\top x \in \{b_F, b'_F\},\ F \in \mathbf{F}. \tag{2.20}$$

Let us introduce the notation

$$a_F = \frac{b_F + b'_F}{2},\ \delta_F = \frac{|b_F - b'_F|}{2}$$

and rewrite (2.20) as

$$|\overline{n}_F^\top x - a_F| = \delta_F,\ F \in \mathbf{F}. \tag{2.21}$$

By normalizing the equation (2.21), we get

$$|\overline{n}_F^{'T} x - a'_F|^k = 1,\ F \in \mathbf{F}, \tag{2.22}$$

where $\overline{n}_F^{'T} = \frac{\overline{n}_F}{||\overline{n}_F||}$, $a'_F = \frac{a_F}{||\overline{n}_F||}$.

Having fixed a certain $k \in \mathbb{R}^1_{>0}$, we add all the equations (2.22) getting:

$$f_1(x,k) = \sum_{F \in \mathbf{F}} |\overline{n}_F^{'T} x - a'_F|^k - |F| = 0. \tag{2.23}$$

Consider the family of surfaces given by the equation (2.23):

$$S^k = \{x \in \mathbb{R}^n :\ f_1(x,k) - |F| = 0\},\ k \in \mathbb{R}^1_{>0} \tag{2.24}$$

The function $f_1(x,k)$ is non-convex for $k \in (0,1)$ and convex for $k \geq 1$, in particular, it is strictly convex for $k > 1$ and strongly convex for $k \geq 2$.

Let $k > 1$. By construction, the point x^0 formed at the intersection of any n hyperfaces from the family $\overline{n}_F^\top x = a_F,\ F \in \mathbf{F}$, whose normal vectors are linearly independent, is an

interior point of the spatial body $C^k = \{x \in \mathbb{R}^n : \ f_1(x,k) - |F| \leq 0\}$, $k \in \mathbb{R}^n_{>0}$. This means the body is strictly convex, while the surface (2.24) is strictly convex. Therefore, for any $k \in (1, \infty)$, E is S^k-surface-located set.

□

Note that the surfaces of the family (2.24) are bounded. In particular, S^1 is a polyhedral surface, S^2 is an ellipsoid, $S^\infty = \lim\limits_{k\to\infty} S^k = P$ is a polytope.

Corollary 2.2. *Every two-level finite point configuration is a vertex-located set.*

This corollary says that a two-level FPC coincides with the vertex set of its convex hull, which is a two-level polytope. On the other hand, all two-level polytopes are combinatorically equivalent to a certain $0-1$-polytope. This means that in the study of two-levels FPCs, without loss of generality, we can assume that they are all subsets of the binary set, i.e. they are $0-1$ sets. This allows certifying that the number of facets of a d-dimensional two-level polytope varies within the range $[d+1, 2^d]$.

Remark 2.4. The family of surfaces (2.24) was constructed based on utilizing an irredundant H-representation of the polytope P, which is supposed to be known. Complementing it with the equation $f_1(x,k) - |F| = 0$, $k \in (1, \infty)$, we obtain a family of polyhedral-surface representations of the two-level set E, which can further be examined for irredundancy.

The equation of the surface S^k can also be complemented with an H-representation of the polytope P obtained from the constraints (2.22) by weakening the equal sign as follows:

$$P = \{x \in \mathbb{R}^n : \ |\overline{n}_F^{'T} x - a'_F|^k \leq 1, \ F \in \mathbf{F}\}.$$

Selecting a subfamily of surfaces S^k, $k = 2k'$, $k' \in \mathbb{N}$ from the family (2.24) allows deriving a family of polynomial polyhedral-surface representations of E.

Corollary 2.3. *An arbitrary two-level finite point configuration is ellipsoidally-located set.*

Indeed, as the circumscribed ellipsoid, one can choose the surface

$$S^2 = \{x \in \mathbb{R}^n : \ f_1(x,2) = 0\} \tag{2.25}$$

from the family (2.24). Its equation is

$$\sum_{F\in\mathbf{F}} (\overline{n}_F^{'\top} x - a'_F)^2 = |F|.$$

Let us rewrite it in the form:

$$x^\top \left(\sum_{F\in\mathbf{F}} \overline{n}'_F \overline{n}_F^{'\top} \right) x - 2 \left(\sum_{F\in\mathbf{F}} a'_F \overline{n}_F^{'\top} \right) x + \sum_{F\in\mathbf{F}} a_F^{'2} = |F|. \tag{2.26}$$

and represent it in the form (1.25). We rewrite the formula (2.25) as (1.23):

$$x^\top A x + b^\top x + c = 0,$$

where

$$A = \sum_{F\in\mathbf{F}} \overline{n}'_F \overline{n}_F^{'\top},\ B = 2 \sum_{F\in\mathbf{F}} a'_F \overline{n}_F^{'\top},\ c = \sum_{F\in\mathbf{F}} a_F^{'2} - |F|.$$

The matrix A is positive definite. Hence, S^2 is an ellipsoid circumscribed about E.

2.3 Operations on finite point configurations

Let us explore properties of some operations on FPCs leading to the formation of new FPCs. E вида (1.8) как результат некоторых теоретико-множественных операций над этими множествами.

Let E be a finite point configuration of the form (1.8) obtained by logical operations on these sets.

Let us partite the number n into L numbers $n^1, ..., n^L \in J_n$, i.e. $n = \sum\limits_{l=1}^{L} n^l$, and introduce

$$E^l \subset \mathbb{R}^{n^l},\ l \in J_L, \tag{2.27}$$

such that $1 \le n_{E^l} = |E^l| < \infty,\ l \in J_L$.

Let an element of E^l be denoted as x^l, in particular, in the coordinate form, $x^l = (x_{il})_{i\in J_{n^l}}$ $(l \in J_L)$.

In formula (2.27), the superscript specifies the index of the corresponding set. We will use similar indexing for its convex hulls, faces in their H-representation, circumscribed surfaces, etc. For example, the notation

$$P^l = conv\ E^l,\ l \in J_L \tag{2.28}$$

defines a family of polytopes associated with (2.27).

For a fixed $l \in J_L$, the system of linear constraints describing P^l is

$$A'^l x \leq a'^l_0,\ A''^l x = a''^l_0,$$

where

$$x^l \in \mathbb{R}^{n^l},\ A'^l = (a'^l_{ij}) \in \mathbb{R}^{n'^l \times n},\ A''^l = (a''^l_{ij}) \in \mathbb{R}^{n''^l \times n},$$
$$a'^l_0 = (a'^l_{i0}) \in \mathbb{R}^{n'^l},\ a''^l_0 = (a''^l_{i0}) \in \mathbb{R}^{n''^l}$$

is the H-representation of P^l; $\mathbf{F}^l$ and $\mathbf{H}^l$ are the sets of its facets and the corresponding supporting hyperplanes. Also, $V^l = vert\ P^l$ is a set of vertices of P^l ($l \in J^L$). The degree of a vertex $x^l \in V^l$ of the polytope P^l is denoted by $\mathcal{R}^l(x^l)$. If all vertices of P^l are regular, then $\mathcal{R}^l$ denotes the regularity degree for P^l-vertices. Let also $\mathcal{A}^l$ be the generating set of E^l; $m(E^l), m'(E^l)$ be the levelness of E^l as a whole and along coordinates, respectively; $m''(P^l)$ be the levelness of the polytope P^l. The dimension of the polytope P^l is denoted by $d_l = \dim\ P^l$, and its f-vector is $f(P^l) = (f_0(P^l), ..., f_{d_l-1}(P^l))$, $l \in J_L$.

In some cases, sets E_l, $l \in J^l$, may possess specific properties underlying single outing the following classes:

- E_l, $l \in J_l$ are vertex-located sets:

$$E^l = V^l,\ l \in J_L; \tag{2.29}$$

- for $l \in J_l$, E_l lie on the surface S^l given by the corresponding equations

$$\exists f^l(x^l) : \mathbb{R}^{n_l} \to \mathbb{R}^1 : \; f^l(x^l) \underset{E^l}{=} 0, \; l \in J_L. \tag{2.30}$$

So, we have

$$E^l \subseteq S^l, \tag{2.31}$$

$$S^l = \{x \in \mathbb{R}^{n^l} : f^l(x^l) = 0\}, \; l \in J_L; \tag{2.32}$$

- E_l, $l \in J_l$ are inscribed in the surfaces (2.32) implying that the conditions (2.31) are met:

$$E^l = P^l \cap S^l, \tag{2.33}$$

$$f^l(x^l) \underset{P^l}{\leq} 0, \; l \in J_L; \tag{2.34}$$

- E_l, $l \in J_l$ are surface-located sets, which implies the fulfillment of three conditions (2.31), (2.32), as well as

$$f^l(x^l), \; l \in J_L \text{ are strictly convex functions.} \tag{2.35}$$

This involves inclusion (2.31), and also $\forall l \in J_L \; \exists \, a^l \in \mathbb{R}^{n^l}, \; r^l \in \mathbb{R}^1$

- E_l, $l \in J_l$ are spherically-located sets. For the minimal hypersphere circumscribed about E^l, we will use the notation $S_{r^{\min,l}}(a^{\min,l})$, $l \in J_L$;

- E_l, $l \in J_l$ are ellipsoidal-located sets. This means the inclusion (2.31) holds along with

$$\exists x^{0l} \in \mathbb{R}^{n^l}, \; \exists \, C^l \in \mathbb{R}^{n^l \times n^l}, \; C^l \succ 0 :$$
$$S^l = \{x^l \in \mathbb{R}^{n^l} : \; (x^l - x^{0l})^\top C^l (x^l - x^{0l}) = 0\}, \; l \in J_L.$$

Next, explore properties of the set E formed due to logical operations on E_l, $l \in J_l$.

In addition, we will study the properties of the polytope $P = conv\ E$ and polytopes (2.28) participating in constructing P. In this case, for the f-vector of the polytope, we will use the notation $f(P) = (f_0(P), f_1(P), ..., f_{d-1}(P))$.

Let us consider the basic logical operations on finite point configurations and derive their main properties.

2.3.1 Subsets of finite point configurations

Let $L = 1$, i.e. $n^1 = n$ and

$$E \subset E^1. \tag{2.36}$$

As proper subset of E^1, the set E has the following features:

1. $|E| < |E^1|$;
2. $P \subseteq P^1$, i.e. P^1 either coincides with P or is its relaxation;
3. dim $P \leq$ dim P^1;
4. $V \subseteq V^1$. Moreover, $V = V^1$ if the operation (2.36) does not affect the vertices; otherwise, $V \subset V^1$. The latter particularly concerns the vertex-located set E^1;
5. $\mathcal{A} \subseteq \mathcal{A}^1$, $m'(E) \leq m'(E^1)$;
6. a) if E^1 lies on the surface S^1, then E also lies on it; b) if E^1 is inscribed in the surface S^1, then E is also inscribed in it; c) if E^1 is S^1-surface-located, then E is also surface-located (for example, if E is a sphericaly-located set, then the same is true for E^1, while $r^{\min} \leq r^{\min,1}$);
7. if E^1 is vertex-located, then E is also vertex-located. Moreover, for P, the strict inclusion $P \subset P^1$ is satisfied.

The subset (2.36) has extended properties comparing E-ones.

Example 2.1. Consider the sets $E = \{(1,2,3),(1,3,2),(2,1,3),(2,3,1),(3,1,2),(3,2,1)\}$ and $E^1 = J_3^3$. They are depicted in Figures 2.1 and 2.2, and their convex hulls, being the

regular hexagons P and cube P^1 are shown in Figures 2.3 and 2.4. One can see that E^1 is not vertex-located set, while its proper subset E is vertex-located.

The levelness of the sets E and E^1 coincides, namely:

$$m(E) = m(E^1) = m'(E) = m'(E^1) = 3.$$

The generating sets are also identical:

$$\mathcal{A} = \mathcal{A}^1 = \{1, 2, 3\}.$$

Both E and E^1 are centrally symmetric about the point $O' = (2, 2, 2)$.

Let us list some new properties of E compared to E^1. In addition to vertex locality, the set E lies in the hyperplane (see Figure 2.3). It is spherically-located (see Figure 2.5), while the original set E^1 did not have these features, as seen from Figure 2.4.

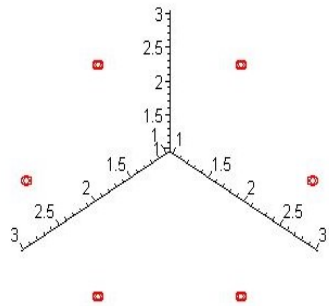

Figure 2.1: The set E

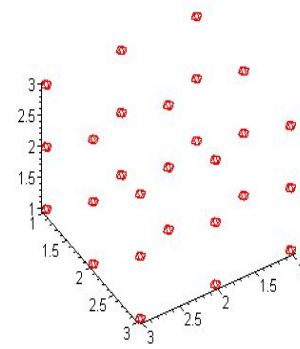

Figure 2.2: The set E^1

2.3.2 Intersection of finite point configurations

Let $L = 2$, and the sets E^1 and E^2 be such that

$$n = n^1 = n^2, \tag{2.37}$$

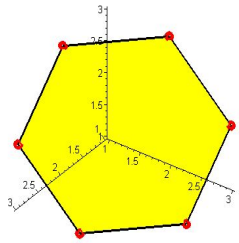

Figure 2.3: The set E and corresponding polytope P

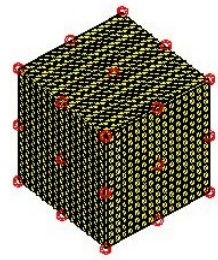

Figure 2.4: The set E^1 and corresponding polytope P

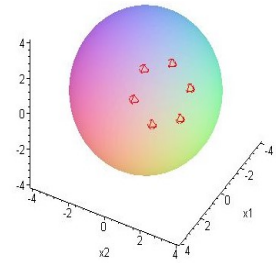

Figure 2.5: The set E and hypersphere $S_{\sqrt{14}}(\mathbf{0})$

while E is

$$E = E^1 \cap E^2.$$

Suppose E is a proper subset of E^1 and E^2:

$$E \subset E^1,\ E \subset E^2. \tag{2.38}$$

Then we have

1. $|E| < \min\{|E^1|, |E^2|\}$.
2. $P \subseteq P^1,\ P \subseteq P^2$.
3. $\dim\ P \leq \min\{\dim\ P^1, \dim\ P^2\}$.
4. $V \subseteq V^1,\ V \subseteq V^2$.
5. $\mathcal{A} = \mathcal{A}^1 \cap \mathcal{A}^2$, whence it follows that $m'(E) \leq \min\{m'(E^1), m'(E^2)\}$.
6. If E^1 or E^2 is vertex-located set, then E is also vertex-located. In this case, due to (2.38), $P \subset P^1,\ P \subset P^2$ holds, i.e. both polytopes P^1 and P^2 are relaxation polytopes of P.
7. If E^1 and E^2 are centrally symmetric about the origin O', then E also has a center of symmetry at O'.
8. If P^1 and P^2 are centrally symmetric about O', then P is also centrally symmetric.

9. A set E will belong to the surface S if E^1 or E^2 lies on this surface.

If E^1 and E^2 lie on the surfaces S^1 and S^2 respectively, and

$$S^1 \neq S^2, \tag{2.39}$$

then E lies on both of these surfaces:

$$E \subset S^1 \cap S^2, \tag{2.40}$$

and inscribed in them in the second:

$$E = S^1 \cap P^1,\ E = S^2 \cap P^2.$$

If, in addition to (2.40), the condition (2.35) holds, i.e. E^1 and E^2 are surface-located sets, then E is also a surface-located set:

$$E = S^1 \cap P,\ E = S^2 \cap P.$$

Moreover, there exists a family

$$S(\alpha),\ \alpha \in [0, 1] \tag{2.41}$$

of strictly convex surfaces circumscribed about E such that

$$S = S(\alpha) = \{x \in \mathbb{R}^n : f(x, \alpha) = \alpha f^1(x) + (1 - \alpha) f^2(x) = 0\}. \tag{2.42}$$

When S^1 and S^2 are hyperspheres, then, by (2.39), a set E is spherically-located, lies in the hyperplane β defined by the intersection S^1 and S^2. In this case, formula (2.42) defines a family of hyperspheres centred on the line connecting the points $a^{\min,1}$, $a^{\min,2}$, the centers of the hyperspheres $S^{\min,1}, S^{\min,2}$. The center $S^{\min}$ lies at the intersection of this line and the hyperplane, moreover, $a^{\min} \in [a^{\min,1},\ a^{\min,2}]$. Accordingly, the radii of the circumscribed

hyperspheres of the minimum radius satisfy the relation $r^{\min} \leq \min\{r^{\min,1}, r^{\min,2}\}$.

When S^1 and S^2 are ellipsoids, the set E is ellipsoidally-located like E^1 and E^2. In this case, the formulas (2.41) and (2.42) define a family of ellipsoids, among which one can choose the ellipsoid of the minimum volume. In addition, E belongs to the intersection of the ellipsoids S^1 and S^2.

Remark 2.5. The surface $S' = S^1 \cap S^2$ possesses specific properties. If S^1 and S^2 are hyperspheres, then S' is an $n-2$-sphere. If the surfaces are ellipsoids, then S' is an $n-2$-ellipsoid. This allows considering the orthogonal projection of the set E onto the plane β, where the set formed in the projection will also be a spherically-located or ellipsoidally-located set, respectively.

2.3.3 Intersection of finite point configurations and surfaces

Let $L = 1$ and a finite point configuration formed at the intersection of E^1 with the surface S br considered, i.e.

$$E = E^1 \cap S, \tag{2.43}$$

where S is the surface (1.19).

Without loss of generality, we can assume that $\exists\, x \in E^1 :\ x \notin S$, i.e. $E \subset E^1$ and all the properties listed above in this section are applicable to E.

Properties of E depend both on the set E^1 and by a function $f_1(x)$ given by (1.19). Let us outline some properties of cuts of finite point configurations by hyperplanes and their intersections with strictly convex surfaces.

Let the function $f_1(x)$ defining the surface S of the form (1.19) be linear, i.e.

$$\exists\, c \in \mathbb{R}^n,\ c_0 \in \mathbb{R}^1 :\ f_1(x) = c^\top x + c_0.$$

Then, the set E given by (2.43) is formed at the intersection of E^1 with the hyperplane $H :\ c^\top x + c_0 = 0$. The set has the following properties:

1. $\dim P < \dim P^1$;

2. the H-representation of the polytope P is the following system of constraints:

$$P = \{x \in \mathbb{R}^n : \ A'^1 x \leq a_0'^1, \ A''^1 x = a_0''^1, \ c^\top x + c_0 = 0\}. \tag{2.44}$$

   Moreover, the H-representation (2.44) can be redundant, regardless of whether the redundant or irredundant H-representation of P' is taken for its construction;

3. if E^1 is a spherically-located set, then E is also spherically-located. The parameters of $S^{\min}$ can be determined from the equation of the hyperplane H, and the parameters of $S^{\min,1}$:

   - $a^{\min,1}$ is the projection of $a^{\min}$ onto S,
   - $r^{\min,1} = ((r^{\min})^2 - h^2)^{1/2}$,

   $h = |a^{\min,1} - a^{\min}| = \dfrac{|c^\top a^{\min,1} + c_0|}{|c|}$ is the distance from $a^{\min,1}$ to H.

2.3.4 Intersection of finite point configurations and strictly convex surfaces

Proposition 2.1. *If the surface* (1.19) *is strictly convex, then E of the form* (2.43) *is a vertex-located set.*

This property allows decomposing an arbitrary FPC into vertex-located ones.

Proposition 2.2. *Suppose $h(x)$ is a strictly convex function. In that case, the formulas* (1.11)-(1.14) *define the decomposition of a finite point configuration E into the family of strictly convex surfaces given by $h(x)$ and the E-decomposition into vertex-located sets.*

From Proposition 2.3.4, it follows that if the surface S given by (1.19) is a hypersphere, then E of the form (2.43) is spherically-located set. If E is an ellipsoid, then E is an ellipsoidally-located set.

2.3.5 Union of finite point configurations

Let $L = 2$, the conditions (2.37) and (2.38) be satisfied, and finite point configuration E be formed as follows:

$$E = E^1 \cup E^2.$$

Then the following properties hold.

1. $|E| \geq \max\{|E^1|, |E^2|\}$.
2. $P \supseteq P^1$, $P \supseteq P^2$;
3. $\dim\ P \geq \max\{\dim\ P^1, \dim\ P^2\}$.
4. $V \subseteq V^1 \cup V^2$.
5. $\mathcal{A} = S(\mathcal{A}^1 \cup \mathcal{A}^2)$, $m'(E) \geq \max\{m'^1(E^1), m'^2(E^2)\}$.
6. if FPCs E^1 and E^2 are vertex-located sets, then at least one of the conditions $P \supset P^1$, $P \supset P^2$ is satisfied. Accordingly, P is a relaxation polytope for at least one of them.
7. if FPCs E^1 and E^2 are centrally symmetric sets about the origin O', then E also has a center of symmetry at O'.
8. if P^1 and P^2 are centrally symmetric polytopes about the origin O', then P is also a centrally symmetric polytope about this point.

When union two finite point configurations, properties appear that are not characteristic of any of the constituent sets. We illustrate this with an example. Let

$$E^1 = \{(0,0,0), (1,0,0), (0,2,0), (0,0,3)\},\ E^2 = -B_3(1).$$

The convex hull of E^1 is the three-dimensional simplex P^1 depicted in Figure 2.6. Regarding levelness, E^1 and the corresponding polytope P^1 are two-level, i.e. $m(E^1) = m'(E^1) = m''(P^1) = 2$. Similarly, the set E^2 and the corresponding polytope P^2 are two-level.

The union of finite point configurations E^1 and E^2 is the set

$$E = \{(0,0,0),(1,0,0),(0,2,0),(0,0,3),(-1,0,0),(0,-2,0),(0,0,-3\},$$

shown in Figure 2.7.

The convex hull of this set is the octahedron shown in Figure 2.8. Since $(0,0,0) \in int\ P$, the set E is not located at vertices of P. Therefore, it is not surface-located set. At the same time, it acquires the property of being centrally symmetric about the origin and symmetric about each coordinate hyperplane. Given this symmetry, to determine the number $m(E)$ it suffices to construct a facet through the points $(1,0,0),(0,2,0),(0,0,3)$, $H:\ h(x) = \frac{x_1}{1} + \frac{x_2}{2} + \frac{x_3}{3} = 1$, and then determine the levelness $m_{h(x)} = 3$ of E toward the normal vector of the facet $h(x)$, hence, $m(E) = 3$. Levelness along coordinates also increases when moving from considering E^1, E^2 to E, yielding $m'(E) = 3$. In order to determine the levelness of the polytope P it suffices to consider the set of vertices $V = E\backslash\{(0,0,0)\}$, where the function $h(x)$ takes only two values. This implies that the polytope P is two-level. An ellipsoid can be circumscribed about it, which is easy to find given the symmetry of the polytope. Its equation is $x_1^2 + \frac{x_2^2}{4} + \frac{x_3^2}{9} = 1$. This demonstrates the possibility of the E-decomposition into two vertex-located sets, the ellipsoidally-located set V and the point $\{(0,0,0)\}$.

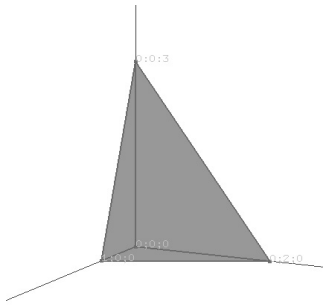

Figure 2.6: The set E^1 and corresponding polytope P

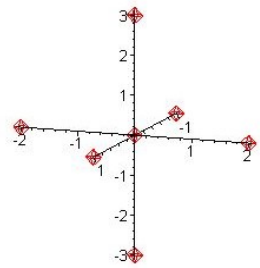

Figure 2.7: The set E

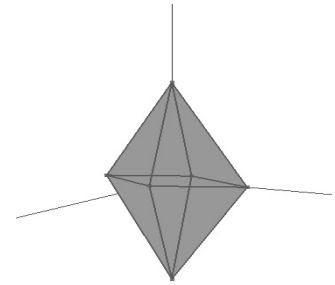

Figure 2.8: The polytope P

Remark 2.6. The same properties of the difference $E^1\backslash E^2$ and the symmetric difference $E^1 \triangle E^2$ can be formulated similarly. We only note that these two operations make it possible

to form vertex-located sets from non-vertex-located sets. It can underlie decompositions of E into vertex-located sets.

2.3.6 Minkowski sum and difference of finite point configurations

Let $L = 2$, the condition (2.37) be satisfied, and the set E be formed by the rule:

$$E = E^1 \oplus E^2, \tag{2.45}$$
$$E = \{x \in \mathbb{R}^n : x = x^1 + x^2,\ x^1 \in E^1,\ x^2 \in E^2\},$$

where $\oplus$ is Minkowski sum operation.

Let us list some properties of the finite point configuration (2.45) and corresponding polytope $P = conv\ E$:

1. $|E| \leq |E_1| \cdot |E_2|$;

2. $\dim\ P \geq \max\{\dim\ P^1, \dim\ P^2\}$;

3. the generating set of E can be found by the rule:

$$\mathcal{A} = S(\{e \in \mathbb{R}^1 :\ e = e^1 + e^2,\ e^l \in \mathcal{A}^l,\ l \in J_2\}),$$

   wherefrom the bound follows that $m'(E) \leq m'(E^1) \cdot m'(E^2)$;

4. if the FPCs E^1 and E^2 are centrally symmetric about O^1 and O^2, correspondingly, then E has the center of symmetry at the origin $O^1 + O^2$. The same applies to the corresponding polytopes P, P^1 and P^2;

5. if the FPCs E^1 and E^2 have parallel axes of symmetry $\gamma^l : a^\top x = a_0^l$, $l = 1, 2$, then the line $\gamma : a^\top x = a_0^1 + a_0^2$ is the axis of symmetry of E. The same applies to the polytope P, which has the axis of symmetry if the polytopes P^1 and P^2 have the same parallel axes of symmetry.

Example 2.2. Let $E^1 = \{(1, 2), (2, 1)\}$ and $E^2 = \{(-1, -1), (-1, 1), (1, -1), (1, 1)\}$.

These sets are shown in Figures 2.9 and 2.10. As one can see, they are centrally symmetric. Particularly, E^1 is symmetric about the bisector of the first coordinate angle, and the line $\gamma^1 : x_1 + x_2 = 3$, while E^2 is symmetric about the coordinate axes and the bisectors of the first and second coordinate angles. As a result, E of the form (2.45) is

$$E = \{(0,1),(1,0),(0,3),(1,2),(2,1),(3,0),(2,3),(3,2)\} \tag{2.46}$$

and is centrally symmetric. Its symmetry axes are the bisector $x_1 - x_2 = 0$ of the first coordinate angle and the line γ is parallel to the bisector of the second coordinate angle (see Figure 2.11). It is also seen that $m'(E^1) = m'(E^2) = 2$, $m'(E) = m'(E^1) + m'(E^2) = 4$.

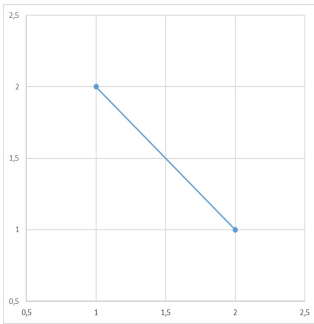

Figure 2.9: The set E^1 and polytope P^1

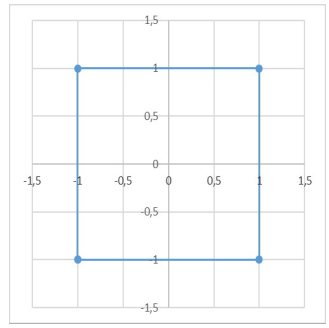

Figure 2.10: The set E^2 and polytope P^2

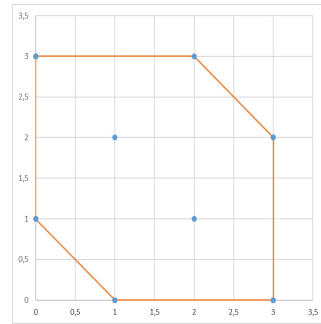

Figure 2.11: The set E and polytope P

Remark 2.7. Similarly, one can formulate some properties of the Minkowski difference

$$E = E^1 \overset{*}{\ominus} E^2, \tag{2.47}$$

We can rewrite (2.47) as

$$E^1 \overset{*}{\ominus} E^2 = E^1 \oplus (-E^2),$$

where $-E^2 = \{-x : \ x \in E^2\}$.

So, for the sets E^1 and E^2 given in Example 2.2, the Minkowski difference (2.47) is the same as the Minkowski sum (2.46) (see Figure 2.11) because, in this case, $E^2 = -E^2$.

2.3.7 Hadamard product of finite point configurations

Let $L = 2$ and the condition (2.37) be satisfied, while the finite point configuration E be constructed as the Hadamard product of the sets E^1 and E^2, i.e.

$$E = E^1 \circ E^2, \tag{2.48}$$

$$E = \{x \in \mathbb{R}^n : x = (x_i)_{i \in J_n}, x_i = x_i^1 x_i^2,\ i \in J_n,\ \forall x^1 = (x_i^1)_{i \in J_n} \in E^1,\ x^2 = (x_i^1)_{i \in J_n} \in E^2\}.$$

Some properties of E and of the corresponding polytope P are

1. $|E| \leq |E_1| \cdot |E_2|$;

2. $\dim\ P \geq \max\{\dim\ P^1, \dim\ P^2\}$;

3. For E, the generating set $\mathcal{A}$ is

$$\mathcal{A} = S(\{e \in \mathbb{R}^1 :\ e = e^1 \cdot e^2,\ e^l \in \mathcal{A}^l,\ l \in J_2\}),$$

   wherefrom we have the bound $m'(E) \leq m'(E^1) \cdot m'(E^2)$ on the levelness of E along coordinates;

4. If E^1 or E^2 is symmetric about a certain coordinate hyperplane, then the finite point configuration (2.48) is also symmetric about this hyperplane.

5. If E^1 or E^2 are centrally symmetric sets, then E is also centrally symmetric. Similarly, the polytope P is centrally symmetric if the polytopes P^1 or P^2 are centrally symmetric.

Remark 2.8. Let the following condition be satisfied for E^1:

$$E^1 = -E^1, \tag{2.49}$$

i.e. this set is centrally symmetric. Then, for every set E^2, the Hadamard product (2.48) has the addition property:

$$\begin{aligned} &E = E^1 \circ E^{2'}, \\ &E^{2'} = \{x \in \mathbb{R}^n : x_i = |y_i|,\ i \in J_n\}_{y \in E^2}. \end{aligned} \tag{2.50}$$

The formula (2.50) says that, under condition (2.49), the second component in the Hadamar product (2.48) belongs to the orthant $\mathbb{R}^n_+$. At the same time, E is formed as the Hadamard product of the sets $E^{2'}$ and $E^{1+} = E^1 \cap \mathbb{R}^n_+$ with subsequent reflection of the formed domain about all coordinate hyperplanes.

Example 2.3. Let E^1 be the vertex set of the rectangle P^1 shown in Figure 2.12, and E^2 be the vertex set of the triangle P^1 shown in Figure 2.13. Their Hadamard product is not a vertex-located set E, but the set is symmetric about the coordinate axes (see Figure 2.14). Its convex hull is the octagon P shown in Figure 2.14.

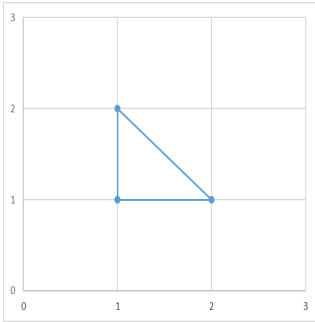

Figure 2.12: The set E^1 and the polytope P^1

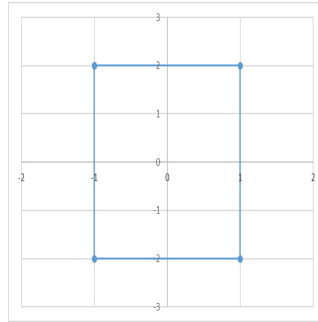

Figure 2.13: The set E^2 and the polytope P^2

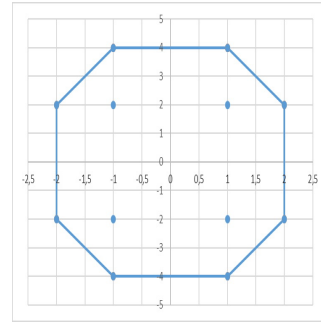

Figure 2.14: The set E and the polytope P

Example 2.4. Let E be formed from the binary set $E^1 = B_3$ and the set E^2 such that E coincides with the one in Example 2.1. The result of applying the formula (2.48) and the set's convex hull are shown in Figure 2.15. In this case, $E^2 \subset \mathbb{R}^3_+$, hence, $E^2 = E^{2'}$ in the formula (2.50). The set E contains $|E| = 34$ elements, including six elements of E^2, eighteen of its projections onto the coordinate hyperplanes, nine of its projections onto the coordinate axes, and one of the projections onto the origin. This set is not centrally symmetric, has no hyperplane of symmetry, and is not vertex-located, while the sets E^1 and E^2 possess all these properties. The generating set of E is $\mathcal{A} = J_3^0$. For E^1 and E^2, we have $\mathcal{A}^1 = \{0, 1\}$ and $\mathcal{A}^2 = J_3$, respectively. The levelness of E along coordinates is $m'(E^1) = 2$, $m'(E^2) = 3$, $m'(E) = 4 > \max\{m'(E^1),\ m'(E^2)\}$.

Example 2.5. Let the set (2.48) is constructed from the set $E^1 = \{-1, 1\}^3$ and the set E^2 given in the previous example. Since these sets satisfy the condition (2.49), the formed

set E and its convex hull P have the center of symmetry at the origin (see Figure 2.15). Also, the coordinate hyperplanes are its hyperplanes of symmetry. The generating set is $\mathcal{A} = -\mathcal{A}^2 \cup \mathcal{A}^2 = \{-3, -2, -1, 1, 2, 3\}$. The levelness along coordinates is $m'(E^1) = 2$, $m'(E^1) = 3$, $m'(E) = 6 = m'(E^1) \cdot m'(E^2)$. In Example 2.1, it was shown that E^2 is spherically-located set, and its elements are equidistant from the origin. The property of spherical locality is preserved for E. The hypersphere circumscribed about this set is given by the equation $x_1^2 + x_2^2 + x_3^2 = 14$. Accordingly, E is also a vertex-located set.

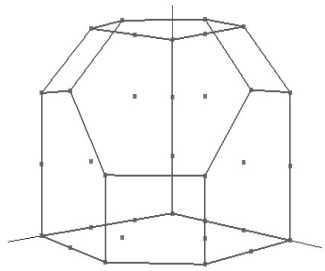

Figure 2.15: The set E and the polytope P (Example 2.4)

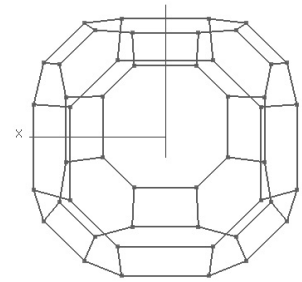

Figure 2.16: The set E and the polytope P (Example 2.5)

Examples 2.4 and 2.5 demonstrate that for spherically located centrally symmetric sets, applying Hadamard product operation can produce both vertex-located and non-vertex-located sets, centrally symmetric and non-centrally symmetric sets, spherically-located and non-spherically-located sets. Let E be the Cartesian product of the sets E^l, $l \in J^L$ formed by (2.27):

$$E = \bigotimes_{l=1}^{L} E^l, \tag{2.51}$$
$$E = \{x = (x^1, ..., x^L) \in \mathbb{R}^n : \ x^l \in E^l, \ l \in J_L\},$$

$x = (x^1, ..., x^L) = (x_{11}, ..., x_{n^1 1}, ..., x_{1L}, ..., x_{n^L L})^\top$.

Let us list some properties of the Cartesian product of the FPCs E^l, $l \in J^L$.

The cardinality of E

$$|E| = \prod_{l=1}^{L} |E^l|.$$

Polytope P is the Cartesian product of the polytopes P^l, $l \in J^L$ given by (2.28):

$$P = \bigotimes_{l=1}^{L} P^l. \tag{2.52}$$

The dimension of P

$$\dim\ P = \sum_{l=1}^{L} \dim\ P^l.$$

It is clear that

$$\dim\ P < n \ \Leftrightarrow\ \exists\, l \in J_L :\ \dim\ P^l < n^l, \tag{2.53}$$

i.e. E lies on a hyperplane if and only if at least one of the sets given by (2.27) lies on a certain hyperplane.

The full-dimensionality criterion for P

The polytope P in (2.52) is full-dimensional if and only if all the polytopes (2.28) are full-dimensional.

Irredundant H-representation of P

Proposition 2.3. *The irredundant H-representation of the polytope P has the form of a linear constraint system* (1.52)*, where:*

- $n' = \sum_{l=1}^{L} n'^l$ *is the number of inequality constraints,* $n'' = \sum_{l=1}^{L} n''^l$ *is the number of equality constraints;*

- *the matrix* A' *has the dimension* $n' \times n$ *and block type such that* $A' = (A'_{ll'})_{l,l' \in J_L}$, *where*

$$A'_{ll} = A'^{l} \in \mathbb{R}^{n'^{l} \times n^{l}},\ l \in J_L;\ A'_{ll'} = \mathbf{0} \in \mathbb{R}^{n'^{l} \times n^{l'}},\ l, l' \in J_L,\ l \neq l';$$

- *the constraint matrix* A'' *has the dimension* $n'' \times n$ *and block type such that* $A'' = (A''_{ll'})_{l,l' \in J_L}$, *where*

$$A''_{ll} = A''^{l} \in \mathbb{R}^{n''^{l} \times n^{l}},\ l \in J_L;\ A''_{ll'} = \mathbf{0} \in \mathbb{R}^{n''^{l} \times n^{l''}},\ l, l' \in J_L,\ l \neq l';$$

- *the right parts' vectors* a'_0, a''_0 *are*

$$a'_0 = (a'^{1}_0, ..., a'^{L}_0) \in \mathbb{R}^{n'}, a''_0 = (a''^{1}_0, ..., a''^{L}_0) \in \mathbb{R}^{n''}.$$

Corollary 2.4. *The set* $\mathbf{F}$ *of facets of the polytope* P *is the union of the set of facets* $\mathbf{F}'^{l}$ *of the polyhedral domains:*

$$P'^{l} = \{x \in \mathbb{R}^n :\ A'^{l} x^l \leq a'^{l}_0,\ A''^{l} x^l = a''^{l}_0\},\ l \in J_L.$$

Thus, we have

$$\mathbf{F} = \bigcup_{l=1}^{L} \mathbf{F}'^{l},$$

and since there is a one-to-one correspondence between the sets of facets P^l and P'^{l}, $l \in J_L$, the following holds for the number of P-facets:

$$|\mathbf{F}| = \sum_{l=1}^{L} |\mathbf{F}^l|.$$

In terms of f-vectors, it is expressed as

$$f_{d-1}(P) = \sum_{l=1}^{L} f_{d^l-1}(P^l).$$

Surface locality of E

For all the sets (2.27) for which (2.31) holds, meaning that there are known surfaces of the form (2.32) where the sets lie, we perform lifting from $\mathbb{R}^{n^l}$ into Euclidean space $\mathbb{R}^n$. To accomplish this, we consider that each point of E is the Cartesian product of certain points of the sets (2.27):

$$\forall x \in E,\ \exists\, x^l \in E^l,\ l \in J_L :\ x = \bigotimes_{l=1}^{L} x^l. \tag{2.54}$$

Then we introduce functions:

$$\forall x \in \mathbb{R}^n\ f'^l(x) = f'^l(x^1, ..., x^l, ..., x^L) = f^l(x^l),\ l \in J_L.$$

Finally, let us introduce cylindrical surfaces:

$$S'^l = \{x \in \mathbb{R}^n : f'^l(x) = 0\},\ l \in J_L. \tag{2.55}$$

Proposition 2.4. *If at least one of the sets* (2.27) *lies on a surface, then the entire set* (2.27) *lies on a certain surface.*

Indeed, if there exists $l \in J_L$ such that $E^l \subseteq S^l$, then $E \subseteq S'^l$, i.e. E lies on the corresponding surface from the family (2.55).

Remark 2.9. If several sets in the collection (2.27) lie on surfaces, i.e. $\exists I \subseteq J_L$, such that $E^l \subseteq S^l$, $l \in I$, $|I| > 1$, then $|I|$ surfaces containing E (S'^l, $l \in I$) can be found in the same way. In addition, such surfaces can also be formed by various combinations (for example, multiplication by scalars and subsequent summation) of the equations:

$$f'^l(x) = 0,\ l \in I. \tag{2.56}$$

Proposition 2.5. *If the conditions* (2.31) *and* (2.33) *are satisfied for the sets* (2.27)*, i.e. all the sets are inscribed in the surfaces* (2.32)*, then there exists a circumscribed surface about* E*.*

Proof. In this case, equations (2.56) that are satisfied at all the points of E have the form:

$$f'^{l}(x) = 0,\ l \in J_L.$$

Let us introduce the surface S by the equation:

$$f(x) = \sum_{l=1}^{L} \lambda_l f'^{l}(x) = 0, \tag{2.57}$$

$$\lambda = (\lambda_l)_{l \in J_L} \in \mathbb{R}^L_{>0},\ |\lambda| = 1. \tag{2.58}$$

By construction and due to (2.34), from the formula (2.57) it follows that at each point $x \in E$, the function (2.57) satisfies the inequality $f(x) \leq 0$ turning into equality $f(x) = 0$ on E. This means that, for the set E, the condition to be inscribed into the surface S given by (2.57) is satisfied. □

Now, we show that if the sets (2.27) are surface-located, then E also inherits this property.

Theorem 2.8. *If the conditions* (2.31), (2.32), *and* (2.35) *are satisfied for the sets* (2.27), *their Cartesian product* (2.51) *is a surface-located set.*

Proof. Consider the surface S given by the equation (2.57) and circumscribed about E. Let us show that it is strictly convex, i.e.

$$\begin{aligned} &\forall \alpha \in (0,1), \forall x', x'' \in \mathbb{R}^n \\ &f(\alpha x' + (1-\alpha) x'') < \alpha f(x') + (1-\alpha) f(x''). \end{aligned} \tag{2.59}$$

Likewise (2.54), from coordinates of the vectors $x', x'' \in \mathbb{R}^n$, we single out the subvectors

$$\begin{aligned} x'^{l} \in \mathbb{R}^{n^l},\ l \in J_L :\ x' = \bigotimes_{l=1}^{L} x'^{l}; \\ x''^{l} \in \mathbb{R}^{n^l},\ l \in J_L :\ x'' = \bigotimes_{l=1}^{L} x''^{l}. \end{aligned} \tag{2.60}$$

Substituting (2.57) into the left-hand side of (2.59) and, taking into account (2.35),

(2.58), and (2.60), we obtain

$$f\left(\alpha x' + (1-\alpha)\, x''\right) = \sum_{l=1}^{L} \lambda_l f'^l(\alpha x' + (1-\alpha)\, x'') =$$
$$= \sum_{l=1}^{L} \lambda_l f^l(\alpha x'^l + (1-\alpha)\, x''^l) < \sum_{l=1}^{L} \lambda_l(\alpha f^l(x'^l) + (1-\alpha)\, f^l(x''^l)) =$$
$$= \alpha \sum_{l=1}^{L} \lambda_l f^l(x'^l) + (1-\alpha) \sum_{l=1}^{L} \lambda_l f^l(x''^l) = \alpha f\left(x'\right) + (1-\alpha)\, f\left(x''\right)).$$

Thus, the condition (2.59) is satisfied for every E', E'', which was to be proved. □

Corollary 2.5. *If the sets* (2.27) *are ellipsoidally-located, then their Cartesian product is also ellipsoidally-located.*

Indeed, the substitution (2.36) into the equation (2.57) yields:

$$f(x) = \sum_{l=1}^{L} \lambda_l (x^l - x^{0l})^\top C^l (x^l - x^{0l}) = 0.$$

As one can see, this is a quadratic surface. On the other hand, according to Theorem 2.8, it is a strictly convex one. Hence, it is an ellipsoid, which was to be proved.

The spherical locality of E

Theorem 2.9. *The set* (2.51) *is spherically-located if and only if the sets* (2.27) *are spherically-located.*

Proof. Necessity. For a fixed $l \in J_L$, consider the hypersphere $S_r(a)$ circumscribed about E and, together with the set E, project it onto the subspace $\{x \in \mathbb{R}^n : x_i = 0,\ i \in \{1, ..., n_0^{l-1} - 1, n_0^l + 1, ..., n\}$, where $n_0^l = \sum_{i=1}^{l} n^l$, thus carrying out projection into the space $\mathbb{R}^{n^l}$. As a result, we obtain an $n_l - 1$-sphere whose equation is satisfied by all points of E^l. It implies the spherical locality of E^l.

Expanding the results onto $l \in J_L$, we conclude that all the sets (2.27) are spherically-located.

Sufficiency. Suppose that all the sets (2.27) are spherically-located. Let us construct the surface S given by the equation (2.57) with the coefficients (2.58) and obtain $f(x) =$

$\sum\limits_{l=1}^{L} \lambda_l((x^l - x^{0l})^2 - r^{l2}) = 0$. Clearly, this equation defines a family of ellipsoids. We select a hypersphere in the family by setting equal coefficients in (2.58) and getting the equation $f(x) = \frac{1}{L}\sum\limits_{l=1}^{L}((x^l - a^l)^2 - r^{l2}) = 0$, which can also be represented as

$$\sum_{l=1}^{L}(x^l - a^l)^2 = \sum_{l=1}^{L} r^{l2},$$

i.e. by the equation of a hypersphere centred at the point:

$$a = \bigotimes_{l=1}^{L} a^l. \tag{2.61}$$

Thus, E is spherically-located, and its circumscribed hypersphere (that can be not unique) has the parameters (2.61),

$$r = (\sum_{l=1}^{L} r^{l2})^{1/2}.$$

□

Remark 2.10. If the condition (2.53) is also satisfied, then the hypersphere $S_r(a)$ is not uniquely. In the family of hyperspheres circumscribed about E, the hypersphere $S^{\min}$ has the following parameters:

$$a^{\min} = \bigotimes_{l=1}^{L} a^{\min,l}; \quad r^{\min} = (\sum_{l=1}^{L} (r^{\min,l})^2)^{1/2}.$$

<u>**Vertex locality of E**</u>

Theorem 2.10. *The set* (2.51) *is vertex-located if and only if all the sets* (2.27) *are vertex-located.*

In other words, E satisfies the condition (1.50) if and only if the sets (2.27) satisfy the condition (2.29).

Vertex criterion of P

$$x = (x^1, ..., x^L) \in V \Leftrightarrow x^l \in V^l,\ l \in J_L.$$

Number of vertices of P

$$|V| = \prod_{l=1}^{L} |V^l|$$

or in terms of f-vectors

$$f_0(P) = \prod_{l=1}^{L} f_0(P^l).$$

This formula directly follows from the fact that, for the vertex set of P, the expression

$$V = \bigotimes_{i=1}^{L} V^l$$

similar to (2.51) holds.

Vertex adjacency criterion of P

Theorem 2.11. *Two vertices $x = \bigotimes_{i=1}^{L} x^l$ and $y = \bigotimes_{i=1}^{L} y^l$ of the polytope P are adjacent if and only if*

$$\exists l^* \in J_L :\ y^{l^*} \in N_{P_l}(x^{l^*});\ y^l = x^l,\ l \in J_L \backslash \{l^*\}.$$

This also directly follows from the fact that P is the Cartesian product of the polytopes (2.28).

Number of adjacent vertices of P

$$\forall x \in V \quad \mathcal{R}(x) = \sum_{i=1}^{L} \mathcal{R}^l(x^l).$$

Corollary 2.6. *(from Theorem 2.11) If all vertices of the polytopes* (2.28) *are regular, then*

the vertices of P are also regular, and the degree of their regularity is

$$\mathcal{R} = \sum_{l=1}^{L} \mathcal{R}^l.$$

Hyperplanes of symmetry of E and P

Proposition 2.6. *If for some $l \in J_L$, $c^{l\top} x^l = c_0^l$ is the hyperplane of symmetry of the set E^l or polytope P^l, then the hyperplane $c^\top x = c_0^l$, where*

$$c_i = \begin{cases} c^1_{i-n^{l-1,0}}, & \text{if } i \in J_{n^{l,0}} \backslash J_{n^{l-1,0}}, \\ 0, & \text{if } i \notin J_{n^{l,0}} \backslash J_{n^{l-1,0}} \end{cases}$$

is the hyperplane of symmetry of the set E of the form (2.51) *or the polytope P of the form* (2.52)*, respectively.*

Central symmetry of E and P

Proposition 2.7. *The set E of the form* (2.51) *is centrally symmetric about the point*

$$x^0 = \bigotimes_{i=1}^{L} x^{0l} \tag{2.62}$$

if and only if the sets (2.27) *are centrally symmetric and $x^{0l} \in \mathbb{R}^{n^l}$ is the center of symmetry of E^l ($l \in J_L$).*

Proposition 2.8. *The polytope P given by* (2.52) *is centrally symmetric about the point x^0 given by* (2.62) *if and only if the polytopes* (2.28) *are centrally symmetric about the points $x^{0l} \in \mathbb{R}^{n^l}$, $l \in J_L$, the centers of symmetry of polytopes P^l ($l \in J_L$).*

E-levelness

$$m(E) = \max_{l \in J_L} m(E^l).$$

E-levelness along coordinates

$$m'(E) = \max_{l \in J_L} m'(E^l).$$

P-levelness

$$m''(P) = \max_{l \in J_L} m''(P^l).$$

In particular, if none of the sets (2.27) is a singleton, i.e.

$$1 < |E^l| < \infty, \ l \in J_L, \tag{2.63}$$

then the below statement holds.

Proposition 2.9. *The Cartesian product of the sets* (2.27) *and* (2.63) *is*

- *two-level, and the polytope* (2.52) *is two-level if and only if the sets* (2.27) *are two-level:*

$$m(E) = m''(P) = 2 \ \Leftrightarrow \ m(E^l) = 2, \ l \in J_L;$$

- *two-level set along coordinates if and only if all the sets* (2.27) *are two-level along coordinates:*

$$m'(E) = 2 \ \Leftrightarrow \ m'(E^l) = 2, \ l \in J_L.$$

Simplicity criterion of P

Proposition 2.10. *The polytope P of the form* (2.52) *is simple if and only if all the polytopes* (2.28) *are simple:*

$$\mathcal{R} = d \ \Leftrightarrow \ \mathcal{R}^l = d^l, \ l \in J_L.$$

2.3.8 Direct sum of finite point configurations

Let the sets E^l, $l \in J_L$ in (2.27) be such that the origins $\mathbf{0} \in P^l$, $l \in J_L$ belong to the corresponding polytope (2.28). The set E is the direct sum of the sets (2.27):

$$E = \bigoplus_{l=1}^{L} E^l. \tag{2.64}$$

The set can be represented as

$$E = \bigcup_{l=1}^{L} E'^l,$$

where

$$E'^l = \{x = (x^1, ..., x^L) \in \mathbb{R}^n : \ x^l \in E^l, \ x^{l'} = \mathbf{0} \in \mathbb{R}^{n^l}, \ l' \neq l\}, \ l \in J_L.$$

The cardinality of E

$$|E| = \sum_{l=1}^{L} |E^l|.$$

The polytope P is the convex hull of the direct sum of the polytopes (2.28):

$$P = conv \bigoplus_{l=1}^{L} P^l. \tag{2.65}$$

The dimension of P

$$\dim \ P = \sum_{l=1}^{L} \dim \ P^l,$$

i.e. for the dimension of the polytope (2.65) the same formula as for the Cartesian product holds.

Full-dimensionality criterion of P

The polytope P given by (2.65) is full-dimensional if and only if the polytopes (2.28) are full-dimensional.

Relaxation of H-representations of P

The irredundant H-representation of the Cartesian product (2.27) given in Proposition 2.3 can serve as a relaxation of the H-representation of the polytope (2.65) of the direct sum of the corresponding sets.

Irredundant H-representation of P

An irredundant H-representation of the polytope P can be found by combining its constraints, namely. adding their left- and right-hand sides. It results in the H-representation of the form (1.52), where:

$$n'' = \sum_{l<l'} n''^{l} n''^{l'}, n' = \sum_{l<l'} (n'^{l} + n''^{l})(n'^{l} + n''^{l}) - n''.$$

For the number of facets of P the formula:

$$|\mathbf{F}| = \sum_{l<l'} |\mathbf{F}^{l}| \cdot |\mathbf{F}^{l'}|$$

is valid and can be rewritten in terms of f-vectors as

$$f_{d-1}(P) = \sum_{l<l'} f_{d^{l}-1}(P^{l}) f_{d^{l'}-1}(P^{l'}).$$

Surface locality of E

Suppose that some sets (2.27) satisfy the condition (2.31), i.e. it is known on which surfaces these sets are located:

$$\exists I \subseteq J_L : \ E^{l} \in S^{l}, \ l \in I.$$

Let us lift into the space $\mathbb{R}^n$ and construct the surfaces (2.55). Then, the points of the set E formed by (2.64) will lie on the surfaces

$$E \in S'^l,\ l \in I.$$

Proposition 2.11. *If the sets* (2.27) *are inscribed into certain surfaces, then the set E of the form* (2.64) *is also inscribed into a certain surface.*

Proof. Let the sets (2.27) satisfy the conditions (2.31) and (2.33) and are inscribed into the surfaces (2.32).

The functions (2.30) which define these circumscribed surfaces can be represented as follows:

$$f^l(x^l) = h^l(x^l) - h^l(\mathbf{0}) = 0,\ l \in J_L. \tag{2.66}$$

We distinguish two groups of the functions: if $h^l(x^l)$ takes a nonzero value at the origin (Group 1, and if this value is null (Group 2), i.e.

$$\text{Group 1: } l \in I' \subseteq J_L \qquad h^l(\mathbf{0}) \neq 0,$$

$$\text{Group 2: } l \in J_L \backslash I' \qquad h^l(\mathbf{0}) = 0.$$

Here, without loss of generality, we can assume that

$$h^l(\mathbf{0}) = 1,\ l \in I'.$$

Lifting into $\mathbb{R}^n$ according to the above rule, we perform the transition $h^l(x^l) \to h'^l(x)$ from the function $h^l(x^l)$ to a new function $h'^l(x)$ defined in $\mathbb{R}^n$ ($l \in J_L$).

Constructing the surface S, we consider two cases:

- if $I' \neq \emptyset$, then S is given by the equation

$$f(x) = \sum_{l=1}^{L} h'^l(x) - 1 = 0; \tag{2.67}$$

where λ satisfies (2.58);

- if $I' = \emptyset$, then S is defined as follows:

$$f(x) = \sum_{l=1}^{L} h'^{l}(x) = 0. \tag{2.68}$$

It is easy to see that every point of E satisfies the equation $f(x) = 0$ since an arbitrary point $x \in E$ has $L - 1$ groups of zero coordinates and one group of coordinates corresponding to x^l that belongs to a certain E^l. Thus, E lies on the constructed surface S, and for every point $x \in P$ the inequality $f(x) \leq 0$ is satisfied due to (2.33) and the method of constructing $f(x)$. As a result, S will be the surface circumscribed about E. □

In conclusion, we show that if the sets (2.27) are surface-located, then E also inherits the same property.

Theorem 2.12. *If the sets formed by* (2.27) *satisfy the conditions* (2.31), (2.32), *and* (2.35), *then their direct sum* (2.64) *is a surface-located set.*

The proof is similar to the proof of Theorem 2.8, where either the function (2.67) or (2.68) is chosen as $f(x)$ depending on whether the condition holds:

$$I' \neq \emptyset. \tag{2.69}$$

As it turns out, such a construction of the function $f(x)$ ensures its strict convexity.

Corollary 2.7. *If the sets formed by* (2.27) *are ellisoidally-located, then the set* (2.64) *is ellipsoidally-located.*

Ellipsoidal and spherical locality of E

Proposition 2.12. *If the sets* (2.27) *are spherically-located, then their direct sum is ellipsoidally-located set.*

Proof. Indeed, suppose that all the sets (2.27) are spherically-located. Let us show that there exists an ellipsoid circumscribed about the set (2.64).

The functions $f^l(x^l) = (x^l - x^{0l})^2 - r^{l2}$ defining hyperspheres circumscribed about sets (2.27) can be represented in the form (2.66),

- $\forall l \in I'\ h^l(x^l) = \frac{1}{a^{l2}-r^{l2}}(x^{l2} - x^{lT}a^l)$;
- $\forall l \notin I'\ h^l(x^l) = (x^{l2} - x^{lT}a^l)$,

respectively, and single out Group 1 and Group 2 in them.

Let us construct the equation (2.67) if the condition (2.69) is met or the equation (2.68) otherwise. In the first case, the surface given by this equation, generally, is an ellipsoid.

Consider these two cases and derive the equations of the resulting quadratic surfaces:

1. if

$$\exists\, \alpha \in \mathbb{R}^1 :\ \forall l \in I'\ a^{l2} - r^{l2} = \alpha, \tag{2.70}$$

then we have

$$\forall l \in I', h^l(x^l) = \frac{1}{\alpha}(x^{l2} - x^{lT}a^l) \underset{E^l}{=} 1, h'^l(x^l) = \alpha \cdot h^l(x^l) = x^{l2} - x^{lT}a^l \underset{E^l}{=} \alpha.$$

We define the function

$$f(x) = \sum_{l=1}^{L} h'^l(x) - \alpha = 0.$$

The equation of the surface S will be

$$f(x) = \sum_{l=1}^{L}(x^{l2} - x^{lT}a^l) - \alpha = 0,$$

rewritable as

$$\sum_{l=1}^{L}(x^{l2} - x^{lT}a^l + a^{l2}) = \alpha + \sum_{l=1}^{L} a^{l2}$$

or

$$(x - a)^2 = a^2 + \alpha,$$

where x is the vector (2.54) and a is the vector (2.61). Thus, E is a spherically-located

set, and its circumscribed hypersphere has the center at a and the radius $\sqrt{a^2+\alpha}$;

2. if (2.70) is violated, then we build the surface

$$\sum_{l\in I'} \frac{1}{a^{l2}-r^{l2}}(x^{l2}-x^{lT}a^l)+\sum_{l\notin I'}(x^{l2}-x^{lT}a^l)=1,$$

which is an ellipsoid centered at the point a. To determine its semiaxes, we represent this equation in the canonical form:

$$\sum_{l\in I'} \frac{1}{a^{l2}-r^{l2}}(x^{l2}-x^{lT}a^l+a^{l2})+\sum_{l\notin I'}(x^{l2}-x^{lT}a^l+a^{l2})=$$
$$=1+\sum_{l\in I'} \frac{a^{l2}}{a^{l2}-r^{l2}}+\sum_{l\notin I'} a^{l2}.$$

Wherefrom,

$$\sum_{l=1}^{L} \frac{(x^{l2}-a^l)^2}{b^{l2}}=1,$$

where

$$b^{l2}=\begin{cases} B(a^{l2}-r^{l2}), & \text{if } l\notin I', \\ B, & \text{if } l\notin I' \end{cases} \qquad (l\in J_L);$$

$$B=1+\sum_{l\in I'} \frac{a^{l2}}{a^{l2}-r^{l2}}+\sum_{l\notin I'} a^{l2}.$$

□

Vertex locality of E

Theorem 2.13. *The set* (2.64) *is vertex-located if and only if all the sets given by* (2.27) *are vertex-located.*

For proof, we can use Theorem 2.12 and the connection between vertex-located and surface-located sets established in Theorem 2.1.

Vertex criterion of P

$$x = (x^1, ..., x^L) \in V \;\Leftrightarrow\; \exists\, l^* \in J_L \qquad x^{l^*} \in V^{l^*}.$$

Number of vertices of P

$$|V| = \sum_{l=1}^{L} |V^l|$$

rewritable in terms f-vectors as

$$f_0(P) = \prod_{l=1}^{L} f_0(P^l).$$

Symmetry hyperplanes of E, P

The direct sum (2.64) of the sets (2.27) satisfies Proposition 2.6.

Central symmetry of E, P

Likewise, the Cartesian product of the sets (2.27), for the set (2.64) and corresponding polytope P, Proposition 2.7 is valid and establishes the central symmetry of E and P, given that the sets (2.27) and/or polytopes (2.28) are centrally symmetric.

E-levelness along coordinates

$$m'(E) = \max_{l \in J_L} m'(E^l).$$

Levelness along coordinates for the direct sum of sets is identical to their Cartesian product. In particular, if the condition (2.63) is satisfied, then the criterion for two-levelness along coordinates for the set (2.64) is the following.

Proposition 2.13. *The direct sum of the sets* (2.27) *and* (2.63) *is a two-level set along coordinates if and only if all these sets are two-level along coordinates:*

$$m'(E) = 2 \;\Leftrightarrow\; m'(E^l) = 2,\; l \in J_L.$$

Chapter 3

Combinatorial point configurations

In the realm of discrete mathematics, specifically within the domains of combinatorial analysis and combinatorial optimization, a pivotal focus is dedicated to the formalization of fundamental concepts such as combinatorial sets and combinatorial objects, as well as singling outing distinctive classes of discrete structures. Within this context, a fundamental issue is establishing a precise definition for combinatorial configurations. This introductory discussion concisely examines the current state of research in this area.

This chapter highlights the combinatorial point configuration class. For its formal definition, the concept of Euclidean combinatorial configurations (e-configurations) is introduced as a result of a bijective mapping of combinatorial configurations into Euclidean space. Combinatorial configurations are considered in the sense of C. Berge [6], and their classification and properties are given in accordance with [6, 14, 24, 26, 27, 34, 71, 81, 82]. The description of e-configurations is based on the works [97], taking into account the properties of finite point configurations reflected in the previous chapters. The concept of e-configuration is inextricably linked with the definition of Euclidean combinatorial sets and their bijective mapping into Euclidean space [74, 75]. These mappings generate various classes of combinatorial point configurations that can be seen as the corresponding sets of e-configurations. This fact is illustrated by various classes of permutation and multipermutation point configurations, partial permutation and multipermutation point configurations and unbounded permutation point configurations in accordance with the studies [40, 76, 79, 106].

3.1 Combinatorial configurations and their collections

Further, we use the notion of a combinatorial configuration in the sense of C. Berge [6]. A configuration is understood as mapping an initial set of elements, which can be of an arbitrary nature, onto a finite abstract resulting set possessing a specific structure. This mapping is subject to constraints that determine the positions of the elements and their mutual relations. When studying such configurations, the relationship between constraints, structure, and mappings is taken into account based on the combinatorial properties of the resulting set.

Based on the classification proposed by C. Berge [6], combinatorial analysis involves solving several key problems such as

- Study of known configurations, involving the analysis and study of existing configurations to understand their properties, characteristics and relationships.
- Formation of new configuration classes with predefined properties allows exploring such classes using various construction methods or transformation techniques.
- Enumerative combinatorics dealing with determining the number of configurations within a given class or under certain restrictions. Enumerative combinatorics provides methods and tools for efficient configuration counting.
- Approximate formulas for the number of configurations when obtaining exact formulas is difficult or infeasible. Such an approximation provides valuable information about the composition and behavior of configuration classes.
- Generation of configurations focuses on systematically generating or listing all configurations within a given class or satisfying specific constraints. Efficient algorithms and techniques are developed to enumerate configurations comprehensively.
- Optimization on multiple configurations involves optimizing certain properties or objectives over a set of configurations. For example, finding a configuration that maximizes or minimizes a specific parameter or optimizing a configuration for a given problem.

These problems are fundamental in combinatorial analysis, contributing to the understanding, exploring, and applying configurations in various fields of mathematics and beyond.

We formalize the notion of a combinatorial configuration in terms of mappings of sets.

Definition 3.1. *By a combinatorial configuration, we mean a mapping χ of some B of arbitrary nature items into a finite abstract set $A = \{a_1, ..., a_k\}$ of a certain structure under a given family of constraints Ω, i.e.*

$$\chi : B \to A. \tag{3.1}$$

The sets B and A are called the <u>initial</u> and <u>resulting sets</u>, respectively. Although formally, there are no constraints on the cardinality of the set B, in fact, a finite set $B = \{b_1, ..., b_n\}$ is usually under consideration.

It is worth noting that relaxing the finiteness condition of the resulting set A does not alter the core essence of the combinatorial configuration concept. Nevertheless, this adjustment naturally influences the cardinality of the resulting set, its key characteristics, and broadens the scope of research problems. On this basis, L. Gulyanitskyy introduced the concept of a combinatorial objects [24, 25] adding the assumption that the set A can be countable.

Mapping (3.1) determines the structure of the resulting set A and specifies an ordered sequence π of elements from A:

$$\pi = \begin{pmatrix} b_1 & \dots & b_n \\ a_{j_1} & \dots & a_{j_n} \end{pmatrix} = [a_{j_1} a_{j_2} ... a_{j_n}], \tag{3.2}$$

where $\{j_1, ..., j_n\} \in J_k$.

Further, for the configuration π of the form (3.2), we will use the notation

$$\pi = [a_{j_1}, a_{j_2}, ..., a_{j_n}] .$$

Thus, every combinatorial configuration is entirely defined by the quaternion:

"mapping - initial set - resulting set - constraints" , i.e.

$$\langle \chi, B, A, \Omega \rangle .$$

In many instances, the original set B can be amalgamated by simply renumbering its elements, and this reordering information can be leveraged to construct various combinatorial configurations. To facilitate this process, we establish a bijection between the initial set B and the set J_n, representing the indices of its elements. Consequently, the mapping (3.1) can be reformulated as follows:

$$\psi : J_n \to A. \tag{3.3}$$

Combinatorial configuration does not change under such a mapping, i.e.

$$\pi = \begin{pmatrix} b_1 & \dots & b_n \\ a_{j_1} & \dots & a_{j_n} \end{pmatrix} = \begin{pmatrix} 1 & \dots & n \\ a_{j_1} & \dots & a_{j_n} \end{pmatrix} = [a_{j_1}, a_{j_2}, ..., a_{j_n}]. \tag{3.4}$$

The set J_n is called the <u>numbering set</u>, and the structuring of the set A is understood as its strict ordering, i.e.

$$a_i \prec a_{i+1},\ i \in J_{k-1}.$$

In this case, elements of the numbering set J_n indicate the positions of the elements of the set A, and the combinatorial configuration itself can be represented by the triad:

$$\langle \psi, A, \Lambda \rangle ,$$

where A is the strictly ordered resulting set, ψ is a mapping of the form (3.3), Λ is a certain collection of constraints on the mapping ψ.

By considering the properties of mappings (3.3) and the interconnection between the cardinalities of the initial numbering and resulting sets, we can identify the primary categories of combinatorial configurations and sets of combinatorial configurations that correspond to well-established combinatorial structures.

Let Λ be a bijective mapping and $n = k$. Then mapping ψ defines a permutation configuration and forms a set of permutation configurations, which in combinatorics is called a <u>set of permutations without repetitions</u>.

Let Λ be an injective mapping and $n < k$. Then mapping ψ defines a partial permutation configuration and forms a set of permutation configurations called a set of partial permutations without repetitions in combinatorics.

Let $\Lambda = \emptyset$, that is, there are no constraints on the mapping (3.3). Then we obtain a set of permutations with unbounded repetitions.

In the subsequent sections, we will explore sets of combinatorial configurations that align with established combinatorial structures, including sets of permutations and partial permutations with repetitions, along with their respective subsets. Throughout our discussion, we will introduce the requisite definitions and rules for the formation of the constraints family Λ in the context of mapping (3.3).

3.2 Euclidean combinatorial configurations

For any combinatorial configuration $\langle \psi, A, \Lambda \rangle$, one can define a bijective mapping φ into Euclidean space $\mathbb{R}^N$:

$$\varphi : \langle \psi, A, \Lambda \rangle \to \mathbb{R}^N. \tag{3.5}$$

As a result of such a mapping, to the point $\pi = [a_{j_1}, a_{j_2}, ..., a_{j_N}]$ formed by (3.4), the point $x = (x_1, ..., x_N) \in \mathbb{R}^N$ corresponds, i.e.

$$x = \varphi(\pi).$$

Definition 3.2. *The image of combinatorial configuration*

$$\langle \psi, A, \Lambda \rangle \tag{3.6}$$

under bijective mapping φ formed by (3.5) *will be called an Euclidean combinatorial configuration, and abbreviations e-configuration and ECC will be used.*

By employing the term "Euclidean", we underscore that the mapping (3.5) yields a point in Euclidean space $\mathbb{R}^N$, and the dimension of an ECC is determined by the dimension N of this space.

Let us consider the set Π of combinatorial configurations $\langle \psi, A, \Lambda \rangle$ induced by feasible mappings ψ subject to certain constraints Λ.

Under a bijective mapping φ formed by (3.5), we introduce the set

$$E = \varphi(\Pi) \subset \mathbb{R}^N, \tag{3.7}$$

which is image of a set Π under the mapping φ, i.e. a set of the corresponding Euclidean combinatorial configurations.

Definition 3.3. *A set E of Euclidean combinatorial configurations formed by* (3.7) *will be called a* <u>*combinatorial point configuration*</u> *(*<u>*CPC*</u>*).*

So, the mapping (3.7) determines an image E of the set Π into Euclidean space $\mathbb{R}^N$, and, depending on the mapping φ, allows the forming various CPCs.

Taking into account bijectivity of the mapping φ, we can specify a preimage Π of a set E as

$$\Pi = \varphi^{-1}(E). \tag{3.8}$$

For an e-configuration $x = (x_1, ..., x_N) \in E$ and a combinatorial configuration $\pi = (\pi_1, ..., \pi_N) \in \Pi$, we can write:

$$x = \varphi(\pi), \pi = \varphi^{-1}(x).$$

A multitude of mappings φ as defined in (3.5) are available for each dimension N. It is essential to designate the specific type of such mapping based on the particular tasks at hand. Given the combinatorial analysis problems enumerated earlier, our emphasis will be on optimizing combinatorial configurations. Hence, the distinctive properties of combinatorial point configurations generated through the bijective mapping φ and their associated combinatorial polytopes hold significant relevance. Primarily, our focus lies in the decompositions of FPCs into hyperplanes and surfaces, vertex- and surface-located sets including spherical-located and ellipsoidally-lovated ones.

Taking into account the above, we define the mapping φ as follows. Let us consider the formation of combinatorial configurations $\langle \psi, A, \Lambda \rangle$. Each combinatorial configuration

corresponds to an ordered sample of items from the set $A = \{a_1, ..., a_k\}$.

We single out a class of combinatorial configurations consisting of ordered N-samples for the fixed $N = n$. Let $G = \{e_1, ..., e_k\}$ be a finite set of k real numbers. We perform a bijection ξ between the sets A and G, $G = \xi(A)$, as follows:

$$e_i = a_i, i \in J_k. \tag{3.9}$$

Thus one can write

$$\langle \psi, G, \Lambda \rangle = \langle \psi, \xi(A), \Lambda \rangle. \tag{3.10}$$

Each element $\pi = [a_{j_1}, a_{j_2}, ..., a_{j_n}]$ of the combinatorial configuration $\langle \psi, A, \Lambda \rangle$ is associated with the following point of n-dimension Euclidean space

$$x = (x_1, ..., x_n) \in \mathbb{R}^n, \tag{3.11}$$

where

$$x_i = \xi(a_{j_i}), i \in J_n. \tag{3.12}$$

Thus, we have established a one-to-one correspondence between the combinatorial configuration $\pi = [a_{j_1}, ..., a_{j_n}]$ and the point $x = (e_{j_1}, ..., e_{j_n}) \in \mathbb{R}^n$.

As a result, we have formed a bijective mapping

$$\varphi : \langle \psi, A, \Lambda \rangle \to (x_1, ..., x_n) \in \mathbb{R}^n. \tag{3.13}$$

In Definition 3.2, we introduced an e-configration (3.6) by the formulas (3.11) and (3.12).

Applying the mapping (3.13) to all combinatorial configurations forming Π, we obtain a CPC E given by the formula (3.7).

Every point $x = (x_1, ..., x_n) \in E$ is representable as

$$x = (e_{j_1}, ..., e_{j_n}), \tag{3.14}$$

where $e_{j_i} \in G \; \forall j_i \in J_n$, $i \in J_n$, and $j_p \neq j_q$, $\forall p, q \in J_n$, $p \neq q$.

The mapping process utilizing the rules (3.10)-(3.14) is called an <u>immersion</u> of the set Π into Euclidean space $\mathbb{R}^n$. The dimension of the resulting CPC (further referred to as E), is contingent upon the dimension of the space into which the immersion is performed. Following the approach described for constructing the mapping (3.13), this dimension aligns with the cardinality of the numbering set J_n.

It is worth highlighting that Yu. Stoyan introduced a category of combinatorial sets where distinctions arise from the constituent elements or their arrangement [75]. These sets are referred to as "Euclidean combinatorial sets" or e-sets. Their images in Euclidean space are termed "special combinatorial sets" or s-sets, as per Stoyan's work [75].

In this case, if $\mathcal{P}$ is an e-set, and $\mathcal{E}$ is a s-set obtained by mapping $\phi : \mathcal{P} \to \mathcal{E}$, then:

$$\exists\, \phi \; : \; \mathcal{E} = \phi(\mathcal{P}) \subset \mathbb{R}^n, \; \mathcal{P} = \phi^{-1}(\mathcal{E}). \tag{3.15}$$

It is easy to see that Π is an e-set. Indeed, for a pair of configurations $\pi, \pi' \in \Pi$ such that for arbitrary

$$\pi = [a_{j_1}, a_{j_2}, ..., a_{j_n}] \, , \; \pi' = \left[a_{j'_1}, ..., a_{j'_n}\right] ,$$

it is true

$$\pi \neq \pi' \Leftrightarrow [j_1, j_2, ..., j_n] \neq [j'_1, j'_2, ..., j'_n],$$

which ensures the difference between the combinations $\{a_{j_i}\}_{i \in J_n}$, $\{a_{j'_i}\}_{i \in J_n}$ of these two configurations or the order of their constituent elements. Then $E = \varphi\,(\Pi)$ is an s-set utilizing the following notations:

$$\phi = \varphi, \; \mathcal{P} = \Pi, \; \mathcal{E} = E.$$

In terms of the method of construction, an e-set Π coincides with the set of combinatorial configurations that we describe through the mapping $\varphi : \Pi \to E$. Moreover, the combinatorial point configuration E as the image of the set Π is an s-set. In this context, the notions of a special combinatorial set and a combinatorial point configuration are equivalent. Hence, the examination of the classes of Euclidean combinatorial sets that are investigated and the

methods by which they are mapped into Euclidean space becomes of paramount importance.

The category of combinatorial point configurations is notably extensive. It hinges on the initial selection of a set of combinatorial configurations, denoted as Π. When forming the corresponding CPC E, one can specify numerous mappings $\varphi(\Pi) \subset \mathbb{R}^n$ for any natural n. Therefore, we will narrow our focus to specific categories of combinatorial point configurations, delineating both the set Π and the mapping φ in the form of (3.13).

3.3 Typology of combinatorial point configurations

Let us offer an approach to classifying a combinatorial point configuration $E \subset \mathbb{R}^n$ using the concept of a multiset.

For any e-configuration $x \in E$ given by (3.15), we form the multiset

$$G(x) = \{e_{j_1}, e_{j_2}, ..., e_{j_n}\} \tag{3.16}$$

with the underlying set $S(G(x))$ and primary specification $[G(x)]$.

Forming the union of the multisets $G(x)$ over all $x \in E$, we come to the multiset

$$G(E) = \bigcup_{x \in E} G(x). \tag{3.17}$$

with the underlying set $S(G(E))$ and primary specification $[G(E)]$.

Definition 3.4. *Multiset $G(x)$ of the form* (3.16) *will be called an* <u>*inducing multiset of an e-configuration*</u> *$x \in E$.*

Definition 3.5. *The multiset $G(E)$ formed by* (3.17) *will be called an* <u>*inducing multiset of a combinatorial point configuration*</u> *$E \subset \mathbb{R}^n$ generated by a set of e-configurations $x \in E$.*

Let G be the multiset of the form

$$G = \{g_1, g_2, ..., g_\eta\} = \{e_1^{n_1}, e_2^{n_2}, ..., e_k^{n_k}\} \tag{3.18}$$

with the underlying set

$$S(G) = \{e_1, e_2, ..., e_k\} \tag{3.19}$$

and primary specification

$$[G] = (n_1, n_2, ..., n_k). \tag{3.20}$$

For multiplicities of elements of the primary specification $[G]$, the following relation holds

$$\eta = \sum_{i=1}^{k} n_i, 1 \leq n_i \leq n, i \in J_k. \tag{3.21}$$

Further, we utilize the multiset G as an inducing multiset $G(E)$ of the combinatorial point configuration $E \subset \mathbb{R}^n$, i.e. we suppose that $G(E) = G$. Now, we can introduce specific classes of combinatorial point configurations characterized by a combinations of the numbers n, η and k.

Definition 3.6. *A combinatorial point configuration $E \subset \mathbb{R}^n$ will be called a permutation point configuration (PPC) if $G(x) = S(G)$ for any $x \in E$ and $\eta = n = k$.*

Definition 3.7. *A combinatorial point configuration $E \subset \mathbb{R}^n$ will be called a multipermutation point configuration (MPC) if $G(x) = G$ for any $x \in E$ and $\eta = n \geq k$.*

If $\eta = n = k$, an MPC will be a PPC. Thus, Definition 3.7 generalizes the concept of a PPC. Respectively, all properties of MPC are also valid for PPC. At the same time, PPCs possess specific properties. Therefore, if necessary, we will emphasize how n and k are related.

Definition 3.8. *A combinatorial point configuration $E \subset \mathbb{R}^n$ will be called a partial permutation point configuration (PPPC) if $G(x) = S\left(G(x)\right) \subset G$ for any $x \in E$ and $n < \eta$, $\eta = k$.*

Definition 3.9. *A combinatorial point configuration $E \subset \mathbb{R}^n$ will be called a partial multipermutation point configuration (PMPC) if $G(x) \subset G$ for any $x \in E$ and $n < \eta, k \leq \eta$.*

A PMPC is a PPPC if $\eta = k$, and Definition 3.7 generalizes the concept of a PPPC. All properties of a PMPC are valid for a PPPC. To point out specific properties of a PPPC, we will additionally emphasize that $n = k$.

Analyzing Definitions 3.4-3.8, we can see that there are no restrictions on the cardinality of the combinatorial point configuration $E \subset \mathbb{R}^n$. Moreover, it makes sense to consider a set of all possible e-configurations generated by the inducing multiset $G(E) = G$. In this regard, we introduce additional concepts.

Definition 3.10. *A multipermutation point configuration $E \subset \mathbb{R}^n$ is called the entire MPC (EMPC) if E consists of all possible e-configurations $x \in E$ such as $G(x) \subset G$.*

If $G = S(G)$, we say that we deal with the entire PPC (EPPC).

Definition 3.11. *A partial multipermutation point configuration $E \subset \mathbb{R}^n$ will be called the entire PMPC (EPMPC) if E consists of all possible e-configurations $x \in E$ such as $G(x) \subset G$.*

When $G = S(G)$, we operate with the entire PPPC (EPPPC).

Definition 3.12. *A partial multipermutation point configuration $E \subset \mathbb{R}^n$ will be called the entire unbounded partial permutation point configuration (EUPPPC) if E consists of all possible e-configurations $x \in E$ such as $G(x) \subset G(E) = G = \{g_1, g_2, ..., g_\eta\} = \{e_1^n, e_2^n, ..., e_k^n\}$.*

For the entire combinatorial point configurations defined above, we introduce special notations. An entire point multipermutation configuration E with the inducing multiset G of the form (3.18)-(3.21) is denoted by $E_{nk}(G)$. If $\eta = n = k$, we use the notation $E_n(G) = E_{nn}(G)$.

By analogy, the entire partial multipermutation point configuration is denoted by $E^n_{\eta k}(G)$ and the entire unbounded partial multipermutation point configuration is denoted by $\bar{E}^n_k(G)$. When $\eta = k$, we use the notation $E^n_k(G)$.

The classification of combinatorial point configurations is associated with their distinctive characteristics. These peculiarities are shaped, on one hand, by the properties of the preimages of CPCs as combinatorial structures and, on the other hand, by their attributes as finite point configurations. Therefore, for CPCs, all the results described in Chapters 1 and 2

are applicable. First of all, we discuss vertex- and surface-located finite point configurations, as well as their decompositions into hyperplanes and surfaces. In this regard, we emphasize that:

- a CPC E is vertex-located if it satisfies condition (1.51);
- if a CPC E satisfies condition (1.19) for certain strictly convex surface S of the form (1.19), then E is referred to as a surface-located CPC;
- if the circumsurface S is a hypersphere, then E is called a spherically-located CPC;
- if the circumscribed surface S is an ellipsoid, then E is called an ellipsoidally-located CPC;
- if the surface S is a supersphere, then E is called a superpherically-located CPC;
- if the levelness of E is $m(E)$, then E is called a $m(E)$-level CPC and so on.

Every known combinatorial structure is an preimage Π of a certain combinatorial point configurations E, namely, $\Pi = \varphi^{-1}(E)$. Following (3.7) and (3.8), let us consider these configurations as the result of immersing $E = \varphi(\Pi)$ of the combinatorial set Π into Euclidean space $\mathbb{R}^n$. Then we get

$$P_{nk} = \varphi^{-1}\left(E_{nk}(G)\right), \tag{3.22}$$

$$P_n = \varphi^{-1}\left(E_n(G)\right), \tag{3.23}$$

$$P^n_{\eta k} = \varphi^{-1}\left(E^n_{\eta k}(G)\right), \tag{3.24}$$

$$P^n_k = \varphi^{-1}\left(E^n_k(G)\right), \tag{3.25}$$

$$\bar{P}^n_k = \varphi^{-1}\left(\bar{E}^n_k(G)\right). \tag{3.26}$$

Taking into account the mapping (3.9), we come to

- P_n of the form (3.23) is the set of permutations without repetitions from elements of A;
- if $k < n$, then P_{nk} of the form (3.22) is the set of multipermutations with repetitions from elements of A;

- P_k^n of the form (3.25) is the set of partial permutations without repetitions from elements of A;

- if $k < \eta$, then $P_{\eta k}^n$ of the form (3.24) is the set of partial multipermutations with repetitions from elements of A;

- $\bar{P}_k^n$ of the form (3.26) is the set of partial multipermutations with unbounded repetitions from elements of A.

Some properties of these combinatorial sets, mostly related to the enumeration problem, will be applied further when combinatorial point configurations are explored.

3.4 Illustrative examples

We illustrate the above concepts with several examples aiming to illustrate the combinatorial structure of a CPC $E \subset \mathbb{R}^n$.

- Let $n = 3$ and $G(E) = \{0, 5, 7\}$ be the multiset with the underlying set $S(G(E)) = G(E) = \{0, 5, 7\}$. In this case, $\eta = k = 3$.

  Consider e-configurations $x^1 = (0, 7, 5)$, $x^2 = (0, 5, 7)$, $x^3 = (5, 0, 7)$, $x^4 = (5, 7, 0)$, $x^5 = (7, 5, 0)$, $x^6 = (7, 0, 5)$. Then, the CPC $E = \{x^i\}_{i \in J_6}$ is an EPPC, while its subsets are PPCs. Moreover, $E = \{x^i\}_{i \in J_6}$ is an EPPC.

- Let $n = 4$ and $G(E) = \{1, 3, 3, 6\}$ be the multiset with the underlying set $S(G(E)) = G(E) = \{1, 3, 6\}$. Here, $\eta = 4,\ k = 3$.

  Consider the following e-configurations: $x^7 = (1, 3, 3, 6)$, $x^8 = (1, 3, 6, 3)$, $x^9 = (1, 6, 3, 3)$, $x^{10} = (3, 3, 1, 6)$, $x^{11} = (3, 3, 6, 1)$, $x^{12} = (3, 1, 3, 6)$, $x^{13} = (3, 1, 6, 3)$, $x^{14} = (3, 6, 1, 3)$, $x^{15} = (3, 6, 3, 1)$, $x^{16} = (6, 1, 3, 3)$, $x^{17} = (6, 3, 1, 3)$, $x^{18} = (6, 3, 3, 1)$.

  Then the CPC $E = \{x^i\}_{i \in J_{18} \setminus J_6}$ is an EMPC, while every its subset is a MPC.

- Let $n = 2$ and $G(E) = \{-1, 0, 2\}$ be the multiset with the underlying set $S(G(E)) = G(E) = \{-1, 0, 2\}$. In this case, $\eta = k = 3$.

Consider the e-configurations $x^{19} = (-1, 0)$, $x^{20} = (-1, 2)$, $x^{21} = (0, -1)$, $x^{22} = (0, 2)$, $x^{23} = (2, -1)$, $x^{24} = (2, 0)$. Then the CPC $E = \{x^i\}_{i \in J_{24} \setminus J_{18}}$ is an EPPPC, and any of its subsets is a PPPC.

- Let $n = 3$ and $G(E) = \{0, 2, 2, 5, 5\}$ be the multiset with the underlying set $S(G(E)) = \{0, 2, 5\}$. In this case, $\eta = 5,\ k = 3$.

  Introduce the following e-configurations: $x^{25} = (0, 2, 2)$, $x^{26} = (0, 2, 5)$, $x^{27} = (0, 5, 2)$, $x^{28} = (0, 5, 5)$, $x^{29} = (2, 0, 2)$, $x^{30} = (2, 0, 5)$, $x^{31} = (2, 2, 0)$, $x^{32} = (2, 2, 5)$, $x^{33} = (2, 5, 0)$, $x^{34} = (2, 5, 2)$, $x^{35} = (2, 5, 5)$, $x^{36} = (5, 0, 2)$, $x^{37} = (5, 0, 5)$, $x^{38} = (5, 2, 0)$, $x^{39} = (5, 2, 2)$, $x^{40} = (5, 2, 5)$.

  Then the CPC $E = \{x^i\}_{i \in J_{40} \setminus J_{24}}$ is an EPMPC, while every its subset is a PMPC.

- Let $n = 3$ and $G(E) = \{-1, 1\}$ be the multiset with the underlying set $G(E) = S(G(E)) = \{-1, 1\}$. In this case, $\eta = k = 2$.

  Consider the e-configurations $x^{41} = (-1, -1, -1)$, $x^{42} = (-1, -1, 1)$, $x^{43} = (-1, 1, -1)$, $x^{44} = (-1, 1, 1)$, $x^{45} = (1, -1, -1)$, $x^{46} = (1, -1, 1)$, $x^{47} = (1, 1, -1)$, $x^{48} = (1, 1, 1)$.

  Then the CPC $E = \{x^i\}_{i \in J_{48} \setminus J_{40}}$ is an EUPMPC.

Exploring various CPCs given by their elements, it is an issue to classify them. In this case, initially, the inducing set $G(E)$ of a certain CPC E is not given and needs to be derived. This task is easily solvable since the conditions that the CPC E must satisfy depending on its class are listed above. We illustrate this approach to classifying CPCs with the following examples.

We set $n = 4$ and analyze the following e-configurations: $x^{49} = (0, 1, 5, 7)$; $x^{50} = (0, 5, 1, 7)$; $x^{51} = (1, 5, 0, 7)$; $x^{52} = (0, 5, 7, 1)$; $x^{53} = (7, 1, 0, 5)$; $x^{54} = (5, 1, 0, 7)$; $x^{55} = (7, 1, 5, 0)$; $x^{56} = (5, 7, 0, 1)$.

- Suppose that $E^{(1)} = \{x^i\}_{i \in J_{56} \setminus J_{48}}$. Then $G(x^{49}) = G(x^{50}) = G(x^{51}) = G(x^{52}) = G(x^{53}) = G(x^{54}) = G(x^{55}) = G(x^{56}) = \{0, 1, 5, 7\}$, $G\left(E^{(1)}\right) = \bigcup\limits_{x^i \in E^{(1)}} G(x^i) = S\left(G\left(E^{(1)}\right)\right) =$ $=\{0, 1, 5, 7\}$.

  Since $G(x^i) = S\left(G\left(E^{(1)}\right)\right)$ for any $x^i \in E^{(1)},\ i \in J_{56} \setminus J_{48}$, then the CPC $E^{(1)}$ is a PPC.

- Let us add the e-configurations $x^{57} = (3,1,5,7)$, $x^{58} = (0,5,1,3)$, $x^{59} = (1,5,3,7)$, $x^{60} = (0,3,7,1)$ to the above CPC and explore the CPC $E^{(2)} = \{x^i\}_{i \in J_{60} \setminus J_{48}}$.

  We have $G(x^{57}) = \{1,3,5,7\}$, $G(x^{58}) = \{0,1,3,5\}$, $G(x^{59}) = \{0,3,5,7\}$, $G(x^{60}) = \{0,1,3,7\}$ and

$$G\left(E^{(2)}\right) = \bigcup_{x^i \in E^{(2)}} G(x^i) = S\left(G\left(E^{(2)}\right)\right) = \{0,1,3,5,7\}.$$

  Since $G(x^i) \subset S\left(G\left(E^{(2)}\right)\right)$ for any $x^i \in E^{(2)}$, $i \in J_{60} \setminus J_{48}$, then the CPC $E^{(2)}$ is a PMPC.

- Let $x^{61} = (1,1,5,7)$; $x^{62} = (1,5,1,7)$; $x^{63} = (1,7,1,5)$; $x^{64} = (1,5,7,1)$; $x^{65} = (7,1,1,5)$; $x^{66} = (5,1,1,7)$; $x^{67} = (7,1,5,1)$; $x^{68} = (5,7,1,1)$.

  For the CPC $E^{(3)} = \{x^i\}_{i \in J_{68} \setminus J_{60}}$, we have

$$G(x^{61}) = G(x^{62}) = G(x^{62}) = G(x^{64}) = G(x^{65}) = G(x^{66}) = G(x^{67}) = G(x^{68}) = \{1,1,5,7\}$$

$$G\left(E^{(3)}\right) = \bigcup_{x^i \in E^{(3)}} G(x^i) = \{1,1,5,7\}.$$

  Since $G(x^i) = G\left(E^{(3)}\right)$ for each $x^i \in E^{(3)}$, then the CPC $E^{(3)}$ is an MPC.

- Let us consider the CPC $E^{(4)} = \{x^i\}_{i \in J_{72} \setminus J_{48}}$, where $x^{69} = (3,3,5,7)$; $x^{70} = (3,5,1,3)$; $x^{71} = (1,3,3,7)$; $x^{72} = (0,3,3,1)$.

  In this case, $S\left(G(x^{69})\right) = \{3,5,7\}$; $S\left(G(x^{70})\right) = \{1,3,5\}$; $S\left(G(x^{71})\right) = \{1,3,7\}$; $S\left(G(x^{72})\right) = \{0,1,3\}$

$$G\left(E^{(4)}\right) = \bigcup_{x^i \in E^{(4)}} G(x^i) = \{0,1,3,3,5,7\}, S\left(G\left(E^{(4)}\right)\right) = \{0,1,3,5,7\}.$$

  Therefore, $E^{(4)}$ is a PMPC.

In the following chapters, we investigate important properties of multipermutation and partial multipermutation point configurations.

3.5 Special cases of EMPCs and EPMPCs

When for a FPC E,

$$k = 2,$$

we will call it a set of special e-configurations.

This class is singled out from the classes EMPC and EPMPC due to its specific properties such as vertex locality, polyhedral sphericity and others.

In particular,

1. $E_{n2}(G)$ is called the entire special multipermutation point configuration (ESPC);
2. $E^n_{\eta 2}(G)$ is the entire special partial multipermutation point configuration (ESPPC);
3. $\overline{E}^n_2(G)$ is the entire unbounded special partial multipermutation point configuration (EUSPPC).

The induced multiset of an ESPC is

$$G = \{e_1^{n_1}, e_1^{n_2}\}, e_1 < e_2, n_1 + n_2 = n, n_1, n_2 \geq 1, \tag{3.27}$$

while the induced multiset of an ESPPC is

$$G = \{e_1^{\eta_1}, e_1^{\eta_2}\}, e_1 < e_2, 1 \leq \eta_1, \eta_2 \leq n, \eta_1 + \eta_2 = \eta > n. \tag{3.28}$$

Let us introduce the parameter m for the multiplicity of e_2 among coordinates of special e-configurations and the parameters $m_1, m_2 : m_1 \leq m_2$ expressing a lower bound and an upper bound for the multiplicity of e_2 among the configurations' coordinates, i.e. $m \in [m_1, m_2]$.

For every ESPC, $m = m_1 = m_2$, and (3.27) can be rewritten in the compact form

$$G = \{e_1^{n-m}, e_2^m\}, \tag{3.29}$$

where $1 < m < n$.

Similarly, for every ESPPC, $1 < m_1 < m_< n$, and (3.27) can be rewritten in the compact form

$$G = \{e_1^{\eta - m_2}, e_2^{m_2}\},$$

where

$$m_1 = n - \eta_1, m_2 = \eta_2. \tag{3.30}$$

Respectively, every $E \subset E_{n2}(G)$ will be a special multipermutation point configuration (SPC). At the same time, every $E' \subset E_{\eta 2}^n(G)$ will be a special partial multipermutation point configuration (SPPC) if E' is induced by a multiset $G' \subseteq G$ with $|G'| > n$, otherwise E' will be a SPC.

Clearly, for sets of special e-configurations,

$$S(G) = \{e_1, e_2\}.$$

Depending on values e_1, e_2, some classes of the sets can be introduced such as

- if

$$S(G) = \{0, 1\}, \tag{3.31}$$

  we use term "binary" instead of "special";

- if

$$S(G) = \{-1, 1\}. \tag{3.32}$$

Respectively,

1. the ESPC $E_{n2}(G)$ satisfying (3.31) is called the entire binary multipermutation point

configuration (EBPC) and is denoted as

$$B_n\left(m\right), \tag{3.33}$$

where m is given by (3.29);

2. similarly, the ESPC $E_{n2}(G)$ satisfying (3.32) is denoted as $B'_n(m)$

3. the EBPPC $E^n_{\eta 2}(G)$ satisfying (3.31) is called the entire binary partial multipermutation point configuration (EBPPC) and is denoted

$$B_n\left(m_1, m_2\right), \tag{3.34}$$

where m_1, m_2 are given by (3.30);

4. the EBPPC $E^n_{\eta 2}(G)$ satisfying (3.32) is denoted

$$B'_n(m_1, m_2),$$

where m_1, m_2 are given by (3.30);

5. the EUSPPC $\overline{E}^n_2(G)$ satisfying (3.31) is the entire unbounded binary multipermutation point configuration (EUBPPC) denoted as B_n;

6. the EUSPPC $\overline{E}^n_2(G)$ satisfying (3.32) is denoted as B'_n.

Chapter 4

Multipermutation point configurations

The chapter is dedicated to studying the properties of sets of the class EMPC in general and exploring the peculiarities of its special cases.

The description of the EMPC $E_{nk}(G)$ and its convex hull is based on the works [28, 31, 64, 76, 79, 109]. For studying the EPPC $E_n(G)$ and its convex hull, the publications [9, 13, 14, 76, 106, 108] are used. When presenting the properties of the EBPC $B_n(m)$ and its convex hull, the papers [5, 10, 40, 66, 112] are utilized.

The study of the properties of EMPC, such as surface locality, decomposition, levelness, and symmetry, is based on the approaches described in Chapters 1 and 2 of this monograph. The same applies to properties of the corresponding polytopes, such as dimension, adjacency of polytopes' vertices, combinatorial equivalence, irredundancy of H-representations, single-outing simple polytopes, and so on.

4.1 The entire multipermutation point configuration

In this section, we explore properties of the EMPC $E_{nk}(G)$ along features of its convex hull called the multipermutohedron and denoted by

$$\Pi_{nk}(G) = conv\ E_{nk}(G). \tag{4.1}$$

Let us exclude from consideration the singleton set $E_{n1}(G)$, further assuming that the

condition $k \geq 2$ is satisfied. Respectively, $n > 1$.

Remark 4.1. Due too $1 \leq \eta_i \leq n,\ i \in J_k$ and $[G] = (n_1, ..., n_k)$, for the multiplicities (3.14), it holds: $1 \leq n_i \leq n - k + 1,\ i \in J_k$.

Below, we list the main properties of the EMPC and permutohedron.

The cardinality of $E_{nk}(G)$

$$|E_{nk}(G)| = \frac{n!}{n_1! \cdot ... \cdot n_k!}. \tag{4.2}$$

The plane locality of $E_{nk}(G)$

The set $E_{nk}(G)$ lies in the hyperplane:

$$\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} g_i. \tag{4.3}$$

The spherical locality of $E_{nk}(G)$

It is easy to see that all points of $E_{nk}(G)$ satisfy the equation:

$$\sum_{i=1}^{n} x_i^2 = \sum_{i=1}^{n} g_i^2, \tag{4.4}$$

that is, this set is inscribed in a hypersphere centered at the origin, hence, the set $E_{nk}(G)$ is spherically-located.

Since $E_{nk}(G)$ also lies in the hyperplane, such hypersphere is not unique, and circumscribed hyperspheres about it form a family. We write it out using the notation:

$$S_j = \sum_{i=1}^{n} g_i^j,\ j \in \mathbb{N}. \tag{4.5}$$

Given (3.14), the expressions (4.5) can be represented in compact form:

$$S_j = \sum_{i=1}^{k} n_i e_i^j,\ j \in \mathbb{N}.$$

In these new notations, we rewrite (4.3) and (4.4):

$$\sum_{i=1}^{n} x_i = S_1; \tag{4.6}$$

$$\sum_{i=1}^{n} x_i^2 = S_2. \tag{4.7}$$

Theorem 4.1. *The points $E_{nk}(G)$ lie on the family of hyperspheres $S_{r(a)}(\mathbf{a})$:*

$$\sum_{i=1}^{n} (x_i - a)^2 = r^2(a), \text{ where } r(a) = \sqrt{S_2 - 2aS_1 + na^2}, \tag{4.8}$$

where $a \in \mathbb{R}^1$ is a parameter, and S_1, S_2 is given by the expressions (4.6) *and* (4.7).

In the family (4.8),

- *S^0, the hypersphere centered at the origin, is given by the equation* (4.7) *and has radius:*

$$r^0 = \sqrt{S_2}; \tag{4.9}$$

- *the circumsphere $S^{\min}$ of the minimum radius has the parameters:*

$$a^{\min} = \tfrac{S_1}{n}; \tag{4.10}$$

$$r^{\min} = r(a^{\min}) = \sqrt{S_2 - \tfrac{S_1^2}{n}}. \tag{4.11}$$

The hypersphere $S^{\min}$ is given by the equation:

$$\sum_{i=1}^{n} \left(x_i - \tfrac{S_1}{n}\right)^2 = S_2 - \tfrac{S_1^2}{n}. \tag{4.12}$$

It can be represented in the form:

$$\sum_{i=1}^{n} x_i^2 - \frac{2S_1}{n} \sum_{i=1}^{n} x_i^2 = S_2 - \frac{2S_1^2}{n}.$$

Finally, we formulate the property of the set $E_{nk}(G)$, which is highly important for further presentation.

Polyhedral-sphericity of $E_{nk}(G)$

Proposition 4.1. *The set $E_{nk}(G)$ is polyhedral-spherical and can be represented as*

$$E_{nk}(G) = S_{r(a)}(\mathbf{a}) \cap \Pi_{nk}(G), \tag{4.13}$$

where the point $\mathbf{a}$ *is defined by the parameter $a \in \mathbb{R}^1$, while the radius $r(a)$ is given by the formula* (4.8).

The proof follows directly from Theorem 2.4. Since $E_{nk}(G)$ is spherically-located, it belongs to the class of surface-located sets. Therefore, it is a polyhedral-surface set, and the formula (2.4) becomes (4.13). Thus, $E_{nk}(G)$ is polyhedral-spherical.

Depending on the choice of the parameter a in the equation of the circumscribed hypersphere and the H-representation of the polytope $\Pi_{nk}(G)$, various polyhedral-spherical representations of the set $E_{nk}(G)$ are formed based on the formula (4.13).

$E_{nk}(G)$-decomposition into families of parallel hyperplanes

We present two types of decompositions of $E_{nk}(G)$ into parallel hyperplanes:

1. into families of hyperplanes parallel to coordinate ones;
2. into parallel hyperplanes orthogonal to the hyperplanes (4.3).

Theorem 4.2. *The set $E_{nk}(G)$ lies on the families $\{H_s^t\}_{t \in J_{T_s}}$ hyperplanes of the form*

$$H_s^t: \qquad \frac{s}{n-s}\sum_{i=1}^{n-s} x_i - \sum_{i=n-s+1}^{n} x_i + a_t^s = 0, \tag{4.14}$$

where $s \in J_{n-1}$, $t \leq T_s \leq C_n^s$,

$$a_t^s = -\frac{s}{n-s} \cdot S_1 + \frac{n}{n-s} \cdot e_t^{G,s}, \ t \in J_{T_s}. \tag{4.15}$$

Proof. The equation (4.14) can be rewritten as

$$\begin{aligned} &s\sum_{i=1}^{n-s} x_i + (s-n)\sum_{i=n-s+1}^{n} x_i + A_t^s = 0, \\ &\text{where } A_t^s = a_t^s\,(n-s)\,. \end{aligned} \tag{4.16}$$

Given (4.3), the condition (4.16) for $E_{nk}(G)$ is equivalent to the following:

$$\begin{aligned} &s\sum_{i=1}^{n} x_i - n\sum_{i=1}^{s} x_{n-i+1} + A_t^s = \\ &= s\cdot S_1 - n\sum_{i=1}^{s} x_{n-i+1} + A_t^s = 0. \end{aligned} \tag{4.17}$$

This shows that the constants $\{A_t^s\}_{s,t}$ are determined by the possible values taken by the function $\sum\limits_{i=1}^{s} x_{n-i+1}$ на $E_{nk}(G)$. So, let $\Sigma_{G,s}$ be the multiset of all such values, respectively, its underlying set $S(\Sigma_{G,s})$ will contain all possible different values:

$$\Sigma_{G,s} = \left\{ \sum_{i\in\omega:\ |\omega|=s} g_{n-i+1} \right\},\ S\left(\Sigma_{G,s}\right) = \left\{ e_t^{G,s} \right\}_{t\in T_s}. \tag{4.18}$$

Then the parameter A_t^s takes T_s different values from the set $S\left(\Sigma_{G,s}\right)$, whose number does not exceed C_n^s.

Let us rewrite (4.17) as $s\cdot S_1 - n\cdot e_t^{G,s} + A_t^s = 0$, where

$$A_t^s = -s\cdot S_1 + n\cdot e_t^{G,s},\ t\in J_{T_s}. \tag{4.19}$$

Returning to the value of a_t^s and taking into account that $a_t^s = \frac{A_t^s}{n-s}$, we see that there exists the decomposition of $E_{nk}(G)$ along the hyperplanes (4.14) and (4.15). □

The resulting decomposition is also the decomposition (1.31) of the set $E_{nk}(G)$ toward the normal vector to the hyperplanes (1.32) of the form

$$H^j(\overline{n}) = \{x\in\mathbb{R}^n : j\sum_{i=1}^{n-j} x_i + (j-n)\sum_{i=n-j+1}^{n} x_i = 0\},\ j\in J_{n-1}. \tag{4.20}$$

We generalize this decomposition to the case of an arbitrary partition of the set of

variables $\{x_i\}_{i\in J_n}$ into s- and $n-s$-element subsets.

Corollary 4.1. *For an arbitrary fixed $\omega \subset J_n$, the set $E_{nk}(G)$ decomposes in the family $\left\{H'^t_\omega\right\}_{t\in J_{T_s}}$ hyperplanes of the form:*

$$H'^t_\omega : \qquad \frac{s}{n-s}\sum_{i\notin\omega} x_i - \sum_{i\in\omega} x_i + a^s_t = 0, \tag{4.21}$$

where $s = |\omega|$, $t \le T_s \le C^s_n$, and quantities a^s_t set using (4.15).

Indeed, (4.14) is represented as (4.21) when choosing $\omega = J_n \backslash J_{n-s}$. Using the expressions (4.16) and (4.17) in $\sum\limits_{i\notin\omega} x_i$, $\sum\limits_{i\in\omega} x_i$ from $\sum\limits_{i=1}^{n-s} x_i$, $\sum\limits_{i=n-s+1}^{n} x_i$, we get the formula (4.21).

Note that the hyperplanes in the family (4.21) are orthogonal to the hyperplanes (4.6) since the normal vector to them is the vector of ones, while the normal vector to H'^t_ω has $n-s$ coordinates equal to $\frac{s}{n-s}$, the rest s coordinates are ones.

Note also that in this transition from partitioning the set J_n into subsets $J_n\backslash J_{n-s}$, J_{n-s} to its partition into arbitrary s- and $n-s$-element subsets, the formula (4.17) becomes

$$s\cdot S_1 - n\sum_{i\in\omega} x_i + A^s_t = 0. \tag{4.22}$$

Separating the variables in (4.22), we get

$$\sum_{i\in\omega} x_i = \frac{1}{n}(s\cdot S_1 - A^s_t).$$

Now, taking into account (4.19), we obtain

$$\sum_{i\in\omega} x_i = \frac{1}{n}(2s\cdot S_1 - n\cdot e^{G,s}_t) = \frac{2s\cdot S_1}{n} - e^{G,s}_t,\ t\in T_s.$$

Thus, we came to one more decomposition of $E_{nk}(G)$ into the family of T_s of parallel hyperplanes given for each $\omega \subset J_n$. We formulate this result as the below corollary.

Corollary 4.2. *$\forall\, \omega \subset J_n$ the decomposition of $E_{nk}(G)$ into parallel hyperplanes toward the*

vector

$$\overline{n}_i(\omega) = \begin{cases} 1, \ i \in \omega, \\ 0, \ i \notin \omega \end{cases} \quad (i \in J_n)$$

is valid and has the form of the decomposition (1.40)-(1.42).

Namely, $\forall \omega \subset J_n, \ s = |\omega|$

$$E_{nk}(G) = \bigcup_{t=1}^{T_s} E^{t,\omega}, \tag{4.23}$$

$$E^{t,\omega} = H^{t,\omega} \cap E_{nk}(G). \tag{4.24}$$

where

$$H^{t,\omega} = \left\{ x \in \mathbb{R}^n : \sum_{i \in \omega} x_i = B_t^s \right\}, \ t \in J_{T_s}, \tag{4.25}$$

$$B_t^s = \tfrac{2sS_1}{n} - e_t^{G,s},$$

while the constants $e_t^{G,s}$ *are given by* (4.18).

If $s = 1$, the decomposition (4.23)-(4.25) becomes a decomposition (1.40)-(1.42) of the set $E_{nk}(G)$ along coordinates. It considers that all coordinates of an arbitrary point of $E_{nk}(G)$ take all k values from $S(G)$. Accordingly, the following statement holds.

Proposition 4.2. *$E_{nk}(G)$ is a k-level set along each coordinate.*

Corollary 4.3. *The set $E_{nk}(G)$ is k-level along coordinates:*

$$m'(E_{nk}(G)) = k. \tag{4.26}$$

Moreover, the condition (1.40) can be represented in the form of (1.43),

$$E'^{ij} = E_{n-1,k_i}(G^i), \tag{4.27}$$

or (1.44), (4.27), where $E = E_{nk}(G)$,

$$G^i = G \backslash \{e_i\},\ k_i = |S(G^i)|,\ i \in J_k. \tag{4.28}$$

Since, by our assumption, the condition $n > 1$ is satisfied, all the sets (4.27) are non-empty EMPCs. Thus, the necessary condition (1.45) is satisfied that (1.40)-(1.42) is a decomposition of the set $E = E_{nk}(G)$ along coordinates.

We also indicate which sets are formed at the intersection of $E_{nk}(G)$ with the hyperplane $H^{t,\omega}$ depending on ω:

- If $s = 1$ or $s = n - 1$, we are dealing with a decomposition (1.40)-(1.42), where the sets (1.40) are formed from the EMPCs of $n - 1$-dimensional permutation e-configurations (4.27) by adding one coordinate from G to the coordinates of its points.

$$E^{'ij} = Pr_{H^{ij}} E^{ij} = Pr_{H^{ij}} E_{nk}(G),\ i \in J_k,\ j \in J_n.$$

  The sets E^{ij} formed in cutting can be represented by the Cartesian product of the EMPC of $n - 1$-dimensional permutation e-configurations $E^{'ij}$ and the one-element EMPC $\{e_i\}$ of 1-dimensional permutation e-configurations ($i \in J_k,\ j \in J_n$).

- A similar situation occurs in other cases. Thus, if $1 < s < n - 1$, then, at the intersection of $E_{nk}(G)$ and $H^{t,\omega}$, one or more Cartesian products of the EMPCs of s-dimensional and $n - s$-dimensional permutation e-configurations induced by the multiset G are formed.

Vertex criterion of $\Pi_{nk}(G)$

Theorem 4.3. *The set of vertices of the multipermutohedron* (4.1) *coincides with the EMPC* $E_{nk}(G)$*:*

$$vert\, \Pi_{nk}(G) = E_{nk}(G). \tag{4.29}$$

The proof of this theorem for a set of an EPPC is given in [106], for an EMPC in [79]. We present a new short proof based on the spherical locality of $E_{nk}(G)$. Thus, according to Theorem 4.1, $E_{nk}(G)$ is polyhedral-spherical. Hence, by Corollary 2.1, it is vertex-located.

Vertex adjacency criterion for $\Pi_{nk}(G)$

Theorem 4.4. *The vertices of the polytope $\Pi_{nk}(G)$ adjacent to the vertex $x \in vert\,\Pi_{nk}(G)$ are all the points obtained from x by permuting the components of the underlying set* (1.3) *of the multiset G equal to e_i, e_{i+1} $(i \in J_{k-1})$ (further referred to as $e_i \leftrightarrow e_{i+1}$-transposition), and only they are.*

Vertex regularity degree of $\Pi_{nk}(G)$

Theorem 4.5. *The number $\mathcal{R}$ of adjacent vertices to an arbitrary vertex of the polytope $\Pi_{nk}(G)$ is determined by the formula:*

$$\mathcal{R} = \prod_{i=1}^{k-1} n_i n_{i+1}. \tag{4.30}$$

Remark 4.2. It is easy to see that the value $\mathcal{R}$ in (4.30) is bounded from below by $n-1$:

$$\mathcal{R} \geq n-1. \tag{4.31}$$

Moreover, for $k = n$, the inequality (4.31) becomes equality. It will be shown below that, in class $\Pi_{nk}(G)$, there exist other polytopes for which this inequality holds as equality. Note that (4.31) also implies that the dimension of $\Pi_{nk}(G)$ does not exceed $n-1$.

The irredundant H-representation of $\Pi_{nk}(G)$

Theorem 4.6. *The polytope $\Pi_{nk}(G)$ is given by the following linear constraints, including the equation* (4.3) *and inequalities*

$$\sum_{j=1}^{i} x_{\alpha_j} \geq \sum_{j=1}^{i} g_j,\ \{\alpha_j\}_{j \in J_i} \subset\ J_n,\ i \in J_{n-1}. \tag{4.32}$$

The constraint system (4.32) consists of $n-1$ collections of inequalities corresponding to a fixed value i, called the i-union $(i \in J_{n-1})$.

Remark 4.3. The inequalities (4.32) can also be rewritten as

$$\sum_{j\in\omega} x_j \geq \sum_{j=1}^{|\omega|} g_j,\ \omega \subset J_n. \tag{4.33}$$

Thus, the polytope $\Pi_{nk}(G)$ is described analytically by the constraints (4.3) and (4.33) (further referred to as $(\underline{\Pi_{nk}(G).\mathrm{HR}})$).

The system $(\Pi_{nk}(G).\mathrm{HR})$ can also be rewritten in an equivalent form that includes the equation (4.32) and the following inequalities:

- lower-bound constraints on variables:

$$x_i \geq e_1,\ i \in J_n; \tag{4.34}$$

- upper-bound constraints on variables:

$$x_i \leq e_k,\ i \in J_n; \tag{4.35}$$

- the remaining constraints of (4.33):

$$\sum_{j\in\omega} x_j \geq \sum_{j=1}^{|\omega|} g_j,\ \omega \subset J_n,\ 1 < |\omega| < n-1. \tag{4.36}$$

Theorem 4.7. *The system* (4.3) *and* (4.33) *of constraints of the polytope* $\Pi_{nk}(G)$ *is redundant if and only the minimum and/or maximum element of* G *is a multiple, i.e. if:*

$$n_1 + n_k > 2. \tag{4.37}$$

From (4.33)*, excluding the unions with indexes*

$$i \in J = \overline{i^{\min}, n_1} \cup \overline{n - n_k, i^{\max}}, \tag{4.38}$$

where

$$i^{\min} = \min\{2, n - n_k\},\ i^{\max} = \max\{n-2, n_1\}, \tag{4.39}$$

converts the system (4.3) *and* (4.33) *into the irredundant constraint system for the polytope* $\Pi_{nk}(G)$.

Remark 4.4. This theorem is a refinement of the theorem presented in [109]. It allows to foresee not only cases of redundant unions of constraints in the system (4.36), but also one of the unions of inequalities (4.34) or (4.35). And this, in turn, makes it possible to formulate this refinement as a criterion for the irredundancy of the H-representation of the multipermutohedron.

Corollary 4.4. *The irredundant H-representation of the polytope $\Pi_{nk}(G)$ (further referred to as ($\underline{\Pi_{nk}(G).IHR}$)) is the constraint system that includes the equation* (4.32) *and unions of inequalities* (4.33) *with numbers $i = |\omega| \in \overline{J}$, where $\overline{J}$ is the complement to the set J given by the formula* (4.38)*, i.e. $\overline{J} = J_{n-1} \backslash J$.*

The dimension of $\Pi_{nk}(G)$

Theorem 4.8. *The polytope $\Pi_{nk}(G)$ is $n-1$-dimensional:*

$$\dim \Pi_{nk}(G) = n - 1. \tag{4.40}$$

Proof. The proof of this theorem for the permutohedron is given in [106]. We generalize it to the EMPC using Remark 1.2. (4.3) and (4.32) forms an H-representation of $\Pi_{nk}(G)$, including one equation, i.e. the rank (1.55) of the corresponding matrix-row $\rho = 1$. On the other hand, the center $\mathbf{a}^{\min}$ of the circumscribed hypersphere $S^{\min}$, whose parameter $a^{\min}$ is given by (4.10)), is an interior point of $\Pi_{nk}(G)$ in the affine subspace (4.3). This is because all the inequalities (4.32) are satisfied strictly at the point $\mathbf{a}^{\min}$. According to (1.57), this means $\dim \Pi_{nk}(G) = n - \rho = n - 1$, which is needed to be proven. □

Symmetry hyperplanes of $E_{nk}(G)$ and $\Pi_{nk}(G)$

Theorem 4.9. *The set $E_{nk}(G)$ and the polytope $\Pi_{nk}(G)$ are symmetric about each of the C_n^2 hyperplanes:*

$$x_i - x_j = 0, \ 1 \le i < j \le n. \tag{4.41}$$

Remark 4.5. Adding the equation (4.6) to the conditions of Theorems 4.1, 4.2, and 4.9 allows establishing the existence of: a) families (4.6) and (4.8) of $n-2$-spheres circumscribed about $E_{nk}(G)$; b) $E_{nk}(G)$-decompositions (4.6), (4.14) and (1.39), (4.6) into families of parallel $n-2-$planes; c) symmetry of $E_{nk}(G)$ and $\Pi_{nk}(G)$ about $n-2$-planes give by (4.6), (4.41).

Centrally symmetric sets among $E_{nk}(G)$

Theorem 4.9 says that $E_{nk}(G)$ possesses certain symmetry. However, not every set in this class is centrally symmetric.

Now, we single out centrally symmetric sets in the class $E_{nk}(G)$ and, accordingly, centrally symmetric polytopes among $\Pi_{nk}(G)$.

Theorem 4.10. *The set $E_{nk}(G)$ is centrally symmetric if and only if the elements of the inducing multiset G satisfy the condition:*

$$\frac{g_i + g_{n-i+1}}{2} = \frac{S_1}{n},\ i \in J_{[\frac{n+1}{2}]}, \tag{4.42}$$

i.e. the center of symmetry $E_{nk}(G)$ can only be the point $\mathbf{a}^{\min}$, which is the center of $S^{\min}$.

Proof. Diametrically opposite point to the point $x \in E_{nk}(G)$ is the point $y \in E_{nk}(G)$ satisfying the condition:

$$\text{if } x_i = g_j, \text{ then } y_i = g_{n-j+1},\ i \in J_n.$$

In this case, the only center of symmetry of $E_{nk}(G)$ can be the midpoint z of the segment $[x, y]$. Namely, $z = \frac{x+y}{2}$, where $z_i = \frac{g_j+g_{n-j+1}}{2}$, $i \in J_n$. This condition must be satisfied for an arbitrary point $x \in E_{nk}(G)$, which is possible only if all coordinates of the point z are equal, namely,

$$b = \frac{g_i + g_{n-i+1}}{2},\ i \in J_n.$$

At the same time, in this case, if n is even, then $S_1 = \sum_{i=1}^{\frac{n}{2}}(g_i + g_{n-i+1}) = \frac{n}{2} \cdot 2b = n \cdot b$, which implies (4.42). If n is odd, then $S_1 = \sum_{i=1}^{\frac{n-1}{2}}(g_i + g_{n-i+1}) + g_{\frac{n+1}{2}} = (n-1)b + b = n \cdot b$. Therefore, in any case, $b = \frac{S_1}{n} = a^{\min}$. □

Theorem 4.10 says that symmetric $E_{nk}(G)$-sets are induced only by those multisets

whose elements are symmetric about their mean. The same applies to elements of the underlying set and primary specification of G.

Corollary 4.5. *The polytope* $\Pi_{nk}(G)$ *is centrally symmetric if and only if the multiset* G *satisfies the condition* (4.42).

Corollary 4.6. *If the set* $E_{nk}(G)$ *is centrally symmetric, then its underlying set and primary specification satisfy conditions:*

$$|e_i - a^{\min}| = |e_{k-i+1} - a^{\min}|,\ i \in J_{\left[\frac{k+1}{2}\right]}; \tag{4.43}$$

$$n_i = n_{k-i+1},\ i \in J_{\left[\frac{n+1}{2}\right]}. \tag{4.44}$$

Example 4.1. The EMPC $E_{53}(G)$ induced by the multiset $G = \{1^2, 3, 4^2\}$ has the following parameters: $S(G) = \{1, 3, 4\}$, $[G] = \{2, 1, 2\}$, $a^{\min} = \frac{13}{5}$. The condition (4.44) is met, while the condition (4.43) is violated because $|e_1 - a^{\min}| = \frac{8}{5} \neq |e_3 - a^{\min}| = \frac{7}{5}$. Thus, the set $E_{53}(G)$ is not centrally symmetric.

In this section, some properties of the EMPC and multipermutohedron have been established. In the following sections, we consider subclasses of $E_{nk}(G)$, adapt the main properties given in this section to them and derive new features specific to these specific subclasses.

Let us move to the consideration of particular cases of $E_{nk}(G)$, namely, to the extreme cases in this class corresponding to the maximum ($k = n$) and the minimum ($k = 2$) values of k. The first case corresponds to the EPPC $E_n(G)$ and is considered in detail in Section 4.2. The latter case corresponds to the EMPC $E_{n2}(G)$ that is induced by two different numbers and studied in Section 4.4. Let us derive properties of the sets $E_n(G)$ and $E_{n2}(G)$ and their convex hulls.

4.2 The entire permutation point configuration

Consider the entire permutation point configuration $E_n(G)$ and its convex hull called the permutohedron:

$$\Pi_n(G) = conv(E_n(G)).$$

Taking into account that, in this case, the condition

$$G = \mathcal{A} \tag{4.45}$$

is satisfied and, accordingly,

$$n = k, \tag{4.46}$$

in all the formulas given in Sec. 4.1, elements of the multiset G can be replaced by elements of its underlying set

$$g_i \to e_i,\ i \in J_n. \tag{4.47}$$

For example, the expression (4.5) becomes

$$S_j = \sum_{i=1}^{n} e_i^j,\ j \in \mathbb{N}. \tag{4.48}$$

As a result, we get the following properties of $E_n(G)$ and $P_n(G)$.

The cardinality of $E_n(G)$

$$|E_n(G)| = n!. \tag{4.49}$$

The dimension of $\Pi_n(G)$

$$\dim\ \Pi_n(G) = n - 1. \tag{4.50}$$

Vertex locality of $E_n(G)$

$$vert\,\Pi_n(G) = E_n(G).$$

Spherical and hyperplane locality of $E_n(G)$

The hyperplane, where $E_n(G)$ lies, is given by the equation

$$\sum_{i=1}^{n} x_i = S_1.$$

The family

$$\{S_{r(a)}(\mathbf{a})\}_{a\in\mathbb{R}^1} \tag{4.51}$$

of circumscribed hyperspheres about $E_n(G)$ is given by the equation (4.8), where S_1 and S_2 are given by the formula (4.48), and $a \in \mathbb{R}^1$ is a parameter.

Polyhedral-sphericity of $E_n(G)$

$$E_n(G) = S_{r(a)}(\mathbf{a}) \cap \Pi_n(G),$$

where $S_{r(a)}(\mathbf{a})$ is arbitrary hypersphere from the family (4.51), and $a \in \mathbb{R}$ is a parameter.

Vertex adjacency criterion for $\Pi_n(G)$

The set of adjacent vertices to each vertex $e \in E_n(G)$ is formed from e by a single $e_i \leftrightarrow e_{i+1}$-transposition ($i \in J_{n-1}$).

Vertex regularity degree of $\Pi_n(G)$

$$\mathcal{R} = n - 1. \tag{4.52}$$

The simplicity of $\Pi_n(G)$

According to Remark 4.2, the condition (4.52) means that the degree of regularity of vertices on the polytope $\Pi_n(G)$ reaches its lower bound $n-1$. Due to (4.50), this bound coincides with the dimension of $\Pi_n(G)$, i.e.

$$\mathcal{R} = \dim\ \Pi_n(G),$$

implying that $\Pi_n(G)$ is simple.

The irredundant representation of $\Pi_n(G)$

The above H-representations of $\Pi_{nk}(G)$ can easily be adapted for $\Pi_n(G)$, taking into account (4.48). Since there are no multiple elements in G, the condition (4.37) is violated, and this is sufficient for the H-representation ($\Pi_{nk}(G)$.HR) to be irredundant. We introduce the notation ($\Pi_n(G)$.IHR) for the representation and write it taking into account (4.45):

$$\sum_{j=1}^{n} x_j = \sum_{j=1}^{n} e_j,$$
$$\sum_{j\in\omega} x_j \geq \sum_{j=1}^{|\omega|} e_j,\ \omega \subset J_n.$$

The number of constraints in ($\Pi_n(G)$.HR) is

$$|\mathbf{H}| = 2^n - 1,$$

2^{n-1} of which are inequalities and one equation (here, $\mathbf{H}$ is the set of facets $\Pi_n(G)$).

$E_n(G)$ is n-level set along coordinates

n-levelness of $E_n(G)$ along coordinates follows directly from Corollary 4.3. Substituting (4.46) in (4.26), we get

$$m'(E_n(G)) = n.$$

In addition, $E_n(G)$ allows the decomposition (1.40)-(1.42), where the formula (1.39)

becomes $\forall\, j \in J_n$,

$$H^{ij} = \{x \in \mathbb{R}^n :\ x_j = e_i\}\,,\ i \in J_n, \tag{4.53}$$

while the sets (1.40) are formed from auxiliary sets (4.27) having the form of

$$E^{'ij} = E_{n-1,k-1}(G \backslash \{e_i\}),\ i, j \in J_n.$$

Decompositions of $E_n\,(G)$ into hyperplanes parallel to $\Pi_n(G)$-facets

Since the H-representation ($\Pi_n(G)$.HR) is irredundant, then, for each $\omega \subset J_n$, the formulas (4.23)-(4.25) define the decomposition of $E_n\,(G)$ toward the normal vector of the facet $H^{1,\omega}$ from the family (4.25):

$$H^{1,\omega} = \left\{x \in \mathbb{R}^n :\ \sum_{i \in \omega} x_i = S_{|\omega|}\right\}.$$

Considering (4.53) and that $\Pi_n(G)$ has facets parallel to the coordinate hyperplanes, the levelness $m(E_n\,(G))$ of $E_n\,(G)$lies in the range:

$$n \leq m(E_n\,(G)) \leq \max_{s \in J_{n-1}} C_n^s = C_n^{\left[\frac{n}{2}\right]}.$$

Centrally symmetric $E_n(G)$ and $\Pi_n(G)$

Theorem 4.10 implies that $E_n(G)$, $\Pi_n(G)$ are symmetric if

$$\frac{e_i + e_{n-i+1}}{2} = \frac{S_1}{n},\ i \in J_{\left[\frac{n+1}{2}\right]}, \tag{4.54}$$

which is also equivalent to satisfying the following condition:

$$|e_i - a^{\min}| = |e_{n-i+1} - a^{\min}|,\ i \in J_{\left[\frac{n+1}{2}\right]}.$$

4.3 The EPPC E_n

Now, we suppose that $G = J_n$, i.e. G is the set of the first n natural numbers, while the EPPC is $E_n(J_n)$. This set is denoted as E_n, i.e. $E_n = E_n(J_n)$, and the corresponding permutohedron is

$$\Pi_n = conv\ E_n.$$

In addition to the properties directly following from the fact that E_n and Π_n belong to EPPCs and permutohedra, respectively, such as

$$|E_n| = n!,$$
$$\dim\ \Pi_n = \mathcal{R} = n - 1,$$
$$vert\, \Pi_n = E_n$$

and others, the EPPC E_n and polytope Π_n have peculiarities caused by the specifics of their inducing set J_n.

For example, in the decomposition (1.40)-(1.42), the hyperplanes (1.39) form a set of families (4.53) of equidistant hyperplanes: $\forall j \in J_n$

$$H^{ij} = \{x \in \mathbb{R}^n :\ x_j = i\}\,,\ i \in J_n. \tag{4.55}$$

The formula (4.48) becomes

$$S_j = \sum_{i=1}^{n} i^j,\ \ j \in \mathbb{N}.$$

Accordingly, the expressions (4.6) and (4.7) take the form:

$$S_1 = \sum_{i=1}^{n} i = \tfrac{n(n+1)}{2}; \tag{4.56}$$
$$S_2 = \sum_{i=1}^{n} i^2 = \tfrac{n(n+1)(2n+1)}{6} \tag{4.57}$$
.

Applying the relations (4.56) and (4.57), we obtain the following properties of E_n, Π_n as a consequence of the listed above characteristics of $E_n(G)$ and $\Pi_n(G)$.

The plane locality of E_n, Π_n

E_n and Π_n lie on the hyperplane:

$$\sum_{i=1}^{n} x_i = \frac{n(n+1)}{2}.$$

The spherical locality of E_n

E_n lies on the family of hyperspheres (4.51) of radius:

$$r(a) = \sqrt{\frac{n(n+1)(2n+1)}{6} - a \cdot n(n+1) + a^2}, \tag{4.58}$$

where $a \in \mathbb{R}^1$ is a parameter.

In the family (4.51) and (4.58):

- the hypersphere of minimum radius $S^{\min}$ has the following parameters:

$$a^{\min} = \tfrac{n+1}{2}, \tag{4.59}$$
$$r^{\min} = \tfrac{1}{6}\sqrt{3n(n+1)(n-1)};$$

- the hypersphere S^0 centered at the origin has the radius:

$$r^0 = \sqrt{\frac{n(n+1)(2n+1)}{6}}.$$

Polyhedral-sphericity of E_n

E_n is a polyhedral-spherical set, namely,

$$E_n = S_{r(a)}(\mathbf{a}) \cap \Pi_n,$$

where $S_{r(a)}(\mathbf{a})$ is a hypersphere from the family (4.51) given by the parameter $a \in \mathbb{R}^1$ and having the radius (4.58).

The H-representation of Π_n

An irredundant representation of $\Pi_n(G)$ (further referred to as (Π_n.IHR)) is obtained directly from ($\Pi_n\,(G)$.IHR) by substitution:

$$\sum_{j=1}^{i} g_j = \sum_{j=1}^{i} j = \frac{i(i+1)}{2}.$$

It results in (Π_n.IHR) having the form of

$$\begin{gathered} \sum_{j=1}^{n} x_j = \tfrac{n(n+1)}{2}, \\ \sum_{j\in\omega} x_j \geq \tfrac{i(i+1)}{2},\ \omega \subset J_n,\ i = |\omega|\,. \end{gathered}$$

Vertex adjacency criterion for Π_n

The vertices of Π_n adjacent to every $x \in E_n$ are the ones obtained from x by the $i \leftrightarrow i+1$-transposition ($i \in J_{n-1}$), and only they are.

E_n-decompositions into families of hyperplanes parallel to Π_n-facets

Since the H-representation(Π_n.IHR) is irredundant, replacing its inequalities by equalities defines the set $\mathbf{H}$ of facets of Π_n:

$$\begin{aligned} &\mathbf{H} = \{H^\omega\}_{\omega\subset J_n}: \\ &H^\omega = \{x \in \mathbb{R}^n : \sum_{i\in\omega} x_i = \frac{j(j+1)}{2}\},\ \omega \subset J_n,\ j = |\omega|. \end{aligned} \tag{4.60}$$

So, for example, if $\omega = \{i\} \in J_n$, (4.60) defines the set of hyperplanes in the decomposition (4.55) corresponding to the decomposition of E_n along the coordinate x_i and into hyperplanes parallel to the facet

$$H^{1,\omega} = \{x_i = e_1\}.$$

Theorem 4.11. *For $\omega \subset J_n$ and $j = |\omega|$, the decompositions of the set E_n into hyperplanes parallel to the facet $H^{1,\omega}$ have the form:*

$$\begin{aligned} &H^{a_j,\omega} = \{x \in \mathbb{R}^n : \sum_{i\in\omega} x_i = a_j\},\ a_j \in J_{a_j^{\max}} \backslash J_{a_j^{\min}-1}, \\ &\textit{where } a_j^{\min} = \frac{j(j+1)}{2},\ a_j^{\max} = \frac{j(2n-j+1)}{2}. \end{aligned} \tag{4.61}$$

Proof. For every $j \in J_{n-1}$ and all $\omega \subset J_n$ such that $|\omega| = j$, the function $h(\omega, x) = \sum_{i\in\omega} x_i$ runs through all integer values from the range $[a_j^{\min}, a_j^{\max}]$, where $a_j^{\min}, a_j^{\max}$ are given by (4.61). Indeed, this function takes the value $a_j^{\min}$ on permutation e-configurations $x^{\min,\omega} \in E_n : \{x_i^{\min,\omega}\}_{i\in\omega} = J_j,\ \{x_i^{\min,\omega}\}_{i\notin\omega} = J_n \backslash J_j$. The value $a_j^{\max}$ is reached on the permutation e-configurations $x^{\max,\omega} \in E_n : \{x_i^{\max,\omega}\}_{i\in\omega} = J_n \backslash J_{n-j},\ \{x_i^{\max,\omega}\}_{i\notin\omega} = J_{n-j}$. At other points of E_n the function $h(\omega, x)$ takes an integer value from the interval $(a_j^{\min},\ a_j^{\max})$.

Let $x \in E_n$ be a point such that $h(\omega, x) = a \in (a_j^{\min}, a_j^{\max})$. Consider the set $N(x)$ of vertices adjacent to it. Since they are all formed from x by $x_i \leftrightarrow x_j$-transpositions such that $|x_i - x_j| = 1$, three subsets can be distinguished in $N(x)$: a) the set of points of $N^+(x)$ c) the set $N^0(x)$ consisting of those vertices adjacent to x, where the value of $h(.)$ remains unchanged. It is easy to see that $N^0(x)$ is formed by the mentioned transpositions of x-coordinates within ω, i.e. $|N^0(x)| \le |\omega| - 1 = j - 1 \le n - 2$, at the remaining $n - j$ points, where $n - j > 0$, $h(.)$ decreases or increases exactly by one.

Thus, it is shown that, for all $\omega \subset J_n$, $j = |\omega|$ and $a_j \in J_{a_j^{\max}} \backslash J_{a_j^{\min}-1}$, it is true that $E_n \cap H^{a_j,\omega} \neq \emptyset$, i.e. there exists the decomposition (4.23)-(4.25) of E_n into hyperplanes parallel to the facets of Π_n. It is

$$\begin{aligned} E_n &= \bigcup_{t=a_j^{\min}}^{a_j^{\max}} E^{t,\omega}, \\ E^{t,\omega} &= H^{t,\omega} \cap E_n, \end{aligned}$$

where

$$H^{t,\omega} = \left\{x \in \mathbb{R}^n : \sum_{i\in\omega} x_i = t\right\},\ t \in J_{a_j^{\max}} \backslash J_{a_j^{\min}-1}.$$

□

E_n-levelness

Lemma 4.1. *The levelness of the set E_n is given by the formula:*

$$m(E_n(G)) = \left\lfloor \frac{n}{2} \right\rfloor \left\lceil \frac{n}{2} \right\rceil + 1. \tag{4.62}$$

Proof. Based on Theorem 4.11, from the formula (4.61) it is easy to derive the exact number of levels of the E_n-decomposition into hyperplanes parallel to Π_n-facets. Thus, if $H^{1,\omega}$ is a facet of Π_n corresponding to certain inequality of the $j = |\omega|$-th union of inequalities of this polytope, then the value

$$\begin{aligned} L_j = \frac{j(2n-j+1)}{2} - \frac{j(j+1)}{2} + 1 = \\ \frac{j}{2}(2n-j+1-j-1)+1 = j(n-j)+1 \end{aligned} \tag{4.63}$$

yields the levelness of E_n toward the normal vector of the facet $H^{1,\omega}$.

We are interested in the maximum of the function (4.63) since the levelness of the set E_n can be found by the formula $m(E_n) = \max\limits_{j \in J_{n-1}} L_j$. As j increases, the value of L_j increases, then decreases. It reaches the maximum:

- $m(E_n) = (\frac{n}{2})^2 + 1$ for $j = \frac{n}{2}$, if j is even;
- $m(E_n) = \left\lfloor \frac{n}{2} \right\rfloor \left\lceil \frac{n}{2} \right\rceil + 1$ for $j = \left\lfloor \frac{n}{2} \right\rfloor, \left\lceil \frac{n}{2} \right\rceil$, if j odd.

Combining both of these cases, we get the formula (4.62). □

Centrally symmetric E_n and Π_n

Since, in E_n, the neighboring elements of G are located at a distance of ones, the condition (4.54) is satisfied. Therefore, E_n and Π_n are centrally symmetric, with the point $\mathbf{a}^{\min}$ as their center of symmetry and given by the parameter (4.59).

4.4 The entire special multipermutation point configuration

The entire special multipermutation point configuration $E_{n2}(G)$ possesses specifics compared to the entire class $E_{nk}(G)$.

The convex hull of $E_{n2}(G)$ is called the special permutohedron and denoted by

$$\Pi_{n2}(G) = conv\, E_{n2}(G).$$

Let us list the properties of $E_{n2}(G)$ and $\Pi_{n2}(G)$ following from the listed above properties of the EMPC $E_{nk}(G)$ and multipermutohedron $P_{nk}(G)$. Suppose that the condition (3.27) is satisfied.

The cardinality of $E_{n2}(G)$

$$|E_{n2}(G)| = C_n^{n_1} = C_n^{n_2}. \tag{4.64}$$

The formula (4.64) follows from the formulas (4.2) and (3.27).

As seen, the cardinality of the ESPC is determined by binomial coefficients. It varies within the range:

$$n \leq |E_{n2}(G)| \leq C_n^{\left[\frac{n}{2}\right]}$$

and reaches a minimum and a maximum at

$$\min\{n_1, n_2\} = 1, \tag{4.65}$$

$$\max\{n_1, n_2\} = \left[\frac{n}{2}\right], \tag{4.66}$$

respectively.

The dimension of $\Pi_{n2}(G)$

$$\dim\ \Pi_{n2}(G) = n - 1.$$

Vertex locality of $E_{n2}(G)$

$$vert\, \Pi_{n2}(G) = E_{n2}(G).$$

Since, for the set $E_{n2}(G)$, the formula (4.5) becomes

$$S_j = n_1 e_1^j + n_2 e_2^j,\ j \in \mathbb{N}, \tag{4.67}$$

the properties of its plane- and spherical locality are formulated as follows.

The plane locality of $E_{n2}(G)$

The set $E_{n2}(G)$ lies on the hyperplane:

$$\sum_{i=1}^{n} x_i = n_1 e_1 + n_2 e_2 = n e_1 + n_2 (e_2 - e_1) = n e_2 + n_1 (e_1 - e_2). \tag{4.68}$$

The spherical locality of $E_{n2}(G)$

The set $E_{n2}(G)$ is inscribed in the family of hyperspheres (4.51) centered at the point $\mathbf{a}$ of radius:

$$r(a) = \sqrt{n_1 (e_1 - a)^2 + n_2 (e_2 - a)^2}, \tag{4.69}$$

defined by the parameter $a \in \mathbb{R}^1$.

Hyperspheres $S^{\min}$, S^0 for $E_{n2}(G)$

Let us determine the parameters of the minimal hypersphere circumscribed about $E_{n2}(G)$. For that, we introduce the value:

$$\Delta = e_2 - e_1. \tag{4.70}$$

Given the expression (4.68), the parameter (4.10) is

$$a^{\min} = \frac{n_1 e_1 + n_2 e_2}{n} = e_1 + \frac{n_2}{n}\Delta = e_2 - \frac{n_1}{n}\Delta. \tag{4.71}$$

Substituting (4.70) and (4.71) into the relation (4.69) yields:

$$r^{\min} = \sqrt{n_1\left(\frac{n_2}{n}\Delta\right)^2 + n_2\left(\frac{n_1}{n}\Delta\right)^2} = \frac{\Delta}{n}\sqrt{n_1 n_2 (n_1 + n_2)} = \frac{\Delta}{n}\sqrt{n_1 n_2 n},$$

where

$$r^{\min} = \Delta\sqrt{\frac{n_1 n_2}{n}}. \tag{4.72}$$

Thus, $S^{\min}$ has the parameters (4.71) and (4.72).

From (4.72), it is also seen that $r^{\min}$ varies within

$$\Delta\sqrt{1 - \frac{1}{n}} \leq r^{\min} \leq \frac{\Delta}{2}\sqrt{n}, \tag{4.73}$$

reaching the minimum and maximum under satisfying the conditions (4.65) or (4.66), respectively.

Note that $r^{\min}$ reaches the upper bound given in (4.73) only if n is even and $n_1 = n_2 = \frac{n}{2}$. In this case, the formulas (4.71) and (4.72), yielding the parameters of $S^{\min}$:

$$r^{\min} = \frac{\Delta}{2}\sqrt{n}, \tag{4.74}$$

$$a^{\min} = \frac{e_1 + e_2}{2}. \tag{4.75}$$

For $E_{n2}(G)$, the circumscribed hypersphere S^0 centered at the origin has the radius $r^0 = \sqrt{S_2} = \sqrt{n_1 e_1^2 + n_2 e_2^2}$.

Polyhedral-sphericity of $E_{n2}(G)$

The set $E_{n2}(G)$ is polyhedral-spherical, namely,

$$E_{n2}(G) = S_{r(a)}(\mathbf{a}) \cap \Pi_{n2}(G),$$

where $S_{r(a)}(\mathbf{a})$ is the family (4.51) and (4.69) of circumspheres given by the parameter $a \in \mathbb{R}^1$.

The irredundant H-representation of $\Pi_{n2}(G)$

Due to the assumption $n > 1$, $n_1 + n_2 = n$ holds, hence, the necessary and sufficient condition (4.37) for the irredundancy of the H-representation ($\Pi_{n2}(G)$.HR) is certainly satisfied for $n > 2$. Moreover, the group of constraints (4.36) is always redundant. As a result, the irredundant H-representation of the special permutohedron ($\Pi_{n2}(G)$.IHR) is a subsystem of the constraint system (4.32), (4.34), and (4.35), which can be represented as (4.68) and

$$e_1 \leq x_i \leq e_2,\ i \in J_n \tag{4.76}$$

(further referred to as ($\underline{\Pi_{n2}(G).\text{HR1}}$)).

Let us also rewrite the formula (4.35), taking into account (3.27) and getting:

$$x_i \leq e_2,\ i \in J_n. \tag{4.77}$$

As one can be seen from ($\Pi_{n2}(G)$.HR1), the polytope $\Pi_{n2}(G)$ is a cut of the hypercube (4.76) by the hyperplane (4.68).

Based on Theorem 4.7, we formulate a theorem, establishing the irredundant H-representation of $\Pi_{n2}(G)$.

Theorem 4.12. *($\Pi_{n2}(G)$.HR1) is the redundant H-representation of the polytope $\Pi_{n2}(G)$ if and only if the condition* (4.65) *holds for G.*

If the condition (4.65) *is met, we have*

- *if*

$$n_1 = 1,$$

($\Pi_{n2}(G)$.IHR) is (4.68) *and* (4.77)*;*

- *if*

$$n_k = 1,$$

then $(\Pi_{n2}(G).IHR)$ *involves the constraints* (4.34) *and* (4.68).

If the condition (4.65) *is violated, the representations* $(\Pi_{n2}(G).IHR)$ *and* $(\Pi_{n2}(G).HR1)$ *coincide.*

Vertex adjacency criterion for $\Pi_{n2}(G)$

Vertices of the polytope $\Pi_{n2}(G)$, adjacent to the point $x \in E_{n2}(G)$, are formed from x by its single $e_1 \leftrightarrow e_2$-transposition of its coordinates.

Vertex regularity degree of $\Pi_{n2}(G)$

The regularity degree of $\Pi_{n2}(G)$-vertices is

$$\mathcal{R} = n_1 n_2. \tag{4.78}$$

As you can see from (4.78), $\mathcal{R}$ varies within

$$n - 1 \leq \mathcal{R} \leq \left\lfloor \frac{n}{2} \right\rfloor \cdot \left\lceil \frac{n}{2} \right\rceil \leq \frac{n^2}{4}, \tag{4.79}$$

reaching the minimum and maximum under the conditions (4.65) and (4.66), respectively.

Remark 4.6. The formula (4.79) allows singling out one more class of simple polytopes in class $\Pi_{nk}(G)$, which is special permutohedra satisfying the condition (4.65). It is shown below that they are $n-1$-simplices.

Two-levelness of $E_{n2}(G)$

$E_{n2}(G)$ is a two-level set, i.e.

$$m(E_{n2}(G)) = 2.$$

Applying Proposition 4.2 to the case (3.27), we obtain that the set $E_{n2}(G)$ is two-level along every coordinate, wherefrom $m'(E_{n2}(G)) = 2$. The family of hyperplanes (1.39) in the

decomposition (1.40)-(1.42) is

$$H^{ij} = \{x \in \mathbb{R}^n : x_j = e_i\},\ i \in J_2,\ j \in J_n, \tag{4.80}$$

and the sets (4.27) formed in the projection $E_{n2}\,(G)$ onto the hyperplane (4.80) are either ESPCs of the dimension lower by one or singleton sets: $\forall\, j \in J_n$,

$$E^{'1j} = E_{n-1,k_1}(\{e^{n_1-1}, e^{n_2}\}),\ E^{'2j} = E_{n-1,k_2}(\{e^{n_1}, e^{n_2-1}\}), k_1, k_2 \in \{0, 1\}.$$

On the other hand, based on (4.76), the set $\mathbf{H}$ of facets $\Pi_{n2}\,(G)$ is a subset of $2n$ facets of the hypercube $[e_1, e_2]^n$, i.e. it is a subset of the family

$$\mathbf{H} \subseteq \{H_{ij}\}_{i \in J_2,\, j \in J_n}.$$

This implies that all the facets of $\Pi_{n2}\,(G)$ are parallel to the coordinate hyperplanes. Hence, $E_{n2}\,(G)$ is two-level toward normal vectors to its facets, and is a two-level set.

Centrally symmetric E_{n2} and Π_{n2}

Since the case $k = 2$ is under consideration, the conditions (4.44) and (4.43) of symmetry E_{n2} and Π_{n2} are simplified to

$$n_1 = n_2 = \frac{n}{2}. \tag{4.81}$$

This implies that, for even n only, there exist centrally symmetric ESPCs and multipermutohedra. Respectively, the dimension of the centrally symmetric polytopes is odd.

So, the following statement holds.

Lemma 4.2. *The set $E_{n2}\,(G)$ and the polytope $\Pi_{n2}\,(G)$ are symmetric if and only if the dimension of $\Pi_{n2}\,(G)$ is odd, and the condition* (4.81) *is satisfied.*

4.5 The EBPC $B_n(m)$

Consider class $B_n(m)$ of the entire binary multipermutation point configurations (EBPC).

Further, we will assume that

$$m \in J_n^0, \tag{4.82}$$

in particular, $B_n(0)$ and $B_n(n)$ are the singletons

$$B_n(0) = \mathbf{0},\ B_n(n) = \mathbf{e} = (1, ..., 1),$$

where $\mathbf{0}$ is a zero vector, and $\mathbf{e}$ is a vector of ones in $\mathbb{R}^n$.

The convex hull of the EBPC $B_n(m)$ is called the hypersimpex [66] and is denoted as

$$\Delta_{n,m} = convB_n(m). \tag{4.83}$$

The choice of such a name is caused by the fact that the hypersimplex is a generalization of the unit $n-1$-simplex

$$\Delta_{n,1} = \Delta_n = \{\mathbf{0} \leq x \leq \mathbf{e} :\ x^\top \mathbf{e} = 1\} \tag{4.84}$$

formed in the cut of the unit hypercube $[0,1]^n$ by the hyperplane $x^\top \mathbf{e} = 1$, in the case of its cut by the hyperplane $x^\top \mathbf{e} = m$, where m satisfies the condition (4.82):

$$\Delta_{n,m} = \{\mathbf{0} \leq x \leq \mathbf{e} :\ x^\top \mathbf{e} = m\}.$$

Taking into account (3.33) and

$$G = \left\{0^{n-m}, 1^m\right\}, \tag{4.85}$$

we reformulate the properties of $E_{n2}(G)$, $\Pi_{n2}(G)$ derived in Section 4.4 for the set $B_n(m)$ and

polytope $\Delta_{n,m}$. For that, we make the substitutions:

$$e_1 \to 0,\ e_2 \to 1,\ n_1 \to n-m,\ n_2 \to m. \tag{4.86}$$

As a result, we have the following.

The cardinality of $B_n(m)$

The cardinality of $B_n(m)$ is

$$|B_n(m)| = C_n^m.$$

It ranges in:

$$n \leq |B_n(m)| \leq C_n^{\left[\frac{n}{2}\right]}.$$

The dimension of $\Delta_{n,m}$

$$\dim\ \Delta_{n,m} = n-1.$$

Vertex locality of $B_n(m)$

$$vert\ \Delta_{n,m} = B_n(m).$$

Spherical and hyperplane locality of $B_n(m)$

For $B_n(m)$, the formula (4.67) is simplified to

$$S_j = m,\ j \in \mathbb{N}.$$

Substituting the formula (4.85) into (4.68)-(4.75), we get

- $\Delta = 1$;

- $B_n(m)$ lies in the hyperplane:

$$\sum_{i=1}^{n} x_i = m; \tag{4.87}$$

- $B_n(m)$ is inscribed in the family (4.51) of hyperspheres with the center $\mathbf{a}$ and radius:

$$r(a) = \sqrt{m - 2m \cdot a + n \cdot a^2}, \tag{4.88}$$

where $a \in \mathbb{R}^1$ is a parameter.

Hyperspheres $S^{\min}$ and S^0 for $B_n(m)$

In the family (4.87) and (4.88), the hypersphere $S^{\min}$ is given by the parameters:

$$a^{\min} = \frac{m}{n},\ r^{\min} = \sqrt{\frac{m(n-m)}{n}},$$

while the hypersphere S^0 has radius $r^0 = \sqrt{m}$.

Polyhedral-sphericity of $B_n(m)$

The set $B_n(m)$ is polyhedral-spherical, namely,

$$B_n(m) = S_{r(a)}(\mathbf{a}) \cap \Delta_{n,m},$$

where $S_{r(a)}(\mathbf{a})$ is arbitrary hypersphere from the family (4.87) and(4.88).

The irredundant H-representation of $\Delta_{n,m}$

Let us formulate a corollary from Theorem 4.12 about the irredundant H-representation of $\Delta_{n,m}$ denoted by ($\underline{\Delta_{n,m}.\text{IHR}}$). Suppose the inducing multiset is given by (4.85).

Corollary 4.7. *If the condition* (4.65) *is met, then:*

1. *if*

$$1 < m < n, \tag{4.89}$$

then the H-representation ($\Delta_{n,m}$.IHR) has the form of (4.87) *and*

$$0 \leq x_i \leq 1,\ i \in J_n; \tag{4.90}$$

2. if (4.89) *is violated and $m = 1$, then ($\Delta_{n,1}$.IHR) is given by* (4.87),

$$x_i \geq 0,\ i \in J_n; \tag{4.91}$$

3. if (4.89) *is violated, while*

$$m = n - 1,$$

then the H-representation ($\Delta_{n,n-1}$.IHR) is given by (4.87) *and*

$$x_i \leq 1,\ i \in J_n. \tag{4.92}$$

Case 2 in this corollary corresponds to the unit $n-1$-simplex $\Delta_{n,1}$. Case 3 corresponds to another $n-1$-simplex among the $(0-1)$-permutohedron, which is $\Delta_{n,n-1}$.

In general, polytopes in the family $\{\Delta_{n,m}\}_m$ are cuts of the unit hypercube by the hyperplane (4.87). Their irredundant H-representations can be given in the form (4.84) for all the above cases, namely:

$$\Delta_{n,1} = \{x \geq \mathbf{0} :\ x^\top \mathbf{e} = 1\};$$

$$\Delta_{n,n-1} = \{x \leq \mathbf{e} :\ x^\top \mathbf{e} = n-1\};$$

$$\Delta_{n,m} = \{\mathbf{0} \leq x \leq \mathbf{e} :\ x^\top \mathbf{e} = m\},\ \text{if}\ \ 1 < m < n.$$

Vertex adjacency criterion for $\Delta_{n,m}$

All adjacent vertices of $\Delta_{n,m}$ to an arbitrary vertex $x \in B_n(m)$ can be formed from x by a single $(0-1)$-transposition of its coordinates.

Vertex regularity degree of $\Delta_{n,m}$

For $\Delta_{n,m}$,

$$\mathcal{R} = m(n-m),$$

according to (4.78) and (4.86).

In this case, $\mathcal{R}$ takes values in the range (4.79) as the value of $\min\{m, n-m\}$ increases from 1 to $\left\lfloor \frac{n}{2} \right\rfloor$.

Remark 4.7. The lower bound $\mathcal{R} = n-1$ is reached only on $\Delta_{n,1}$ and $\Delta_{n,n-1}$, i.e. on the unit $n-1$-simplex $\Delta_{n,1}$ and on the $n-1$-simplex induced by the multiset $\{0, 1^{n-1}\}$.

Two-levelness of $B_n(m)$

$B_n(m)$ is a two-level set:

$$m(B_n(m)) = 2.$$

$B_n(m)$ is a two-level as a special case of two-level ESPCs.

Decompositions of $B_n(m)$ into parallel hyperplanes

Due to (3.33) and (4.85), there exists the decomposition (1.42)-(1.40) of the set $B_n(m)$. It can be written as $\forall\, j \in J_n$

$$\begin{aligned} &B_n(m) = E^{0j} \cup E^{1j},\ E^{ij} = \Pi^{ij} \cap B_n(m), \\ &\Pi^{ij} = \{x \in \mathbb{R}^n :\ x_j = i\},\ i = 0, 1. \end{aligned} \tag{4.93}$$

Moreover, the sets (4.27) formed in the projections of $B_n(m)$ onto the hyperplane (1.40) fall into the same class, namely: for every $j \in J_n$,

$$\begin{aligned} E'^{0j} &= B_{n-1}(\{0^{n-m-1}, 1^m\}), \\ E'^{1j} &= B_{n-1}(\{0^{n-m}, 1^{m-1}\}). \end{aligned}$$

The formula (4.93) defines simultaneously the decomposition of the set $B_n(m)$ along coordinates and toward normal vectors of $\Delta_{n,m}$-facets.

Centrally symmetric $B_n(m)$ and $\Delta_{n,m}$

Centrally symmetric $B_n(m)$ and $\Delta_{n,m}$ satisfy the condition:

$$n \text{ is even, } m = \frac{n}{2}. \tag{4.94}$$

The same EBPCs that satisfy the condition (4.94) form a subclass of EBPCs where $\mathcal{R}$ reaches the upper bound $\frac{n^2}{4}$.

4.6 The EMPC $E'_{n3}(G)$

We introduce one more subclass of EMPCs formed from ESPCs by adding a single intermediate element to the inducing multiset.

Let us consider the multiset:

$$G = \{e_1^{n_1}, e_2, e_3^{n_3}\} = \{e_1^{n_1}, e_2, e_3^{n-n_1-1}\} \tag{4.95}$$

and the EMPC induced by G.

To single out these EMPCs from the whole class $E_{nk}(G)$, we will use a special notation $E'_{n3}(G)$. At the same time, the corresponding multipermutohedron is denoted as

$$\Pi'_{n3}(G) = conv\ E'_{n3}(G)$$

Sets of class $E'_{n3}(G)$ are defined for $n \geq 3$, while, for $n = 3$, such a set is an EPPC $E'_{33}(G) = E_3(G)$ due to $S(G) = G$. In other cases, i.e. for $n > 3$, the set is an EMPC.

Let us list some properties of $E'_{n3}(G)$ and $\Pi'_{n3}(G)$.

The cardinality of $E'_{n3}(G)$

Proposition 4.3. *The cardinality of $E'_{n3}(G)$ is determined by the formula:*

$$|E'_{n3}(G)| = C_n^{n_1}(n - n_1).$$

Indeed, substituting the primary specification $[G] = (n_1, 1, n - n_1 - 1)$ of the multiset G given by (4.95) into the formula (4.2), we get

$$|E'_{n3}(G)| = \frac{n!}{n_1!(n - n_1 - 1)!} = \frac{n!(n - n_1)}{n_1!(n - n_1)!} = C_n^{n_1}(n - n_1).$$

The dimension of $\Pi'_{n3}(G)$

$$\dim\ \Pi'_{n3}(G) = n - 1.$$

Vertex regularity degree of $\Pi'_{n3}(G)$

Proposition 4.4. *The polytope $\Pi'_{n3}(G)$ is simple.*

Indeed, in accordance with (4.30), the vertex regularity degree is

$$\mathcal{R} = n_1 n_2 + n_2 n_3 = n_1 + n - n_1 - 1 = n - 1.$$

On the other hand, the dimension of $\Pi'_{n3}(G)$ is also $n - 1$. Thus, this polytope is simple.

Hyperplane locality of $E'_{n3}(G)$

The set $E'_{n3}(G)$ lies on the hyperplane:

$$\sum_{i=1}^{n} x_i = n_1 \cdot e_1 + e_2 + n_3 \cdot e_3. \tag{4.96}$$

The irredundant H-representation of $\Pi'_{n3}(G)$

As already noted, the set $E'_{n3}(G)$ is defined for $n \geq 3$. Moreover, only for $n = 3$ G has no multiple elements, and no one H-representation (4.32) and (4.33) is irredundant.

We introduce the notation ($\underline{\Pi'_{n3}(G).\text{IHR}}$) for the irredundant H-representation of the polytope $\Pi'_{n3}(G)$ and formalize it in the following corollary of Theorem 4.12.

Corollary 4.8. *($\Pi'_{n3}(G)$.IHR) has the form of* (4.96) *and*

$$e_1 \leq x_i \leq e_3,\ i \in J_n. \tag{4.97}$$

As seen from (4.97), likewise $\Pi_{n2}(G)$, the polytope $\Pi'_{n3}(G)$ is a cut of this hypercube by the hyperplane (4.96).

Three-levelness of $E'_{n3}(G)$

The set $E'_{n3}(G)$ is three-level:

$$m(E'_{n3}(G)) = 3.$$

According to Proposition 4.2, the set $E'_{n3}(G)$ is three-level along each coordinate, i.e. $m'(E'_{n3}(G)) = 3$. In this case, there exists the decomposition (1.40)-(1.42), where the family (1.39) is

$$H^{ij} = \{x \in \mathbb{R}^n : x_j = e_i\},\ i \in J_3,\ j \in J_n, \tag{4.98}$$

and the condition (4.27), defining the auxiliary sets formed in the projection $E'_{n3}(G)$ onto the hyperplane (4.98), turns into the following: for every $j \in J_n$,

$$\begin{aligned} E'^{1j} &= E_{n-1,k_1}(\{e_1^{n_1-1}, e_2, e_3^{n_3}\}), k_1 \in \{2,3\}, \\ E'^{2j} &= E_{n-1,2}(\{e_1^{n_1}, e_3^{n_3}\}), \\ E'^{3j} &= E_{n-1,k_3}(\{e_1^{n_1}, e_2, e_3^{n_3-1}\}), k_3 \in \{2,3\}. \end{aligned} \tag{4.99}$$

As you can see, the sets (4.99) belong to class $E'_{n3}(G)$ or ESPCs of the dimension one less than the original set dimension.

In addition to the fact that $E'_{n3}(G)$ is three-level along coordinates, the set $\mathbf{H}$ of facets $\Pi'_{n3}(G)$ satisfies the relation:

$$\mathbf{H} \subseteq \{\{x_i = e_i\}_{i=1,3}\}_{j \in J_n}.$$

This means all $\Pi'_{n3}(G)$-facets are parallel to the coordinate hyperplanes. Therefore, (4.98) defines the decomposition of $E'_{n3}(G)$ toward normal vectors of $\Pi'_{n3}(G)$-facets. Thus, $m(E'_{n3}(G)) = m'(E'_{n3}(G)) = 3$, and $E'_{n3}(G)$ is a three-level set.

Centrally symmetric $E'_{n3}(G)$ and $\Pi'_{n3}(G)$

Combining the conditions (4.42) and (4.44) with (4.95), we get that $E'_{n3}(G)$ and $\Pi'_{n3}(G)$ are centrally symmetric if and only if

$$e_2 = \frac{e_1 + e_3}{2},\ n_1 = n_3. \tag{4.100}$$

Thus, in order for the relation (4.100) to be true, n must be odd. This means symmetric polytopes of the class $\Pi'_{n3}(G)$ have an odd dimension.

Comparing the H-representations ($\Pi_{n2}(G)$.IHR) and ($\Pi'_{n3}(G)$.IHR), one can see that both polytopes $\Pi_{n2}(G)$ and $\Pi'_{n3}(G)$ are cuts of the hypercube by the hyperplane. Moreover, as was shown, the sets of their vertices can differ significantly. This difference in the combinatorial structure of the resulting sets, $E_{n2}(G)$ and $E'_{n3}(G)$, is due to, for the polytope $\Pi_{n2}(G)$, the cut by the hyperplane (4.3) is carried out exactly through the vertices of the hypercube. As a result, no new vertices arise, while $\Pi'_{n3}(G)$ is formed by a cut of the hypercube through the interior points of its edges. Thus, new vertices in the cut arise and form the set $E'_{n3}(G)$.

4.7 Simple multipermutohedra

In conclusion, we list all simple polytopes in class $\Pi_{nk}(G)$.

Recall that three such subclasses have already been identified. These are the $n-1$-simplices $\Delta_{n,1}$ and $\Delta_{n,n-1}$, permutohedron $\Pi_n(G)$, and multipermutohedron $\Pi'_{n3}(G)$.

To enumerate all simple multipermutohedra, one of two methods can be used:

1. Single out those polytopes $\Pi_{nk}(G)$ whose vertex degree satisfies the condition:

$$\mathcal{R} = n - 1.$$

In order to accomplish this, the following integer optimization problem can be solved:

$$\mathcal{R}(x_1, ..., x_n) = \prod_{i=1}^{n-1} x_i x_{i+1} \to \min$$
$$\sum_{i=1}^{n} x_i = n;$$
$$x_i \in \mathbb{N},\ i \in J_n;$$
$$x_i \leq x_{i+1},\ i \in J_{n-1}.$$

The number of different solutions to this problem is essentially the number of simple polytopes $\Pi_{nk}(G)$, and the nonzero coordinates of the solutions will determine the primary specifications of the inducing multisets.

2. Enumerate all polytopes in class $\Pi_{nk}(G)$ satisfying the condition that the number of γ of incident faces to each vertex of which coincides with the dimension of the polytope, i.e. $\gamma = n - 1$.

The second approach is applied in [28], and the following solution is obtained:

Theorem 4.13. *The polytope $\Pi_{nk}(G)$ is simple if and only if:*

- *if $k = 2$ then $1 \in \{n_1, ..., n_k\}$;*
- *if $k > 2$ then*

$$n_i = 1,\ i \in J_{k-1} \backslash \{1\}. \tag{4.101}$$

Among the simple polytopes listed above, the first condition is satisfied by only $\Delta_{n,1}$ and $\Delta_{n,n-1}$. The second is valid for $\Pi_n(G)$ and $\Pi'_{n3}(G)$. In addition, the condition (4.101) is also satisfied for a whole class of polytopes $\Pi'_{n3}(G)$ whose primary specifications of inducing sets contain unit elements, except for the maximum of the first and last elements.

We combine these two conditions into one and reformulate Theorem 4.13 as follows.

Theorem 4.14. *The multipermutohedron $\Pi_{nk}(G)$ is simple if and only if the primary specification of its inducing multiset satisfies the below condition:*

$$n_i \cdot n_{i+1} = \max\{n_i, n_{i+1}\},\ i \in J_{k-1}.$$

4.8 Combinatorically equivalent multipermutohedra

Theorem 4.15. *Two multipermutohedra of the same dimension are combinatorically equivalent if and only if, after ordering their elements non-decreasingly, the primary specifications of their inducing multisets coincide up to their reverse ordering:*

$$\Pi_{nk}(G) \cong \Pi_{nk}(G') \Leftrightarrow [G] = [G'] \ \ or \ \ [G] = [G''],$$

where $G'' = \left\{g'_{n-i+1}\right\}_{i \in J_n}$.

Proof. Let G satisfy the condition

$$g_i \leq g_{i+1},\ i \in J_{n-1}$$

and $G' = \{g'_i\}_{i \in J_n}$ such that

$$g'_i \leq g'_{i+1},\ i \in J_{n-1}.$$

Consider two cases:

- Case 4.8.1 when primary specifications of G, G' are the same, i.e. $[G] = [G']$;
- Case 4.8.2 - $[G]$, $[G']$ match after reverse reordering of elements, т.е. $[G] = [G'']$.

In Case 4.8.1, we establish a bijection between the vertices $\Pi_{nk}(G)$ and $\Pi_{nk}(G')$, in other words, between the points $E_{nk}(G)$ and $E_{nk}(G')$, as follows:

$$\forall\, x = (g_{i_1}, ..., g_{i_n}) \in E_{nk}(G) \to x' = (g'_{i_1}, ..., g'_{i_n}) \in E_{nk}(G'). \tag{4.102}$$

According to the vertex adjacency criterion of the multipermutohedron, for each pair of points $x \in E_{nk}(G)$ and $x' \in E_{nk}(G')$, satisfying the (4.102) condition, the same one-to-one correspondence can be established for adjacent vertices, i.e. between elements of their neighborhoods $N_{\Pi_{nk}(G)}(x)$ and $N_{\Pi_{nk}(G')}(G')(x')$:

$$\forall\, x = (g_{i_1}, ..., g_{i_n}) \in E_{nk}(G),\ y \in N_{\Pi_{nk}(G)}(x) :\ y = (g_{j_1}, ..., g_{j_n}) \leftrightarrow$$
$$x' = (g'_{i_1}, ..., g'_{i_n}) \in E_{nk}(G'),\ y' \in N_{\Pi_{nk}(G')}(x') :\ y' = (g'_{j_1}, ..., g'_{j_n}).$$

Thus, the condition (1.62) is satisfied, i.e. the graphs of the polytopes $\Pi_{nk}(G)$, $\Pi_{nk}(G')$ are isomorphic, and hence, these polytopes themselves are combinatorically equivalent.

In Case 4.8.2, a bijection is as follows:

$$\begin{aligned} &\forall\, x \in E_{nk}(G) : \\ &\qquad x = (g_{i_1}, ..., g_{i_n}) \to x' = (g'_{n-i_1+1}, ..., g'_{n-i_n+1}) \in E_{nk}(G'). \end{aligned} \tag{4.103}$$

The formula (4.103) allows moving from Case 4.8.2 to Case 4.8.1, which implies the combinatorial equivalence of $\Pi_{nk}(G)$ and $\Pi_{nk}(G')$. □

Remark 4.8. Theorem 4.15 implies that the reverse enumeration of elements of a primary specification of an inducing multiset does not change the combinatorial structure of the EMPC and the corresponding multipermutohedron.

Therefore, without loss of generality, we further assume that:

$$\exists\, i \in J_{[\frac{n+1}{2}]} :\ n_j \geq n_{k-j+1},\ j \in J_i;\ n_i > n_{k-i+1}, \tag{4.104}$$

where $n_0 = 0$.

As you can see, the number of all possible combinatorically nonequivalent permutohedra of a certain dimension will be determined by the number of different primary specifications (4.104). Determining this number is an issue required deep consideration.

4.9 Illustration of $E_{nk}(G)$ and $\Pi_{nk}(G)$ $(n = 3, 4)$

Let us classify EMPCs and multipermutohedra for the dimensions two and three. Also, we give them a geometric interpretation as projections onto the hyperplane $x_n = 0$ for $n = 3, 4$.

Example 4.2. Let $n = 3$. According to 4.8 and the assumption that $k \geq 2$, there are two possible primary specifications:

$$[G_1] = \left(1^3\right),\ [G_2] = (2, 1)\,. \tag{4.105}$$

In the first case, $k = n = 3$, i.e. we are dealing with the EPPC $E = E_3(G_1)$. The polytope $P = \Pi_3(G_1)$ is a hexagon whose projection onto $x_3 = 0$ is shown in Figure 4.1.

In the second case, $k = 2$, i.e. we consider the ESPC $E = E_{32}(G_2)$, and its polytope $P = \Pi_{32}(G_2)$ is a triangle (two-simplex) whose binary version, $B_3(1)$, $\Delta_{3,1}$, projected onto $x_3 = 0$ is shown in Figure 4.2.

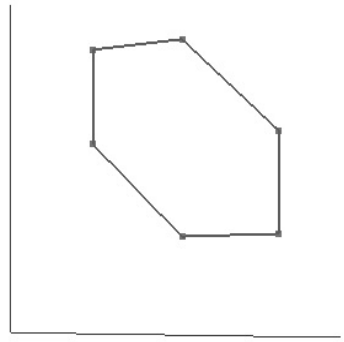

Figure 4.1: The projection of E_3 and Π_3

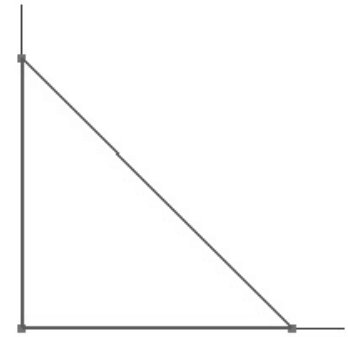

Figure 4.2: The projection of $B_3(1)$ and $\Delta_{3,1}$

Example 4.3. Let us consider the case of $n = 4$, which corresponds to three-dimensional permutohedra. The following options for primary specifications are possible here:

$$\begin{aligned} &[G_1] = \left(1^4\right), \ [G_2] = \left(2, 1^2\right), \ [G_3] = (1, 2, 1), \\ &[G_4] = \left(2^2\right), \ \ [G_5] = (3, 1). \end{aligned} \tag{4.106}$$

Let us introduce the notation: $E'^i = E_{4k_i}(G_i)$, $k_i = |S(G_i)|$, $P'^i = conv E'^i$, $i \in J_5$. Then $k_1 = 4 = n$, $k_2 = k_3 = 3$, $k_4 = k_5 = 2$, i.e. E'^1 is the EPPC, E'^4, E'^5 is the ESPC. E'^3 belongs to class $E'_{n3}(G)$, while E'^2 is just the EPPC. The sets E'^4 and E'^5 are binary. Respectively, P'^4 and P'^5 are the hypersimplex $\Delta_{3,2}$ and unit three-simplex $\Delta_{3,1}$, respectively. Their projections $E^i = \mathrm{Pr}_\alpha E'^i$ onto the hyperplane $\alpha: \ x_4 = 0$, along with the projections of the corresponding polytopes $P^i = \mathrm{Pr}_\alpha P'^i$ $(i \in J_5)$, shown in Figures 4.3-4.7.

As you can see, P^1 is a truncated dodecahedron, P^2 is a truncated tetrahedron, P^3 is a cuboctahedron, P^4 is an octahedron, P^5 is a simplex (three-simplex) (see Appendix A). Moreover, the degree of regularity of the vertices of the polytopes P^1, P^2 and P^5 is three, i.e.

it coincides with their dimension. Respectively, these three polytopes are simple. At the same time, the degree of regularity of vertices of the polytopes P^3 and P^4 is four, i.e. the polytopes are not simple.

Remark 4.9. Note that since the set of equidistant numbers is chosen as the set generator for the constructed sets of e-configurations, in the space $\mathbb{R}^4$ all combinatorically equivalent faces are regular polygons, but after projecting onto the $x_4 = 0$ hyperplane, this property is violated, so a dodecahedron is formed in this projection, but not a regular dodecahedron, a cuboctahedron that is not a regular cuboctahedron, etc. If an orthogonal projection onto the hyperplane of polytopes were constructed, exactly the corresponding Platonic and Archimedean solids would be formed.

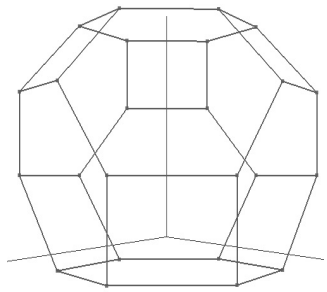

Figure 4.3: E^1 and P^1

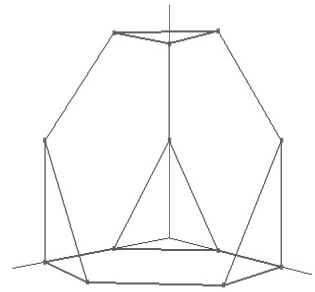

Figure 4.4: E^2 and P^2

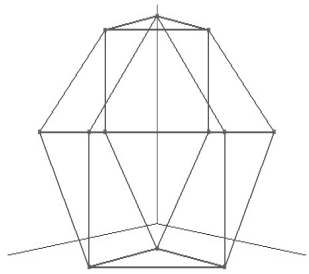

Figure 4.5: E^3 and P^3

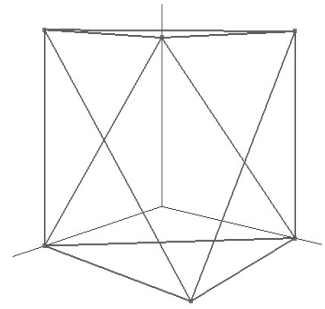

Figure 4.6: E^4 and P^4

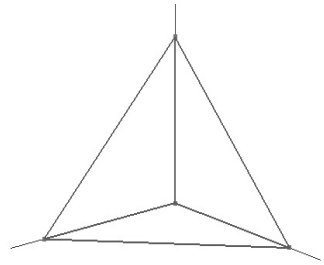

Figure 4.7: E^5 and P^5

Chapter 5

Partial multipermutation point configurations

The chapter describes the properties of the partial multipermutation point configurations and their special cases. The publications [79, 106] were used to study the entire partial permutation point configurations and their convex hulls. The EPPPC and its convex hull are explored based on [15, 28, 79, 85, 107]. The sources [5, 8, 29, 95, 111] are used as a background for studying features of the EBPPC, EUBPPC and the corresponding polytopes [22, 112].

Particular attention is paid to decomposing the EPMPCs into families of vertex-located sets, singling outing a subclass of vertex-located EPPPCs, and constructing their surface-polyhedral functional representations.

5.1 The entire partial multipermutation point configuration

When presenting the properties of the EPPPCs, we follow the scheme proposed in Chapter 4. Namely, first, we consider the EPPPC $E_{\eta k}(G)$, then its special cases: the EPPPC $E_k^n(G)$, EUPPPC $\overline{E}_k^n(G)$, and the special EPPPC $E_{\eta 2}^n(G)$.

These sets of e-configurations are studied jointly with their convex hulls.

The convex hull:

- of the EPMPC $E_{\eta k}^n(G)$ is called the <u>partial multipermutohedron</u>:

$$\Pi_{\eta k}^n(G) = conv E_{\eta k}^n(G);$$

- of the EPPPCs $E_k^n(G)$ is called the partial permutohedron:

$$\Pi_k^n(G) = conv E_k^n(G);$$

- of the EUPPPC $\overline{E}_k^n(G)$ is called the partial multipermutohedron with unbounded repetitions:

$$\overline{\Pi}_k^n(G) = conv \overline{E}_k^n(G).$$

Before proceeding to a detailed presentation of the properties of $E_{\eta k}^n(G)$ and $\Pi_{\eta k}^n(G)$, we recall that for the EPPPCs, the condition $\eta > n$ is satisfied, respectively, the cardinality of the inducing multiset G varies within the range

$$n + 1 \leq \eta \leq n \cdot k. \tag{5.1}$$

Remark 5.1. In contrast to the EMPC, for $E_{\eta k}^n(G)$, we do not require the fulfillment of the condition $n > 1$ because if condition (5.1) is satisfied, then even for $n = 1$, the set $E = E_{\eta k}^n(G)$ does not degenerate into a point. Note that the condition $\eta \geq n + 1$ is necessary for the set E to be an EMPPC, and $\eta \leq n \cdot k$ is necessary for G to be inducing set for such a set. Therefore, we assume that condition (5.1) holds.

The family $E_{\eta k}^n(G)$ covers the whole class EPPPC, among which there are vertex-located and non-vertex-located, spherically-located and ellipsoidally-located sets, sets containing polynomial and exponential on n number of elements. The diversity is also observed among the $\Pi_{\eta k}^n(G)$ polytopes. Irredundant H-representations of partial multipermutohedra can contain polynomial and exponential on n number of constraints.

Now, we explore some properties of the set $E_{\eta k}^n(G)$ and the polytope $\Pi_{\eta k}^n(G)$, given that they are induced by the multiset G of the form (1.4), (1.7).

The cardinality of $E_{\eta k}^n(G)$

Closed formulas for the cardinality of $E_{\eta k}^n(G)$ are known only for particular cases. For example, the cardinality of:

- the EPPPC $E_k^n(G)$ can be found by the formula:

$$|E_k^n(G)| = \frac{k!}{(k-n)!}; \tag{5.2}$$

- the EUPPPC $\overline{E}_k^n(G)$ is defined as follows:

$$\left|\overline{E}_k^n(G)\right| = k^n; \tag{5.3}$$

- the EPMPC $E_{n+1,k}^n(G)$ induced by the multiset G of the cardinality $n+1$:

$$\left|E_{n+1,k}^n(G)\right| = \frac{(n+1)!}{\eta_1! \cdot \ldots \cdot \eta_k!}.$$

As seen from (5.1), the EPPPCs and EUPPPCs are extreme cases in η in the class $E_{\eta k}^n(G)$. The values (5.2) and (5.3) set the upper and lower bounds on $|E_{\eta k}^n(G)|$:

$$\frac{k!}{(k-n)!} \leq \left|E_{\eta k}^n(G)\right| \leq k^n.$$

If the condition $n+1 < \eta < n \cdot k$ is satisfied, then we deal with an EPMPC. Its cardinality can be found by utilizing the possibility of its decomposition into EMPCs. In order to accomplish this, we form a set of n-element subsets of G:

$$\mathbf{G} = \left\{\mathcal{G}^i\right\}_{i \in I}, \tag{5.4}$$

where $\mathcal{G}^i$ such that

$$\begin{aligned} &G = \bigcup_{i \in I} \mathcal{G}^i, \\ &\mathcal{G}^i \subset G,\ \left|\mathcal{G}^i\right| = n,\ \kappa_i = \left|S\left(\mathcal{G}^i\right)\right|,\ i \in I, \\ &\forall i, i' \in I\ i \neq i'\ \mathcal{G}^i \neq \mathcal{G}^j. \end{aligned} \tag{5.5}$$

Lemma 5.1. *The set $E_{\eta k}^n(G)$ is decomposed into the family*

$$\left\{E_{n\kappa_i}\left(\mathcal{G}^i\right)\right\}_{i \in I} \tag{5.6}$$

of at most C_η^n EMPCs of n-dimensional multipermutation e-configurations:

$$E_{\eta k}^n(G) = \bigcup_{i \in I} E_{n\kappa_i}\left(\mathcal{G}^i\right), \tag{5.7}$$

where I and $\mathcal{G}^i$ satisfy the condition (5.5) *for all $i \in I$.*

Indeed, each of the multisets in the family (5.4) induces the EMPC $E_{nk_i}\left(\mathcal{G}^i\right)$, which is a proper subset of $E_{\eta k}^n(G)$. Together, these sets form the whole set $E_{\eta k}^n(G)$. Their number $|I|$ can be bounded from above by the number C_η^n.

Corollary 5.1. *The cardinality of $E_{\eta k}^n(G)$ can be found by the formula:*

$$\left|E_{\eta k}^n(G)\right| = n! \sum_{\mathcal{G}^i \subset \mathbf{G}} \frac{1}{n_1^i! \cdot \ldots \cdot n_{\kappa_i}^i!}, \tag{5.8}$$

where

$$\left[\mathcal{G}^i\right] = \left(n_1^i, \ldots, n_{\kappa_i}^i\right), \ i \in I.$$

Proof. According to (5.5), different multisets from the family (5.4) induce different sets of the family (5.6), i.e. if $i, i' \in I$ are such that

$$i \neq i' \ \Rightarrow E_{n\kappa_i}\left(\mathcal{G}^i\right) \cap E_{n\kappa_{i'}}\left(\mathcal{G}^{i'}\right) = \varnothing. \tag{5.9}$$

Taking into account (5.7) and (5.9), the cardinality of $E_{\eta k}^n(G)$ can be found as follows:

$$\left|E_{\eta k}^n(G)\right| = \sum_{\mathcal{G}^i \in \mathbf{G}} \left|E_{n\kappa_i}\left(\mathcal{G}^i\right)\right|. \tag{5.10}$$

Determining $|E_{n\kappa_i}\left(\mathcal{G}^i\right)|$ by the formula (4.2) and substituting the result into (5.10) we obtain the formula (5.8). □

Vertex criterion of $\Pi_{\eta k}^n(G)$

Theorem 5.1. *The point $x \in E_{\eta k}^n(G)$ is a vertex of the polytope $\Pi_{\eta k}^n(G)$ if and only if*

$$\exists\ s, r \in J_n^0,\ s + r = n, \tag{5.11}$$

such that the coordinates of the point x are formed by permutations of the numbers:

$$g_1, g_2, ..., g_s, g_{\eta-r+1}, ..., g_\eta. \tag{5.12}$$

Vertex adjacency criterion for $\Pi^n_{\eta k}(G)$

Theorem 5.2. *If $x \in vert\, \Pi^n_{\eta k}(G)$, then all vertices adjacent to it are obtained in one of two ways:*

- *a permutation of two x-coordinates equal to*

$$g_i, g_{i+1} \;\; (g_i \neq g_{i+1},\; i \in J_{s-1} \cup J_{\eta-1} \backslash J_{\eta-r}), \tag{5.13}$$

- *the replacement of a component equal to g_s by the value $g_{\eta-r} \neq g_s$ or a component $g_{\eta-r+1}$ by the value $g_{s+1} \neq g_{\eta-r+1}$,*

where s, r determined from (5.11).

Remark 5.2. For *vert* $\Pi^n_{\eta k}(G)$, one can also get a decomposition of type (5.7) by considering in the set $\mathbf{G}$ of the form (5.4) only those multisets that satisfy the conditions (5.11) and (5.12). Let $\mathbf{G}' \subseteq \mathbf{G}$ be the resulting set.

Let us construct $\mathbf{G}'$. For that, first, we introduce the following notation:

$$\begin{aligned} &G'^s = \{g_i\}_{i \in J^0_s} \cup \{g_{\eta-i+1}\}_{i \in J^0_{n-s}},\; s \in J^0_n, \\ &k'_s = \left|S\left(G'^s\right)\right|,\; [G'^s] = (n'^s_1, ..., n'^s_{k'_s}). \end{aligned} \tag{5.14}$$

Among elements of the collection

$$\overline{\mathbf{G}}' = \left\{G'^s\right\}_{s \in J^0_n},$$

there can be identical multisets, i.e. $\overline{\mathbf{G}}'$ is, generally, a multiset whose elements are the multisets (5.14), which underlying set is essentially $\mathbf{G}'$:

$$\mathbf{G}' = S(\overline{\mathbf{G}}').$$

Another way to construct $\mathbf{G}'$ is to combine the common elements of $\mathbf{G}$ and $\overline{\mathbf{G}}'$ assuming that $\mathbf{G}' = \mathbf{G} \cap \overline{\mathbf{G}}'$.

$\overline{\mathbf{G}}'$ is defined by a certain set $I' \subseteq J_n^0$ such that

$$\mathbf{G}' = \left\{G'^s\right\}_{s \in I'} \subseteq \mathbf{G}.$$

Respectively, for the multisets of the family $\mathbf{G}'$, a condition similar to (5.9) holds and looks like: if

$$i, i' \in I' \colon i \neq i' \Rightarrow G'^i \neq G'^{i'}.$$

Hence, the formulas (5.7) and (5.8) can be adapted to (5.14), yielding:

$$vert\, \Pi_{\eta k}^n(G) = \underset{i \in I'}{\cup} E_{nk_i'} \left(G'^i\right). \tag{5.15}$$

Respectively,

$$\left|vert\, \Pi_{\eta k}^n(G)\right| = n! \sum_{i \in I'} \frac{1}{n_1'^i! \cdot \ldots \cdot n_{k_i'}'^i!}. \tag{5.16}$$

Let us formulate conditions under which $\mathbf{G}' = \overline{\mathbf{G}}'$ and, consequently,

$$\overline{\mathbf{G}}' = \left\{G'^s\right\}_{s \in J_n^0}.$$

In this case, G'^0 and G'^n are sets. Correspondingly,

$$n = |S(G'^0)| = |S(G'^n)|. \tag{5.17}$$

If (5.17) holds, the formula (5.16) is simplified to

$$\left|vert\, \Pi_{\eta k}^n(G)\right| = n! \sum_{i \in J_n^0} 1 = n!\,(n+1) = (n+1)!.$$

The irredundant H-representation of $\Pi^n_{\eta k}(G)$

Theorem 5.3. *The partial multipermutohedron $\Pi^n_{\eta k}(G)$ is given by the system of inequalities:*

$$\sum_{j\in\omega} x_j \geq \sum_{j=1}^{|\omega|} g_j,\ \omega \subseteq J_n; \tag{5.18}$$

$$\sum_{j\in\omega'} x_j \leq \sum_{j=1}^{|\omega'|} g_{\eta-j+1},\ \omega' \subseteq J_n. \tag{5.19}$$

The H-representation (5.18) and (5.19) (further referred to as ($\underline{\Pi^n_{\eta k}(G).\mathrm{HR}}$)) consists of $2n$ constraint unions associated with different values of $|\omega|$ or $|\omega'|$. Like $\Pi_{nk}(G)$, the presence of multiple minimal or maximal elements of the inducing multiset G is a necessary and sufficient condition for the redundancy of ($\Pi^n_{\eta k}(G)$.HR).

Theorem 5.4. *The H-representation ($\Pi^n_{\eta k}(G)$.HR) is redundant if and only if the minimum and/or maximum element of G is a multiple, i.e.*

$$\eta_1 + \eta_k > 2. \tag{5.20}$$

Elimination:

- *from* (5.18)*, the unions with indexes*

$$i \in J = \overline{i^{\min}, \eta_1} \cup \overline{\eta - \eta_k, i^{\max}}, \tag{5.21}$$

where

$$i^{\min} = \min\{2, \eta - \eta_k\},\ i^{\max} = \max\{\eta - 2, \eta_1\},$$

- *from* (5.19)*, the unions with indexes*

$$i' \in J' = \overline{i'^{\min}, \eta_k} \cup \overline{\eta - \eta_1, i'^{\max}}, \tag{5.22}$$

where

$$i'^{\min} = \min\{2, \eta - \eta_1\},\ i'^{\max} = \max\{\eta - 2, \eta_k\},$$

turns ($\Pi^n_{\eta k}(G)$.*HR*) *into an irredundant system of constraints of the polytope* $\Pi^n_{\eta k}(G)$.

Remark 5.3. This theorem refines the theorem given in [15] that establishes an irredundant H-representation of the partial multipermutohedron. In addition to [15], Theorem 5.4 establishes the existence of redundant unions of inequalities: a) in (5.18), even for $\eta_1 = 1$; b) in (5.19), even for $\eta_k = 1$. This is possible if the following condition holds:

$$\eta < 2n, \tag{5.23}$$

i.e. the inducing multiset G is relatively small. This allows the formulation of the theorem, establishing a refined criterion of an irredundant H-representation of the polytope $\Pi^n_{\eta k}(G)$.

Corollary 5.2. *The irredundant H-representation of the polytope $\Pi^n_{\eta k}(G)$ (further ($\underline{\Pi^n_{\eta k}(G).IHR}$)) is a system of linear inequalities including*

- *inequalities of the unions* (5.18) *with numbers $i = |\omega| \in \overline{J}$, where $\overline{J} = J_n \backslash J$, and the set J is given by the formula* (5.21)*;*
- *inequalities of the unions* (5.19) *with numbers $i' = |\omega'| \in \overline{J'}$, where $\overline{J'} = J_n \backslash J'$, and J' is given by* (5.22).

<u>The criterion of belongingness of a point to $\Pi^n_{\eta k}(G)$</u>

Despite, in general, the exponential number of constraints in the irredundant H-representation ($\Pi^n_{\eta k}(G)$.IHR), the structure of the partial multipermutohedron is such that there is a simple way to check whether a given point $\mathbb{R}^n$ belongs to this polytope.

Theorem 5.5. *Let $x \in \mathbb{R}^n$ be such that*

$$x_i \leq x_{i+1}, \ i \in J_{n-1}. \tag{5.24}$$

Then from fulfillment at x: for every $i \in J_n$,

- *of the only inequality of ith union of the constraint system* (5.18)*, it follows the validity*

of all inequalities of this union, namely:

$$\sum_{j=1}^{i} x_j \geq \sum_{j=1}^{i} g_j \Rightarrow \sum_{i\in\omega} x_j \geq \sum_{j=1}^{i} g_j,\ \omega \subseteq J_n, |\omega| = i;$$

- *of one inequality $i'-$union of the system* (5.19)*, it follows the fulfillment of the remaining inequalities of this union:*

$$\sum_{j=1}^{i'} x_{n-j+1} \leq \sum_{j=1}^{i'} g_{\eta-j+1} \Rightarrow$$
$$\sum_{j\in\omega'} x_{n-j+1} \leq \sum_{j=1}^{i'} g_{\eta-j+1},\ \omega' \subseteq J_n,\ |\omega'| = i'.$$

Corollary 5.3.

Taking into account Corollary 5.3, we formulate another corollary from this theorem.

Corollary 5.4. *If $x^0 \in \mathbb{R}^n$ is such that*

$$x_i^0 \leq x_{i+1}^0,\ i \in J_{n-1}, \tag{5.25}$$

then $x^0 \in \Pi_{\eta k}^n(G)$ if and only if one of the following unions (5.18) *and* (5.19) *are satisfied at x^0:*

$$\sum_{j=1}^{i} x_j^0 \geq \sum_{j=1}^{i} g_j,\ i \in \{1\} \cup J_n \backslash J_{\eta_1}, \tag{5.26}$$

$$\sum_{j=1}^{i'} x_{n-j+1}^0 \leq \sum_{j=1}^{i'} g_{\eta-j+1},\ i' \in \{1\} \cup J_n \backslash J_{\eta_k}. \tag{5.27}$$

Remark 5.4. This corollary provides a simple way to check whether an arbitrary point $\mathbb{R}^n$ belongs to $\Pi_{\eta k}^n(G)$. It suffices to arrange its coordinates non-decreasingly and check $2n + 2 - \eta_1 - \eta_k$ inequalities of the system (5.26) and (5.27). Moreover, x^0 is an interior point of $\Pi_{\eta k}^n(G)$ if and only if all the constraints (5.26) and (5.27) are satisfied as strict inequalities.

The full-dimensionality of $\Pi^n_{\eta k}(G)$

Theorem 5.6. *The dimension of the partial multipermutohedron coincides with the dimension of Euclidean space:*

$$\dim\ \Pi^n_{\eta k}(G) = n. \tag{5.28}$$

Proof. . Let us apply Remark 1.2. Since the constraint system (5.18) and (5.19) does not contain equations, the rank of the matrix ρ in (1.57) is zero. This implies that to prove the validity of (5.28), it suffices to find an arbitrary interior point of the domain (5.18) and (5.19).

Let us introduce the notation for sums of n minimal and maximal elements of G:

$$S_1^{\min} = \sum_{i=1}^{n} g_i,\ S_1^{\max} = \sum_{i=1}^{n} g_{\eta-i+1}, \tag{5.29}$$

and also for their mean:

$$a_1^{\min} = \frac{S_1^{\min}}{n},\ a_1^{\max} = \frac{S_1^{\max}}{n}. \tag{5.30}$$

It is clear that, due to the ordering requirement (1.7) and the condition (5.1), it holds $S_1^{\min} < S_1^{\max}$. Then for the value a, the mean of (5.30):

$$a = \frac{a_1^{\min} + a_1^{\max}}{2} = \frac{S_1^{\min} + S_1^{\max}}{2n}$$

the relation holds:

$$a \in \left(a_1^{\min},\ a_1^{\max}\right). \tag{5.31}$$

Let us verify that a point corresponding to this parameter $\mathbf{a}$ is an interior point of the polytope $\Pi^n_{\eta k}(G)$. The inclusion $\mathbf{a} \in \operatorname{int} \Pi^n_{\eta k}(G)$ holds if the inequalities (5.18) and (5.19) are strictly satisfied at the point $\mathbf{a}$.

Let us use Remark 5.4 for $x^0 = \mathbf{a}$. Since the condition (5.25) is satisfied, it remains to show that the constraints (5.26) are (5.27) are fulfilled strictly at $\mathbf{a}$.

Let us apply the properties of the mean of numbers ordered non-decreasingly:

$$\forall i < i'\ \frac{\sum\limits_{j=1}^{i} g_j}{i} \le \frac{\sum\limits_{j=1}^{i'} g_j}{i'},\ \frac{\sum\limits_{j=1}^{i} g_{\eta-j+1}}{i} \ge \frac{\sum\limits_{j=1}^{i'} g_{\eta-j+1}}{i'}.$$

Considering (5.31), we substitute the coordinates of the point $\mathbf{a}$ into (5.26) and (5.27) getting:

$$\sum_{j=1}^{i} g_j \leq i\frac{S_1^{\min}}{n} < i\frac{S_1^{\min}+S_1^{\max}}{2n} = i\cdot a,\ i\in J_n,$$
$$\sum_{j=1}^{i'} g_{\eta-j}+1 \geq i'\frac{S_1^{\max}}{n} > i'\frac{S_1^{\min}+S_1^{\max}}{2n} = i'\cdot a,\ i'\in J_n.$$

Thus, $\mathbf{a}\in\ int\,\Pi_{\eta k}^n(G)$, hence, $\Pi_{\eta k}^n(G)$ is a full-dimensional polytope. □

k-levelness of $E_{\eta k}^n(G)$ along coordinates

$$m'(E_{\eta k}^n(G)) = k.$$

Similar to the EMPC, coordinates of points of $E_{\eta k}^n(G)$ take all k values of $S(G)$. Respectively, the set $E_{\eta k}^n(G)$ is k-level along coordinates.

The system of constraints (1.40)-(1.42) defines its decomposition into hyperplanes parallel to the coordinate hyperplanes. For the projections (1.43) of the sets (1.40) formed at the intersection of $E_{\eta k}^n(G)$ with each of these hyperplanes, the condition (1.44) is satisfied, i.e. these projections coincide with the projections of the entire set $E_{\eta k}^n(G)$ onto the hyperplane (1.39). Also, we have

$$E'^{ij} = E_{\eta-1,k_i}^{n-1}(G^i),\ i\in J_k,\ j\in J_n, \tag{5.32}$$

where G^i and k_i are given by the formula (4.28) ($i\in J_k$).

Thus, the set $E_{\eta k}^n(G)$ can be represented as the union of sets whose projections onto the hyperplane parallel to the coordinate ones are the EMPCs of the dimension one less than the original set. In fact, it is the decomposition of the set $E_{\eta k}^n(G)$ since all the sets (5.32) are non-empty according to Remark 5.1 and pairwise disjoint since they lie in parallel hyperplanes.

Decompositions of $E_{\eta k}^n(G)$ toward e

Let us use Lemma 5.1 and unite the sets of the family (5.6) according to the rule of equality of the sums of the coordinates to a given number b, in other words, with respect to

belongingness their points to the hyperplane $x^\top \mathbf{e} = b$.

We form the multiset:

$$B = \{b_i\}_{i \in I},\ b_i = \sum_{g \in \mathcal{G}^i} g,\ i \in I. \tag{5.33}$$

Now, the decomposition (5.7) can be rewritten as

$$E^n_{\eta k}(G) = \bigcup_{b \in S(B)} E^{n,b}_{\eta k}(G), \tag{5.34}$$

where

$$E^{n,b}_{\eta k}(G) = \{x \in E^n_{\eta k}(G) :\ x^\top \mathbf{e} = b\},\ b \in S(B). \tag{5.35}$$

The value $|S(B)|$ is, in fact, the levelness of the decomposition (5.34) of $E^n_{\eta k}(G)$ toward the vector $\mathbf{e}$. Lemma 5.1 estimates the levelness as follows: $|S(B)| \leq |I| \leq C^n_\eta$.

Also, note that the cuts (5.35) of the set $E^n_{\eta k}(G)$ by the hyperplanes $x^\top \mathbf{e} = const$ form a new class of sets of e-configurations, whose properties can be studied. In some cases, all or a part if the sets (5.35) are EPPCs. In particular, if $B = S(B)$, they are all the EMPCs, and then the formula (5.34) defines a decomposition of $E^n_{\eta k}(G)$ into the family of n-dimensional EMPCs.

In the general case, we can say that the set $E_{\eta k}(G)$ is decomposed into the family of sets lying on parallel hyperplanes:

$$\{x \in \mathbb{R}^n :\ x^\top \mathbf{e} = b\},\ b \in S(B)$$

and hence, $E^n_{\eta k}(G)$ is $m_{(\mathbf{e})}$-level toward $\mathbf{e}$, where $m_{(\mathbf{e})} = |S(B)|$.

Remark 5.5. The last two properties of $E^n_{\eta k}(G)$ allow deriving the following bounds on its levelness:

$$m(E^n_{\eta k}(G)) \geq \max\{k, |S(B)|\}, \tag{5.36}$$

where B is given by (5.33).

It is easy to see that the inequality (5.36) turns into equality if ($\Pi^n_{\eta k}(G)$.IHR) contains

nothing but the first and/or last unions in the systems (5.18) and (5.19), and at least one of the last two unions is present. If the irredundant H-representation $\Pi^n_{\eta k}(G)$ contains other unions, the levelness of the set $E^n_{\eta k}(G)$ toward normal vectors of the corresponding facets can be determined in the same way as the multiset B was found, and the underlying set was extracted from it. The cardinality of the underlying set establishes the set's levelness toward normal vectors of facets associated with one constraint union. By combining the results for all valid unions of inequalities, the levelness of the set $E^n_{\eta k}(G)$ can be found.

Centrally symmetric $E^n_{\eta k}(G)$ and $\Pi^n_{\eta k}(G)$

Theorem 5.7. *The set $E^n_{\eta k}(G)$ is centrally symmetric about the point* $\mathbf{a}$ *given by the parameter*

$$a = \frac{S_1}{\eta}, \text{ where } S_1 = \sum_{i=1}^{\eta} g_i \tag{5.37}$$

if and only if

$$g_i = g_{\eta-i+1},\ i \in J_{\left[\frac{\eta+1}{2}\right]}. \tag{5.38}$$

For the central symmetry of the polytope $\Pi^n_{\eta k}(G)$ about the point $\mathbf{a}'$ defined by the parameter:

$$a' = \frac{S_1^{\min} + S_1^{\max}}{2n}, \tag{5.39}$$

where $S_1^{\min}$ and $S_1^{\max}$ are given by the formula (5.29)*, it is necessary and sufficient that the following condition be satisfied:*

$$\frac{g_i + g_{\eta-i+1}}{2} = a',\ i \in J_n. \tag{5.40}$$

Proof. Similar to the proof of Theorem 4.10, we first focus on the symmetry conditions for $E^n_{\eta k}(G)$, and then move to $\Pi^n_{\eta k}(G)$.

1. To an arbitrary point $x \in E^n_{\eta k}(G)$, the below point y will be diametrically opposite to x:

$$\text{if } x_i = g_j, \text{ then } y_i = g_{\eta-j+1},\ i \in J_{\left[\frac{\eta+1}{2}\right]}. \tag{5.41}$$

In this case, y must be a point of $E_{nk}(G)$, i.e. the following condition is met:

$$x \in E^n_{\eta k}(G) \Leftrightarrow y \in E^n_{\eta k}(G). \tag{5.42}$$

The center of symmetry, if it exists, coincides with the midpoint of the segment $[x, y]$. Since the conditions (5.41) and (5.42) must hold for all $x \in E^n_{\eta k}(G)$, the condition (5.38) is obtained. In this case, the midpoints of all formed segments will coincide in only case if all coordinates of their intersection point are equal to each other, and this is possible only at the point **a** given by the parameter (5.37).

2. Since the shape of the partial multipermutohedron is completely determined by n first and n last elements of G, for the symmetry of $\Pi^n_{\eta k}(G)$, it suffices that the conditions (5.41) and (5.42) are satisfied for its vertices, in particular, the condition (5.42) is weakened to

$$x \in vert\, \Pi_{nk}(G) \Leftrightarrow y \in vert\, \Pi_{nk}(G). \tag{5.43}$$

The conditions (5.41) and (5.43) are satisfied for every $x \in vert\, \Pi_{nk}(G)$ if (5.40) holds. Then the center of symmetry of $\Pi^n_{\eta k}(G)$ is determined by n first and n last elements of the inducing set G, i.e. the condition (5.39) will hold. □

In Sections 4.2 and 4.4, two particular cases of the EMPCs are considered in detail, $E_n\,(G)$ and $E_{n2}\,(G)$. They are extreme with respect to k. Similarly, in addition to the sets $E^n_k(G)$ and $\overline{E}^n_k(G)$, in class $E^n_{\eta k}(G)$, the following special cases are explored in more detail:

- Case 5.1.1 η is the least possible;
- Case 5.1.2 k is the least possible.

From (5.1), it follows that Case 5.1.1 corresponds to

$$\eta = n + 1, \tag{5.44}$$

i.e. to considering the EMPPC $E^n_{n+1,k}(G)$. In Case 5.1.2, the condition (3.27) is satisfied, and

the EMPPC $E^n_{\eta 2}(G)$ induced by a generating set of two numbers is under consideration. The intersection of these two classes forms class $E^n_{n+1,2}(G)$, which is considered separately.

We explore the mentioned subclasses of the EMPPCs sequentially.

5.2 The entire partial permutation point configuration

Like the EPPC $E_n(G)$, some properties of the EPPPC $E^n_k(G)$ are a direct consequence of the properties of the EMPPC listed in Sec. 5.1, as well as the result of the change of variables (4.47) and the substitution $\eta = k$.

The remaining properties are specific to the EPPPCs and the corresponding polytopes.

Combinatorial equivalence of $\Pi^n_k(G)$ and $\Pi^n_{n+1}(G')$ и $\Pi_{n+1}(G'')$

For arbitrary sets G, G', G'' such that $|G| = k$, $|G'| = |G''| = n+1$ the relation is true:

$$\Pi^n_k(G) \cong \Pi^n_{n+1}(G') \cong \Pi_{n+1}(G'') \ \forall\, k > n. \tag{5.45}$$

The formula (5.45) says that all partial permutohedra of the same dimension are combinatorically equivalent and establishes a connection between a n-partial permutohedron and a $n+1$-permutohedron. The permutohedron was discussed in Section 4.2, and the properties derived there are easily generalized onto the polytope $\Pi^n_k(G)$. They are listed below.

The cardinality of $E^n_k(G)$

The value $|E^n_k(G)|$ is given by the formula (5.2).

Vertex criterion of $\Pi^n_k(G)$

A corollary from Theorem 5.1 is that a point $x \in E^n_k(G)$ is a vertex of the polytope $\Pi^n_k(G)$ if and only if there exist the numbers s, r satisfying the condition (5.11) such that the

coordinates of x form a permutation of the numbers:

$$e_1, e_2, ..., e_s, e_{k-r+1}, ..., e_k.$$

The number of vertices of $\Pi_k^n(G)$

$$|vert\ E_k^n(G)| = (n+1)!.$$

This formula follows (4.49) and (5.45).

Vertex adjacency criterion for $\Pi_k^n(G)$

From Theorem 5.2, it follows that, for an arbitrary point $x \in vert\ \Pi_k^n(G)$, all vertices of $\Pi_k^n(G)$ adjacent to it are obtained in one of two ways:

1. by an $e_i \leftrightarrow e_{i+1}$-permutation of a pair of x-components;
2. by the replacement $e_s \to e_{k-r}$ or $e_{k-r+1} \to e_s$ of a x-coordinate, where r, s are given by (5.11).

The irredundant H-representation of $\Pi_k^n(G)$

The irredundant H-representation of a partial permutohedron (further referred to as ($\Pi_k^n(G)$.IHR)) is obtained as a consequence of Theorem 5.3, which takes into account that the condition (5.20) violates in this case. Respectively, Respectively, no redundant unions exist in ($\Pi_k^n(G)$.IHR).

Corollary 5.5. *The irredundant H-representation ($\Pi_k^n(G)$.IHR) has the form of*

$$\sum_{j\in\omega} x_j \geq \sum_{j=1}^{|\omega|} e_j,\ \omega \subseteq J_n;$$
$$\sum_{j\in\omega'} x_j \leq \sum_{j=1}^{|\omega'|} e_{k-j+1},\ \omega' \subseteq J_n.$$

The full-dimensionality of $\Pi_k^n(G)$

$$\dim\ \Pi_k^n(G) = n. \tag{5.46}$$

The simplicity of $\Pi_k^n(G)$

Indeed, the degree of regularity of $\Pi_k^n(G)$-vertices satisfies the condition

$$\mathcal{R} = n. \tag{5.47}$$

This formula follows from (4.52) and (5.45). Together, the conditions (5.46) and (5.47) yield $\dim\ \Pi_k^n(G) = \mathcal{R}$, which is the simplicity condition for $\Pi_k^n(G)$.

k-levelness of $E_k^n(G)$ along coordinates

$$m(E_k^n(G)) = k.$$

In addition, since G is a set, for any $i \in J_k$, its subsets G^i of the form (4.28) are sets of cardinality $|G^i| = k-1$. As a result, the formula (5.32) for the projections of $E_k^n(G)$ onto its decomposition hyperplanes, parallel to the coordinate hyperplanes, looks like

$$E^{'ij} = E_{k-1}^{n-1}(G^i),\ i \in J_k,\ j \in J_n.$$

Centrally symmetric $E_k^n(G)$ and $\Pi_k^n(G)$

The cases of centrally symmetric EPPPCs and partial permutohedra are listed in the corollary from Theorem 5.7.

Corollary 5.6. *The set $E_k^n(G)$ is centrally symmetric about the point* **a** *of the form:*

$$a = \frac{S_1}{k},\ \textit{where}\ S_1 = \sum_{i=1}^{k} e_i,$$

if and only if

$$e_i = e_{k-i+1},\ i \in J_{\left[\frac{k+1}{2}\right]}.$$

A necessary and sufficient condition for the central symmetry of the polytope $\Pi_k^n(G)$ *about the point* $\mathbf{a}'$ *given by the parameter*

$$a' = \frac{S_1^{\min} + S_1^{\max}}{2n},$$

where $S_1^{\min}$, $S_1^{\max}$ *given by expressions:*

$$S_1^{\min} = \sum_{i=1}^{n} e_i,\ \ S_1^{\max} = \sum_{i=1}^{n} e_{\eta-i+1},$$

is the following:

$$\frac{e_i + e_{k-i+1}}{2} = a',\ i \in J_{\left[\frac{k+1}{2}\right]}.$$

5.3 The entire unbounded partial permutation point configuration

The EUPPPC $\overline{E}_k^n(G)$ and its convex $\overline{\Pi}_k^n(G)$ are relatively well studied. For example, $\overline{E}_k^n(G)$ is an n-dimensional bounded lattice $\mathcal{A}^n$, and $\overline{\Pi}_k^n(G)$ is the hypercube.

Let us list some of their properties.

$\overline{E}_k^n(G)$ is a discrete lattice

$$\overline{E}_k^n(G) = \mathcal{A}^n, \tag{5.48}$$

i.e. it is the nth degree of its generating set.

The cardinality of $\overline{E}_k^n(G)$

The value $|\overline{E}_k^n(G)|$ is given by the formula (5.3) following from the formula (5.48), the properties of the Cartesian product of sets and the relation $k = |\mathcal{A}|$. Indeed,

$$|\overline{E}_k^n(G)| = |\mathcal{A}^n| = |\mathcal{A}|^n = k^n. \tag{5.49}$$

The irredundant H-representation of $\overline{\Pi}_k^n(G)$

Moving in the equation (5.48) to the convex hull of its left-hand and right-hand parts, we obtain

$$conv\, \overline{E}_k^n(G) = \overline{\Pi}_k^n(G) = (conv\, \mathcal{A})^n = [e_1, e_k]^n, \tag{5.50}$$

wherefrom it can be seen that the constraints, also known as the irredundant H-representation of $\overline{\Pi}_k^n(G)$ (further referred to as ($\overline{\Pi}_k^n(G)$.IHR)), has the form:

$$e_1 \leq x_i \leq e_k,\ i \in J_n, \tag{5.51}$$

It defines a hypercube with side $e_k - e_1$. The same result can be easily obtained from Corollary 5.2.

The criterion of belongingness of a point to $\overline{\Pi}_k^n(G)$

Proposition 5.1. *A point $x \in \overline{E}_k^n(G)$ is a vertex of $\overline{\Pi}_k^n(G)$ if and only if its coordinates are equal to e_1 or e_k:*

$$x \in vert\, \overline{\Pi}_k^n(G) \ \Leftrightarrow\ x_i \in \{e_1, e_k\},\ i \in J_n. \tag{5.52}$$

The number of vertices of $\overline{\Pi}_k^n(G)$

$$|vert\, \overline{\Pi}_k^n(G)| = 2^n,$$

which directly follows from (5.52).

Vertex adjacency criterion for $\overline{\Pi}_k^n(G)$

All vertices adjacent to an arbitrary point $x \in \overline{\Pi}_k^n(G)$ are formed from it by a single replacement $e_1 \to e_k$ or $e_k \to e_1$ of one of its coordinates.

This condition can be represented in terms of the Hamming distance:

$$\forall x, y \in vert\, \overline{\Pi}_k^n(G) \quad x \leftrightarrow y \Leftrightarrow Hd(x, y) = 1. \tag{5.53}$$

Vertex regularity degree of $\overline{\Pi}_k^n(G)$

$$\mathcal{R} = n.$$

The full-dimensionality of $\overline{\Pi}_k^n(G)$

$$\dim\, \overline{\Pi}_k^n(G) = n.$$

The simplicity of $\overline{\Pi}_k^n(G)$

$\overline{\Pi}_k^n(G)$ is a simple polytope.

k-levelness of $\overline{E}_k^n(G)$

$$m(\overline{E}_k^n(G)) = k.$$

Indeed, likewise, in all sets of the class $E_{\eta k}^n(G)$, the set $\overline{E}_k^n(G)$ is k-level along coordinates, i.e. $m(\overline{E}_k^n(G)) = k$. And since all facets of the hypercube $\overline{\Pi}_k^n(G)$ are parallel to the coordinate hyperplanes, the number k will also specify the levelness toward normal vectors of its facets. Overall, $\overline{E}_k^n(G)$ is a k-level set.

Unlike the set $E_k^n(G)$, G is a multiset with the maximum possible multiplicities of elements, so fixing one coordinate does not affect the remaining ones. Respectively, the multisets G^i of the form (4.28) will also be multisets with unbounded multiplicities. Thus,

$G' = G^i = (\mathcal{A})^{n-1}$, $|S(G^i)| = k$ $(i \in J_k)$. In this case, the formula (5.32) for sets formed in the projections of $\overline{E}^n_k(G)$ onto its decomposition hyperplanes, parallel to the coordinate hyperplanes, has the form of

$$E'^{ij} = \overline{E}^{n-1}_k(G'),\ i \in J_k,\ j \in J_n,$$

where G' is formed from G by decreasing the multiplicity of each element by one.

Centrally symmetric $\overline{E}^n_k(G)$

Theorem 5.7 implies that the polytope $\overline{\Pi}^n_k(G)$ is centrally symmetric, which is true since the hypercube is symmetric. Namely, it has a center and hyperplanes of symmetry.

The requirement for the central symmetry of $\overline{E}^n_k(G)$ is the symmetry of the elements of the generating set $\mathcal{A}$ about its mean since, due to $[G] = (k^n)$, the conditions (5.37) and (5.38) in this case can be rewritten in the form:

$$a = \tfrac{S_1}{k},\ \text{where } S_1 = \sum_{i=1}^{k} e_i,$$
$$e_i = e_{k-i+1},\ i \in J_{\left[\frac{k+1}{2}\right]}.$$

5.4 The EPMPC $E^n_{n+1,k}(G)$

Let us consider particular cases of the set $E^n_{\eta k}(G)$ introduced above as Cases 5.1.1 and 5.1.2.

We start with the first of them and outline the properties of the EPMPC $E^n_{n+1,k}(G)$ and the polytope $\Pi^n_{n+1,k}(G)$.

$E^n_{n+1,k}(G)$ is a projection of the entire multipermutation point configuration of $n+1$-elements

It is easy to see that $E^n_{n+1,k}(G)$ is formed from the EMPC $E_{n+1,k}(G)$ by eliminating one coordinate of its points, i.e. is the projection of the latter onto Euclidean space of the dimension lower by one. Without loss of generality, we assume that the last $n+1$th coordinate

is fixed, i.e.

$$E^n_{n+1,k}(G) = Pr_\alpha E_{n+1,k}(G), \tag{5.54}$$

where

$$\alpha = \{x \in \mathbb{R}^{n+1} : x_{n+1} = 0\}. \tag{5.55}$$

When projected onto the hyperplane (5.55), generally, the spherically-located set $E_{n+1,k}(G)$ ceases to be spherically-located since this projection is not orthogonal.

It is easy to see that the polytope $\Pi^n_{n+1,k}(G)$ is the projection of $\Pi_{n+1,k}(G)$ onto the hyperplane α:

$$\Pi^n_{n+1,k}(G) = Pr_\alpha \Pi_{n+1,k}(G).$$

<u>The ellipsoidal locality of $E^n_{n+1,k}(G)$</u>

The following theorem states that $E^n_{n+1,k}(G)$, like $E_{n+1,k}(G)$, is a surface-located set, and it is established which strictly convex surface can be circumscribed about it.

Theorem 5.8. *The set $E^n_{n+1,k}(G)$ is inscribed in the ellipsoid* (1.25) *with the parameters:*

$$A = E + I = [a_{ij}]_{n\times n} : a_{ij} = \begin{cases} 2, \text{ if } i = j, \\ 1, \text{ if } i \neq j; \end{cases} \tag{5.56}$$

$$b = -2S_1\mathbf{e},\ c = S_1^2 - S_2,$$

where I is a matrix of ones, E is an n-order identity matrix,

$$S_j = \sum_{i=1}^{n+1} g_i^j,\ j = 1, 2. \tag{5.57}$$

Proof. We start by considering the EMPC $E_{n+1,k}(G)$, which is spherically-located according to Theorem 4.1. To determine the parameters of its circumsphere, we apply the formulas (4.5), (4.6), and (4.8), getting:

$$\sum_{i=1}^{n+1} x_i = S_1, \tag{5.58}$$

$$\sum_{i=1}^{n+1} (x_i - a)^2 = S_2 - 2aS_1 + (n+1)\,a^2, \tag{5.59}$$

where S_1, S_2 are given by expressions (5.57).

Let us move to the projection onto the hyperplane α, excluding the variable x_{n+1}. To do this, we separate it in the equation (5.58):

$$x_{n+1} = S_1 - \sum_{i=1}^{n} x_i.$$

Let us substitute this expression into (5.59):

$$\begin{aligned} S_2 - 2aS_1 + (n+1)\,a^2 &= \sum_{i=1}^{n} (x_i - a)^2 + (x_{n+1} - a)^2 = \\ &= \sum_{i=1}^{n} (x_i - a)^2 + \left(-\sum_{i=1}^{n} x_i + S_1 - a\right)^2 = \\ &= \sum_{i=1}^{n} (x_i - a)^2 + \left(\sum_{i=1}^{n} x_i\right)^2 + 2\,(a - S_1)\sum_{i=1}^{n} x_i + (S_1 - a)^2. \end{aligned} \tag{5.60}$$

It is easy to see that the equation (5.60) defines an ellipsoid. It is circumscribed about $E^n_{n+1,k}\,(G)$ since, by definition, $E_{n+1,k}\,(G)$-points satisfy the equation (5.59), and by construction, the points of $E^n_{n+1,k}\,(G)$ satisfy the equation (5.60), i.e. they are inscribed in the ellipsoid.

Simplifying (5.60) we get

$$\begin{aligned} S_2 - 2aS_1 + (n+1)\,a^2 &= \sum_{i=1}^{n} x_i^2 - 2a\sum_{i=1}^{n} x_i + na^2 + \\ &+ x^T I x + 2\,(a - S_1)\sum_{i=1}^{n} x_i + S_1^2 - 2aS_1 + a^2. \end{aligned}$$

Wherefrom

$$x^T E x + x^T I x - 2S_1 \sum_{i=1}^{n} x_i + S_1^2 - S_2 = 0. \tag{5.61}$$

As a result, the equation (1.25) of the ellipsoid is obtained, where A, b, c are given by the formula (5.56). □

As can be seen, the parameter a is cancelled throughout the transformations, i.e. in the projection of a family of hyperspheres circumscribed about $E^n_{n+1,k}\,(G)$, the unique circumscribed ellipsoid is obtained.

Vertex locality of $E^n_{n+1,k}(G)$

$$E^n_{n+1,k}(G) = vert\ \Pi^n_{n+1,k}(G).$$

Indeed, according to Theorem 5.8, $E^n_{n+1,k}(G)$ is a surface-located set. Hence, it is vertex-located according to Theorem 2.1.

Polyhedral-ellipsoidality of $E^n_{n+1,k}(G)$

$$E^n_{n+1,k}(G) = \Pi^n_{n+1,k}(G) \cap S,$$

where S is the ellipsoid given by the equation (1.25) with the parameters (5.56).

Combinatorial equivalence of $\Pi^n_{n+1,k}(G)$ and $\Pi_{n+1,k}(G)$

Theorem 5.9.

$$\Pi^n_{n+1,k}(G) \cong \Pi_{n+1,k}(G). \tag{5.62}$$

Proof. If G is a set, the formula (5.62) becomes $\Pi^n_{n+1}(G) \cong \Pi_{n+1}(G)$, and it follows from (5.45). Let us prove that the combinatorial equivalence is also preserved when G is a multiset. Note that when projecting onto Euclidean space of lower dimension, the combinatorial structure of a polytope changes for two reasons. First, some vertices become interior points of a polytope (or its faces of arbitrary dimension) formed as a result of projection. Consequently, the vertex set of the original polytope ceases to be a vertex set of the projection polytope. The second reason is that some vertices are projected onto the same point. As a result, vertices that were not adjacent in the original polytope can be adjacent in the projection polytope. Let us show that, in the case under consideration, neither one nor the other occurs, and therefore, the combinatorial structure of the polytope remains unchanged. The first situation does not arise since, as was shown, $E^n_{n+1,k}(G)$ is vertex-located, similarly to the original set $E_{n+1,k}(G)$. As for the second reason, it is easy to see that the set $E^n_{n+1,k}(G)$ (in other words, the vertex set of $\Pi^n_{n+1,k}(G)$) can be formed in two stages: on the first step, sequentially extract one element from G, forming submultisets (4.28); on the second stage, form all multipermutations from

the obtained n-element multiset. Let, for example, the generated element of $E^n_{n+1,k}(G)$ be x and the inducing multiset be G^i. Complementing x with the $n+1$th coordinate equal to e_i, we get the point $y = (x, e_i) \in E^n_{n+1,k}(G)$. Thus, from the element $E_{n+1,k}(G)$, one can easily determine the element $E^n_{n+1,k}(G)$ as its projection onto the hyperplane (5.55) and vice versa. Thus, there is a one-to-one correspondence between points of $E^n_{n+1,k}(G)$ and $E_{n+1,k}(G)$. This means "gluing" vertices does not occur.

So, when (5.55) is projected, the graph of the polytope does not change. Therefore, the original and projected polytopes are combinatorically equivalent. □

This feature allows generalizing many of the above properties of the EMPC onto the class $E^n_{n+1,k}(G)$ after the following replacements:

$$n \to n+1,\ n_i \to \eta_i,\ i \in J_k.$$

The same applies the polytope $\Pi^n_{n+1,k}(G)$ properties.

<u>Properties of $E^n_{n+1,k}(G)$ and $\Pi^n_{n+1,k}(G)$</u> following from Theorem 5.9

Based on (5.62), many properties of $E_{n+1,k}(G)$ and $\Pi_{n+1,k}(G)$ can be generalized onto $E^n_{n+1,k}(G)$ and $\Pi^n_{n+1,k}(G)$. For example,

- (4.2) is transformed into the formula for the $E^n_{n+1,k}(G)$-cardinality:

$$\left|E^n_{n+1,k}(G)\right| = \left|E_{n+1,k}(G)\right| = \frac{(n+1)!}{\eta_1! \cdot ... \cdot \eta_k!}; \tag{5.63}$$

- (4.30) turns into the formula:

$$\mathcal{R} = \eta_1\eta_2 + \eta_2\eta_3 + ... + \eta_{k-1}\eta_k \tag{5.64}$$

  determining the degrees of vertex regularity of $\Pi^n_{n+1,k}(G)$;

- the formula

$$\dim\ \Pi^n_{n+1,k}(G) = n$$

follows from (4.40) and testifies to the full-dimensionality of $\Pi^n_{n+1,k}(G)$.

- $E^n_{n+1,k}(G)$ is k-level along coordinates and toward the vector $\mathbf{e}$:

$$m'(E^n_{n+1,k}) = m_{(\mathbf{e})} = k.$$

Indeed, according to Proposition 4.2, the set $E_{n+1,k}(G)$ is k-level along its $n+1$-th coordinate. In particular, if it is decomposed along coordinates (see (1.40)-(1.42)) and then is projected onto the corresponding hyperplanes, the EMPCs (4.27) of the form

$$E'^{ij} = E_{nk_i}(G\backslash\{e_i\}),$$
$$k_i = |S(G\backslash\{e_i\})|,\ i \in J_k,\ j \in J_{n+1}$$

are obtained.

Centrally symmetric $E^n_{n+1,k}(G)$ and $\Pi^n_{n+1,k}(G)$

Let us use the formula (5.55) along with the projection property to transform centrally symmetric sets into centrally symmetric ones and formulate a corollary from Theorem 4.10.

Corollary 5.7. *The set $E^n_{n+1,k}(G)$ and the polytope $\Pi^n_{n+1,k}(G)$ are centrally symmetric if and only if the condition holds:*

$$\frac{g_i + g_{n-i+2}}{2} = \frac{S_1}{n+1},\ i \in J_{[\frac{n}{2}]+1},$$

where S_1 given by the expression (5.57).

Simple polytopes among $\Pi^n_{n+1,k}(G)$

Taking into account the relation (5.62), we can reformulate Theorem 4.14 and formulate the simplicity conditions for this class.

Theorem 5.10. *The partial multipermutohedron $\Pi^n_{n+1,k}(G)$ is simple if and only if the primary*

specification of the inducing multiset G satisfies the condition:

$$\eta_i \cdot \eta_{i+1} = \max\{\eta_i, \eta_{i+1}\},\ i \in J_{k-1}.$$

5.5 The entire special partial multipermutation point configuration

Let us consider Case 5.1.2 if $k = 2$, i.e. the generating set consists of two elements.

By analogy with the entire special multipermutation point configuration, the set $E^n_{\eta 2}(G)$ is called the entire special partial multipermutation point configuration (ESPPC), and its convex hull $\Pi^n_{\eta 2}(G)$ is called the special partial permutohedron.

The set $E^n_{\eta 2}(G)$ is induced by the multiset (3.28) such that

$$n < \eta = \eta_1 + \eta_2 \leq 2n \tag{5.65}$$

and according to the terminology introduced in Section 3.5, its elements are called special permutation e-configurations.

As shown below, $E^n_{\eta 2}(G)$ is another class of vertex-located sets in the class EPMPC. Moreover, the sets $E^n_{n+1,k}(G)$ and $E^n_{\eta 2}(G)$, i.e. Cases 5.1.1 and 5.1.2, vertex-located EPMPCs are exhausted.

Let us present some properties of $E^n_{\eta 2}(G)$ and $\Pi^n_{\eta 2}(G)$, which follow both from their belonging to classes of the EPMPC and partial multipermutohedra (see Section 5.1) and from the condition (3.27).

The full-dimensionality of $\Pi^n_{\eta 2}(G)$

$$\dim\ \Pi^n_{\eta 2}(G) = n.$$

The spherical locality of $E^n_{\eta 2}(G)$

Proposition 5.2. *The set $E^n_{\eta 2}(G)$ is spherically-located:*

$$E^n_{\eta 2}(G) \subset S_r(\mathbf{a}),$$

and its circumscribed hypersphere $S_r(\mathbf{a})$ is uniquely defined by the expressions (4.74) *and* (4.75).

Proof. Let us consider a point $x \in E^n_{\eta 2}(G)$. For x, there exists $I_x \subset J^0_n :\ x_i = \begin{cases} e_1, & \text{if } i \in I_x; \\ e_2, & \text{if } i \notin I_x. \end{cases}$

Let us square the distance from x to the point $\mathbf{a}^{\min}$ given by the parameter (4.75):

$$\left(x - \mathbf{a}^{\min}\right)^2 = \sum_{i \in I_x} \left(x_i - a^{\min}\right)^2 + \sum_{i \notin I_x} \left(x_i - a^{\min}\right)^2 =$$
$$\sum_{i \in I_x} \left(e_1 - \frac{e_1 + e_2}{2}\right)^2 + \sum_{i \notin I_x} \left(e_2 - \frac{e_1 + e_2}{2}\right)^2.$$

In the notations (4.70), this expression is rewritten as follows:

$$\left(x - \mathbf{a}^{\min}\right)^2 = \sum_{i=1}^{n} \left(\frac{\Delta}{2}\right)^2 = \frac{\Delta^2 n}{4} = (r^{\min})^2. \tag{5.66}$$

This means the hypersphere equation is obtained with the parameters (4.74) and (4.75). Te equation is satisfied by an arbitrary point $E^n_{\eta 2}(G)$. This is the only hypersphere circumscribed about $E^n_{\eta 2}(G)$ and $\Pi^n_{\eta 2}(G)$ since the latter is a full-dimensional polytope. The equation (5.66) defines the hypersphere $S^{\min}$ for $E^n_{\eta 2}(G)$. □

Vertex locality of $E^n_{\eta 2}(G)$

$$E^n_{\eta 2}(G) = vert\ \Pi^n_{\eta 2}(G). \tag{5.67}$$

Indeed, according to Proposition 5.2, $E^n_{\eta 2}(G)$ is a surface-located set. Therefore, by Theorem 2.1, the set is vertex-located.

Two-levelness of $E^n_{\eta 2}(G)$ along coordinates

$$m'(E^n_{\eta 2}(G)) = 2.$$

By the formulas (3.28), for $E^n_{\eta 2}(G)$, the same decomposition (1.42), (1.40), (4.80), as for $E_{n2}(G)$ exists and decomposes the set into pairs of hyperplanes parallel to the coordinate ones. In this case, in the projection onto the hyperplane (4.80), the ESPPC or singleton sets are formed since the formula (5.32) holds and transforms into

$$E'^{ij} = E^{n-1}_{\eta-1,k_i}(G^i),\ i \in J_k,\ j \in J_n, \tag{5.68}$$

where G^i given by the formula (4.28), $k_i \in \{1,2\},\ i \in J_k$.

Decomposition of $E^n_{\eta 2}(G)$ into ESPCs

Lemma 5.2. *$E^n_{\eta 2}(G)$ decomposes into the family of $\eta - n + 1$ ESPCs:*

$$\begin{aligned} &E^n_{\eta 2}(G) = \bigcup_{i=n-\eta_1}^{\eta_2} E_{nk'_i}\left(G'^i\right), \\ &\textit{where} \qquad G'^i = \left\{e_1^{n-i}, e_2^i\right\},\ i \in\ J_{\eta_2} \backslash J_{n-\eta_1-1}, \\ &k'_i = |S\left(G'^i\right)| = \begin{cases} 1, & \textit{if } i \in \{0,n\}, \\ 2, & i \in J_{n-1}. \end{cases} \end{aligned} \tag{5.69}$$

Proof. Since the formula (5.67) holds, we can use Remark 5.2. In this case, by (3.28) and (5.65), the formulas (5.14) and (5.15) become $n' = \eta - n$,

$$\begin{aligned} &G'^s = \left\{e_1^{n-\eta_2+s}, e_2^{\eta_2-s}\right\},\ s \in J^0_{\eta-n}, \\ &vert\, \Pi^n_{\eta 2}(G) = E^n_{\eta 2}(G) = \bigcup_{i=0}^{\eta-n} E_{nk'_i}\left(G'^i\right), \\ &k'_i = \left|S\left(G'^i\right)\right| \in \{1,2\},\ s \in J^0_{\eta-n}. \end{aligned} \tag{5.70}$$

Now, applying the substitution $s = \eta - i$ in (5.70), we come to the expressions (5.69). □

$\eta - n + 1$-levelness of $E^n_{\eta 2}(G)$ toward $\mathbf{e}$

$$m_{\mathbf{e}}(E^n_{\eta 2}(G)) = \eta - n + 1.$$

We utilize the decomposition (5.34) and (5.35) of the EPPPC toward $\mathbf{e}$.

In this case, the formula (5.33) becomes

$$B = \{b_i\}_{i \in J^0_{\eta - n}},$$
$$b_i = \sum_{j=1}^{n} g'^{i}_{j} = e_1 \cdot (n - \eta_2 + i) + e_2 \cdot (\eta_2 - i),\ i \in J^0_{\eta - n}.$$

Here, $b_i \neq b_j$ if $i \neq j$, i.e. $E^n_{\eta 2}(G)$ decomposes exactly into $\eta - n + 1$ parallel hyperplanes toward $\mathbf{e}$. Respectively, $E^n_{\eta 2}(G)$ is $\eta - n + 1$-level toward $\mathbf{e}$.

In this case, the decomposition (5.34) and (5.35) is

$$E^n_{\eta 2}(G) = \bigcup_{b \in B} E^{n,b}_{\eta 2}(G), \tag{5.71}$$

where

$$E^{n,b}_{\eta 2}(G) = \{x \in E^n_{\eta 2}(G) :\ x^\top \mathbf{e} = b\},\ b \in B. \tag{5.72}$$

connected with the decomposition (5.70) since

$$\forall\, b \in B\ \exists!\, i \in J^0_{\eta - n} :\ E^{n,b}_{\eta 2}(G) = E_{nk'_i}(G'^i).$$

Remark 5.6. These $\eta - n + 1$ levels of $E^n_{\eta 2}(G)$ can be divided into three groups:

- a lower-level E^l corresponding to $b = b^l = \min_i b_i$, and $E^l = \emptyset$ if $b^l = n \cdot e_1$;
- an upper-level E^u corresponding to $b = b^u = \max_i b_i$, and $E^u = \emptyset$ if $b^u = n \cdot e_2$;
- an intermediate-level E^m:

$$E^m = E^n_{\eta 2}(G) \setminus \left\{E^l, E^u\right\}. \tag{5.73}$$

As one can see, each of the sets E^l, E^u, and E^m can be empty. Also, given the relationship between (5.70) and (5.71), and the relation $b_i < b_{i+1}$ $(i \in J^0_{\eta-n-1})$, we have

$$E^l = \begin{cases} E_{n2}(\{e_1^{\eta_1}, e_2^{n-\eta_1}\}), \text{ if } \eta_1 < n; \\ \emptyset, \text{ if } \eta_1 = n; \end{cases} \tag{5.74}$$

$$E^u = \begin{cases} E_{n2}(\{e_1^{n-\eta_1}, e_2^{\eta_2}\}), \text{ if } \eta_2 < n; \\ \emptyset, \text{ if } \eta_2 = n; \end{cases} \tag{5.75}$$

$$E^m = \begin{cases} \overline{E}_{n2}(G), \text{ if } \eta_1 = \eta_2 = n; \\ E_{n-1,k^l}(G\backslash\{e_1\}), \text{ if } \eta_1 < n,\ \eta_2 = n\ (k^l = |S(G\backslash\{e_1\})|); \\ E_{n-1,k^u}(G\backslash\{e_2\}), \text{ if } \eta_1 = n,\ \eta_2 < n\ (k^u = |S(G\backslash\{e_2\})|); \\ E_{n-2,k^{lu}}(G\backslash\{e_1, e_2\}), \text{ if } \eta_1, \eta_2 < n\ (k^{lu} = |S(G\backslash\{e_1, e_2\})|). \end{cases}$$

Moreover, $k^l, k^u, k^{lu} \in \{1, 2\}$, i.e. E^m is the ESPPC, namely, the EUSPPC or degenerates into a singleton. The lower- and upper-levels of the set $E^n_{\eta 2}$ are singled out because they simultaneously possess the properties of the EUSPPC $\overline{E}^n_2(G)$ and ESPC, while intermediate-level points inherit only the properties of $\overline{E}^n_2(G)$.

<u>The cardinality of $E^n_{\eta 2}(G)$</u>

$$\left|E^n_{\eta 2}(G)\right| = \sum_{i=n-\eta_1}^{\eta_2} C_n^i. \tag{5.76}$$

To derive this formula, in the formula (5.69), it suffices to move to the cardinality of the set components:

$$\begin{gathered} \left|E^n_{\eta 2}(G)\right| = \sum_{i=n-\eta_1}^{\eta_2} \left|E_{nk_i}\left(G'^i\right)\right|, \\ k_i \in \{1, 2\}\,,\ i \in J_{\eta_2}\backslash J_{n-\eta_1-1}. \end{gathered} \tag{5.77}$$

Applying now the formula (4.64) to (5.77), which is also valid for the degenerate case $k_i = 1$, we come to the formula (5.76).

The irredundant H-representation of $\Pi^n_{\eta 2}(G)$

Let us formulate the corollary from Theorem 5.4 for the case $k = 2$, noticing that the condition (5.20) of the reducibility of the H-representation ($\Pi^n_{\eta 2}(G)$.HR) is valid for $n \geq 2$. Special cases will be the polytopes of class $\Pi^n_{\eta 2}(G)$ induced by the multiset G whose multiplicity of the minimum/maximum elements reaches its lower bound 1 or the upper bound n. That is why, we list all possible combinations of η_1, η_2 and formulate this corollary with reference to them:

- Case 5.5.1: $1 < \eta_1, \eta_2 < n$;
- Case 5.5.2: $1 < \eta_1, \eta_2 < n$;
- Case 5.5.3: $\eta_1 = 1,\ \eta_2 = n$;
- Case 5.5.4: $\eta_1 = \eta_2 = n$;
- Case 5.5.5: $\eta_1 = n,\ 1 < \eta_2 < n$;
- Case 5.5.6: $1 < \eta_1 < n,\ \eta_2 = n$.

Corollary 5.8. *Depending on the combination η_1 and η_2, the irredundant H-representation ($\Pi^n_{\eta 2}(G)$.IHR) has the form of the below inequalities:*

- *in Case 5.5.1:* (4.76),

$$\sum_{i=1}^{n} x_i \geq \sum_{i=1}^{n} g_i = \eta_1 \cdot e_1 + (n - \eta_1)e_2, \tag{5.78}$$

$$\sum_{i=1}^{n} x_i \leq \sum_{i=1}^{n} g_{\eta - i + 1} = (n - \eta_2)e_1 + \eta_2 \cdot e_2; \tag{5.79}$$

- *in Case 5.5.2:* (4.34) *and* (5.79)*;*
- *in Case 5.5.3:* (4.77) *and* (5.78)*;*
- *in Case 5.5.4:* (4.76)*;*
- *in Case 5.5.5:* (4.76) *and* (5.79)*;*
- *in Case 5.5.6:* (4.34) *and* (4.76)*.*

Vertex adjacency criterion for $\Pi^n_{\eta 2}(G)$

Application of Theorem 5.1 and Remark 5.6 to the case of $k = 2$ allows distinguishing three types of adjacent vertices of $\Pi^n_{\eta 2}(G)$ to a point $x \in E^n_{\eta 2}(G)$:

- $e_1 \to e_2$-vertices obtained from x by replacing the coordinate e_1 by e_2;
- $e_2 \to e_1$-vertices formed from x by replacing the coordinate e_2 by e_1;
- $e_1 \leftrightarrow e_2$-vertices formed from x by transposition of coordinates e_1 and e_2.

As a consequence of this theorem, let us formulate the vertex adjacency criterion for $\Pi^n_{\eta 2}(G)$.

Corollary 5.9. *(from Theorem 5.1). If $x \in E^l$, all its $e_1 \to e_2$- and $e_1 \leftrightarrow e_2$-vertices are adjacent to x, and only they are; if $x \in E^u$, then adjacent ate its $e_2 \to e_1$- and $e_1 \leftrightarrow e_2$-vertices; if $x \in E^m$, then adjacent ate its $e_1 \to e_2$- and $e_2 \to e_1$-vertices.*

Vertex degree of $\Pi^n_{\eta 2}(G)$

Proposition 5.3. *The points of the sets* (5.73)-(5.75) *have the same degree of regularity. Namely:*

$$\begin{aligned}
&\forall x \in E^l \; \mathcal{R}(x) = \mathcal{R}^l = \eta_1 \cdot (n - \eta_1) + \eta_1 = \eta_1 \cdot (n - \eta_1 + 1)\,; \\
&\forall x \in E^u \; \mathcal{R}(x) = \mathcal{R}^u = \eta_2 \cdot (n - \eta_2) + \eta_2 = \eta_2 \cdot (n - \eta_2 + 1)\,; \\
&\forall x \in E^m \; \mathcal{R}(x) = \mathcal{R}^m = n.
\end{aligned} \tag{5.80}$$

This follows directly from Corollary 5.9 as a result of applying the formula (4.78) to the points of E^l and E^u.

Centrally symmetric $E^n_{\eta 2}(G)$ and $\Pi^n_{\eta 2}(G)$

Proposition 5.4. *$E^n_{\eta 2}(G)$ and $\Pi^n_{\eta 2}(G)$ are centrally symmetric if and only if η is even and*

$$\eta_1 = \eta_2 = \frac{\eta}{2}. \tag{5.81}$$

If (5.81) *is satisfied, then the center of symmetry of this set and polytope is the center of the circumscribed hypersphere* (5.66) *given by the parameters* (4.74) *and* (4.75).

Proof. Since $E^n_{\eta 2}(G)$ is vertex-located, the symmetry conditions for $E^n_{\eta 2}(G)$ and $\Pi^n_{\eta 2}(G)$ coincide and are given by the symmetry condition (5.40) of the multiset G about its mean. Since the case of $k = 2$ is under consideration, this condition for the ESPPC is simplified to the expression (5.81), like for the ESPC, it was the condition (4.81).

Taking into account (5.81), the formula (5.39) for the parameter a' defining the center $\mathbf{a}'$ of the circumscribed hypersphere becomes

$$a' = \frac{S_1^{\min} + S_1^{\max}}{2n} = \frac{e_1 + e_2}{2},$$

where

$$S_1^{\min} = \frac{\eta}{2} \cdot e_1 + (n - \frac{\eta}{2})e_2,\ S_1^{\max} = \frac{\eta}{2} \cdot e_2 + (n - \frac{\eta}{2})e_1.$$

This means it is the hypersphere with the parameters (4.74) and (4.75). □

Thus, for every $n > 1$, there exists a family of centrally symmetric ESPPCs $E^n_{\eta 2}(\{e_1^j, e_2^j\}), j \in J_n \backslash J_{[\frac{n}{2}]}$ and the corresponding polytopes. Moreover, it is shown that, for all centrally symmetric ESPCs and ESPPCs generated by the numbers e_1 and e_2, the circumscribed hyperspheres are identical.

Simple polytopes among $\Pi^n_{\eta 2}(G)$

Proposition 5.5. *In class of special partial permutohedra, there are only the following simple polytopes:*

$$\begin{aligned}
P^1 &= \Pi^n_{n+1,2}(\{e_1^n, e_2\});\\
P^2 &= \Pi^n_{n+1,2}(\{e_1, e_2^n\});\\
P^3 &= \Pi^n_{2n-2,2}(\{e_1^{n-1}, e_2^{n-1}\});\\
P^4 &= \Pi^n_{2n-1,2}(\{e_1^{n-1}, e_2^{n}\});\\
P^5 &= \Pi^n_{2n-1,2}(\{e_1^{n}, e_2^{n-1}\});\\
P^6 &= \overline{\Pi}^n_{2}(\{e_1^{n}, e_2^{n}\}).
\end{aligned}$$

Proof. According to Proposition 5.3 and due to the full-dimensionality of $\Pi^n_{\eta 2}(G)$, if the corresponding ESPPC $E^n_{\eta 2}(G)$ contains only the level E^m, the polytope is simple. This applies only to the special partial permutohedron with unbounded repetitions P^6.

In all other cases, at least one of the extreme levels E^l or E^u is present.

In order to combine the expressions for $\mathcal{R}^l$ and $\mathcal{R}^u$ into the formula (5.80), we introduce the auxiliary function $\phi(x) = x(n+1-x)$ and solve the problem $\phi(x) \to \min\limits_{x \in J_{n-1}}$. It is easy to see that there are two solutions $X^{\min} = \{1, n-1\}$ and $\min\limits_{x \in J_{n-1}} \phi(x) = \phi(1) = \phi(n-1) = n$. This means that, for the polytopes remaining after the elimination of P^6, the extreme levels are induced by the multisets of type (3.28) and (5.65), whose element multiplicities are 1 or $n-1$.

In general, simple special partial permutohedra are exhausted by inducing multisets with element's multiplicities such that $\eta_1, \eta_2 \in \{1, n-1, n\}$. Listing all possible combinations of them and, considering $\eta_1 + \eta_2 > n$, we obtain the entire collection $P^1 - P^6$.

□

Remark 5.7. It is easy to see that, in accordance with the above typology of special partial permutohedra, in each of the Cases 5.5.1-5.5.6, there is one simple polytope, namely:

- Case 5.5.1 corresponds to P^3 whose irredundant constraint system can be represented in

vector form as follows:

$$P^3 = \{\mathbf{e_1} \leq x \leq \mathbf{e_2},\ x^\top \mathbf{e} \geq (n-1)e_1 + e_2,\ \ x^\top \mathbf{e} \leq e_1 + (n-1)e_2\},$$

wherefrom it is seen that it is a hypercube with two opposite corners cut off at adjacent vertices;

- Case 5.5.2 corresponds to the polytope P^1, whose irredundant system of constraints is

$$P^1 = \{\mathbf{e_1} \leq x,\ x^\top \mathbf{e} \leq (n-1)e_1 + e_2\}$$

and it is an n-simplex;

- Case 5.5.3 corresponds to P^2 given by the irredundant constraints:

$$P^2 = \{x \leq \mathbf{e_2},\ x^\top \mathbf{e} \geq e_1 + (n-1)e_2\},$$

which also describes the n-simplex;

- Case 5.5.4 corresponds to P^5, and its irredundant system of constraints is

$$P^6 = \{\mathbf{e_1} \leq x \leq \mathbf{e_2}\},$$

while this polytope is a hypercube;

- Case 5.5.5 corresponds to P^5. Its irredundant H-representation is

$$P^5 = \{\mathbf{e_1} \leq x \leq \mathbf{e_2},\ x^\top \mathbf{e} \leq e_1 + (n-1)e_2\}.$$

The polytope itself is a hypercube with the corner cut off;

- Finally, Case 5.5.6 corresponds to P^4, and its irredundant H-representation is

$$P^6 = \{\mathbf{e_1} \leq x \leq \mathbf{e_2},\ x^\top \mathbf{e} \geq (n-1)e_1 + e_2\}.$$

It is also a hypercube with an angle truncated at adjacent vertices.

Among $E^n_{n+1,2}(G)$, let us explore the class $E^n_{n+1,2}(G)$ in more detail, which also belongs to $E^n_{n+1,k}(G)$-sets, on the binary ESPPC (further referred to as the entire binary partial multipermutation point configuration, EBPPC), and on the EUSPPC $\overline{E}^n_2(G)$.

5.6 The ESPPC $E^n_{n+1,2}(G)$

Let us list certain properties of the ESPPC

$$E = E^n_{n+1,2}(G), \tag{5.82}$$

following from its membership both in the classes $E^n_{\eta 2}(G)$ and $E^n_{n+1,k}(G)$.

The cardinality of $E^n_{n+1,2}(G)$

The formula (5.63) is converted to

$$\left|E^n_{n+1,2}(G)\right| = \frac{(n+1)!}{\eta_1! \cdot \eta_2!} = C^{\eta_1}_{n+1} = C^{\eta_2}_{n+1}.$$

Vertex regularity degree of $\Pi^n_{n+1,2}(G)$

$$\mathcal{R} = \eta_1 \eta_2.$$

This follows from (5.64) and says that, in this case, there are no intermediate-levels sets, i.e. the set (5.73) is empty, $E^m = \varnothing$, and the points of the upper-level E^u and lower-level E^l have the same degree of regularity $\mathcal{R} = \mathcal{R}^l = \mathcal{R}^u$.

Two-levelness of $E^n_{n+1,2}(G)$

$$m(E^n_{n+1,2}(G)) = 2.$$

Since $E^n_{n+1,2}(G)$ belongs to the class $E^n_{n+1,k}(G)$, it will be $k = 2$-level by coordinates and toward the vector $\mathbf{e}$.

Based on the form of $(E^n_{n+1,2}(G).\text{IHR})$, the facets $E^n_{n+1,k}(G)$ are parallel to the coordinate hyperplanes or have a normal vector $\mathbf{e}$, wherefrom it follows that the levelness of the set $E^n_{n+1,2}(G)$ is two, and, overall, this set is two-level.

Quadratic surfaces circumscribed about $E^n_{n+1,2}(G)$

The equation (5.61) defines an ellipsoid other than the circumscribed hypersphere with the parameters (4.74) and (4.75). Thus, a quadratic surface circumscribed about $E^n_{n+1,2}(G)$ is not unique.

5.7 Vertex-located EPMPCs

We single out vertex-located sets in the class $E^n_{\eta k}(G)$. Recall that two such sets have already been identified, and they belong to Cases 5.1.1 and 5.1.2. These are the classes $E^n_{n+1,k}(G)$ (see Section 5.4) and $E^n_{\eta 2}(G)$ (see Section 5.5).

As it turns out, no other vertex-located sets among $E^n_{\eta k}(G)$ are established in the next theorem.

Theorem 5.11. *Vertex-located EPMPCs are only the following:*

- *Class 1:* $E^n_{n+1,k}(G)$,
- *Class 2:* $E^n_{\eta 2}(G)$

and only they.

Proof. We will prove the theorem by contradiction in two stages.

Suppose there exists another class, Class 3, among $E^n_{\eta' k'}(G')$ different from Classes 1 and 2:

$$E^n_{\eta' k'}(G') = vert\ \Pi^n_{\eta' k'}(G'). \tag{5.83}$$

Eliminating the conditions (3.27) and (5.44) characterizing Classes 1 and 2, we remain with the case where

$$k' > 2,\ \eta' > n+1. \tag{5.84}$$

Stage 1. Consider a set E belonging to both Class 1 and Class 2. This means $E = E^n_{n+1,2}(G)$, i.e. satisfies (5.82). We form a set $E' = E^n_{\eta' k'}(G')$ from E by adding the only element $e' \in (e_1, e_2)$ to the $n+1$-element inducing multiset G of E. The parameters of the resulting multiset G' are $\eta' = n+2$, $k' = 3$,

$$G' = \{g'_i\}_{i \in J_{\eta'}} = \{e_1^{\eta_1}, e', e_2^{\eta_2}\} = \left\{e_1^{\eta_1}, e', e_2^{n-\eta_1+1}\right\}. \tag{5.85}$$

Let us show that E' is not vertex-located by introducing the polytope $\Pi' = conv\ E'$ and the point $x' \in E' :\ x' = (g'_i)_{i \in J_{n+1} \setminus \{1\}}$.

Let us demonstrate that

$$x' \notin vert\ \Pi', \tag{5.86}$$

based on the fact that, by (5.85),

$$x' = \left(e_1^{\eta_1 - 1}, e', e_2^{n-\eta_1}\right). \tag{5.87}$$

We apply the vertex criterion for the partial multipermutohedron (see Theorem 5.1). Let us take a point $x \in E^n_{\eta' k'}(G')$ and order its coordinates non-decreasingly, then utilizing two auxiliary parameters $r',\ s'$:

$$\begin{aligned} &x_i = g_i,\ i \in J^0_{r'},\ x_{r'+1} > g_{r'+1}; \\ &x_{n-i'+1} = g_{\eta-i'+1},\ i' \in J^0_{s'},\ x_{n-s'} < g_{\eta-s'}. \end{aligned} \tag{5.88}$$

It is easy to see that r', s' are related to the parameters r, s from the formulas (5.11) and (5.12) as follows:

$$r' \geq r,\ s' \geq s,\ s' + r' \geq s + r.$$

Since the coordinates of x' are ordered non-increasingly, one can apply the formula

(5.88) directly to x', G', E'. Hence, $r' = \eta_1 - 1$, $s' = n - \eta_1$ and therefore:

$$n > s' + r' \geq s + r,$$

i.e. the condition (5.11) that x' is a vertex of Π' is violated, which means that (5.86) holds.

Thus, we have shown that adding one element to a multiset inducing a vertex-located ESPPC E of the form (5.82) leads to the formation of a non-vertex EPPPC E', where, in addition to the point (5.87), all permutation e-configurations induced by the multiset $\left\{e_1^{\eta_1-1}, e', e_2^{n-\eta_1}\right\}$ are not vertices of the resulting polytope Π'.

Step 2. Now, we take an arbitrary set $E_{\eta k}^{n}(G)$ satisfying the condition (5.84). By analogy with (5.85), we introduce a submultiset G' formed from three different elements of G, its largest and smallest elements with maximum multiplicity and an arbitrary middle element of multiplicity one:

$$G' = \{e_1^{\eta_1}, e_j, e_k^{\eta_k}\}, \text{ where } 1 < j < k. \tag{5.89}$$

We also consider an auxiliary EPMPC induced by a multiset G':

$$E' = E_{\eta' 3}^{n'}(G'), \ \eta' = \eta_1 + \eta_k + 1, \ n' = \eta' - 2. \tag{5.90}$$

As shown in Step 1, the point $y' = \left(e_1^{\eta_1-1}, e', e_k^{\eta_k-1}\right)$ satisfies relation $y' \notin vert\ \Pi_{\eta' 3}^{n'}(G')$, i.e. it can be represented as a convex linear combination of other points of the set (5.90):

$$\begin{aligned} &\exists\ J,\ Y = \left\{y^j\right\}_{j\in J} \subseteq vert\ \Pi_{\eta' 3}^{n'}(G'), \\ &\{\alpha_j\}_{j\in J} > 0,\ \sum_{j\in J} \alpha_j = 1: \\ &y' = \sum_{j\in J} \alpha_j y_j. \end{aligned} \tag{5.91}$$

We complete the multiset (5.89) to an n-element submultiset G, yielding the multiset:

$$G'' \subseteq G:\ G' \subset G'',\ |G''| = n,\ G'' = G' \cup \{g_{j_i}\}_{i\in J_{n-n'}}.$$

Let us form the point $x' \in \mathbb{R}^n$ and the set $X = \{x^j\}_{j\in J} \subset \mathbb{R}^n$ as the Cartesian products

of y' and $y^j,\ j \in J$ with the vector $\overline{g} = (g_{j_i})_{i \in J_{n-n'}}$ such that

$$x' = (y', \overline{g}),\ x^j = \left(y^j, \overline{g}\right),\ j \in J. \tag{5.92}$$

Due to (5.91) and the construction of (5.92), x' is represented by the following convex linear combination:

$$x' = \sum_{j \in J} \alpha_j x^j, \tag{5.93}$$

with the coefficients given by the formula (5.91).

Regardless of whether the inclusion $X \subseteq vert\ \Pi^n_{\eta k}(G)$ holds, one can see that x' is a convex linear combination of other points $E^n_{\eta k}(G)$, hence, $x' \notin vert\ \Pi^n_{\eta k}(G)$.

Thus, we have found that, for an arbitrary $E^n_{\eta k}(G)$, the fulfillment of the condition (5.84) means that the set is non-vertex-located. The resulting contradiction to the assumption (5.83) proves the theorem. □

5.8 The EBPPC $B_n(m_1, m_2)$

In this section, we list the properties of the EBPPC $B_n(m_1, m_2)$.

By analogy with the hypersimplex (4.83), we introduce the following notation for the convex hull of the set (3.34):

$$\Delta_{n,m_1,m_2} = conv\ B_n(m_1, m_2)$$

and will call it <u>the $(0-1)$-partial permutohedron</u> (the binary partial permutatohedron):

Let us list some properties of the EBPPCs and $(0-1)$-partial permutohedra following from the properties of the ESPPCs and the corresponding special partial permutohedra given in Section 5.5. Taking into account (3.34) and (5.97), we come to the substitutions similar to (4.86):

$$e_1 \to 0,\ e_2 \to 1,\ \eta_1 \to n - m_1,\ \eta_2 \to m_2. \tag{5.94}$$

The full-dimensionality of Δ_{n,m_1,m_2}

$$\dim\ \Delta_{n,m_1,m_2} = n.$$

The spherical locality of $B_n(m_1, m_2)$

$$B_n(m_1, m_2) \subset S_r(\mathbf{a}).$$

The only hypersphere circumscribed about $B_n(m_1, m_2)$ has the parameters:

$$a = a^{\min} = \frac{1}{2},\ r(a) = r^{\min} = \frac{\sqrt{n}}{2}. \tag{5.95}$$

Vertex locality of $B_n(m_1, m_2)$

$$B_n(m_1, m_2) = vert\ \Delta_{n,m_1,m_2}.$$

Two-levelness of $B_n(m_1, m_2)$ along coordinates

$$m'(B_n(m_1, m_2)) = 2.$$

For the set $B_n(m_1, m_2)$ as for a representative of class $E^n_{\eta 2}(G)$, there exists the same decomposition (1.42), (1.40) and (4.80) into pairs of the hyperplanes $x_i = 0, 1,\ i \in J_n$. Moreover, for the projections of the cuts of $B_n(m_1, m_2)$ by these hyperplanes, the formula (5.68) becomes

$$E'^{1j} = B_{n-1}(m_1 - 1, m_2),\ E'^{2j} = B_{n-1}(m_1, m_2 - 1),\ j \in J_n.$$

As one can see, the sets formed in the projection are EBPCs if $m_1, m_2 > 1$, otherwise they degenerate into a singleton $\mathbf{0}$ or $\mathbf{e}$.

The decomposition of $B_n(m_1, m_2)$ toward e

$B_n(m_1, m_2)$ decomposes into the family of the $m_2 - m_1 + 1$ EBPCs:

$$B_n(m_1, m_2) = \bigcup_{m=m_1}^{m_2} B_n(m). \tag{5.96}$$

This formula is the result of substituting

$$G = \left\{0^{n-m_1}, 1^{m_2}\right\}, \ \eta{=}n - m_1 + m_2. \tag{5.97}$$

into the decomposition (5.71) and (5.72) of the ESPPC.

It also demonstrates the decomposition of the set $B_n(m_1, m_2)$ into the family of $m_{\mathbf{e}}$ hyperplanes toward the vector $\mathbf{e}$, where

$$m_{\mathbf{e}} = m_2 - m_1 + 1. \tag{5.98}$$

The cardinality of $B_n(m_1, m_2)$

Moving in (5.96) to cardinality, we derive that the number of elements in the set $B_n(m_1, m_2)$ is equal to the partial binomial sum:

$$|B_n(m_1, m_2)| = \sum_{m=m_1}^{m_2} |B_n(m)| = \sum_{m=m_1}^{m_2} C_n^m. \tag{5.99}$$

The irredundant H-representation of Δ_{n,m_1,m_2}

For this case, we formulate the following proposition based Corollary 5.8.

Proposition 5.6. *The irredundant H-representation of the special partial permutohedron can contain up to four groups of constraints:*

- *the lower-bound constraints* (4.91)*;*
- *the upper-bound constraints* (4.92)*;*

- *the lower-bound constraint on the sum of variables:*

$$\sum_{i=1}^{n} x_i \geq n - m_1; \tag{5.100}$$

- *the upper-bound constraint on the sum of variables:*

$$\sum_{i=1}^{n} x_i \leq m_2. \tag{5.101}$$

Depending on the combination of the parameters m_1, m_2, we come to six different types of $(0-1)$-partial permutohedra:

1. *($\Delta_{n,0,n}$.IHR) has the form of* (4.90)*;*
2. *($\Delta_{n,0,1}$.IHR) is* (4.91)*, $\sum_{i=1}^{n} x_i \leq 1$;*
3. *($\Delta_{n,n-1,n}$.IHR) is* (4.92)*, $\sum_{i=1}^{n} x_i \geq n-1$;*
4. *($\Delta_{n,0,m_2}$.IHR) for $1 < m_2 < n$ is* (4.90)*,* (5.101)*;*
5. *($\Delta_{n,m_1,n}$.IHR) for $1 < m_1 < n$ is* (4.90)*,* (5.100)*;*
6. *(Δ_{n,m_1,m_2}.IHR) for $1 < m_1 < m_2 < n$ is* (4.90)*,* (5.100)*,* (5.101)*.*

Remark 5.8. Finally, we rewrite all possible irredundant H-representations of the polytope Δ_{n,m_1,m_2} in vector form:

$$\Delta_{n,0,n} = \{x \in \mathbb{R}^n : \mathbf{0} \leq x \leq \mathbf{1}\};$$

$$\Delta_{n,0,1} = \{x \in \mathbb{R}^n : x \geq \mathbf{0},\ x^\top \mathbf{e} \leq 1\};$$

$$\Delta_{n,n-1,n} = \{x \in \mathbb{R}^n : x \leq \mathbf{1},\ x^\top \mathbf{e} \geq n-1\};$$

$$\Delta_{n,0,m_2} = \{x \in \mathbb{R}^n : \mathbf{0} \leq x \leq \mathbf{1},\ x^\top \mathbf{e} \leq m_2\} \text{ if } 1 < m_2 < n;$$

$$\Delta_{n,m_1,n} = \{x \in \mathbb{R}^n : \mathbf{0} \leq x \leq \mathbf{1},\ x^\top \mathbf{e} \geq n - m_1\} \text{ if } 1 < m_1 < n;$$

$$\Delta_{n,m_1,m_2} = \{x \in \mathbb{R}^n : \mathbf{0} \leq x \leq \mathbf{1},\ n - m_1 \leq x^\top \mathbf{e} \leq m_2\}, \text{ if } 1 < m_1 < m_2 < n.$$

$B_n(m_1, m_2)$-levelness

$$m(B_n(m_1, m_2)) = \begin{cases} m_2 - m_1 + 1, & \text{if } m_2 - m_1 < n, \\ 2, & \text{если } m_1 = 0, m_2 = n. \end{cases} \tag{5.102}$$

Indeed, (Δ_{n,m_1,m_2}.IHR) says that the hypersimplex Δ_{n,m_1,m_2} has two types of facets: a) facets parallel to coordinate hyperplanes, and, along coordinates, the set $B_n(m_1, m_2)$ is two-level; b) facets with normal vector $\mathbf{e}$. Towards the vector $\mathbf{e}$, the set is $m_{\mathbf{e}}$-level, where the formula (5.98) gives us $m_{\mathbf{e}}$. According to the condition $m_1 < m_2$, the latter is at least two. Accordingly, if there are facets with the normal vector $\mathbf{e}$, then the value of $m_{\mathbf{e}}$ determines the levelness of the set $B_n(m_1, m_2)$.

Taking into account that in all cases except for $\Delta_{n,0,n}$, the second type of facets is present, we derive that the formula (5.102) yields the levelness of an arbitrary set $B_n(m_1, m_2)$.

The formula (5.102) implies that the number $m(B_n(m_1, m_2))$ reaches its lower bound two in two cases only. Namely, if the value $m_2 - m_1$ is 1 or n, thus reaching its lower and upper bounds:

$$m(B_n(m_1, m_2)) = 2 \Leftrightarrow m_2 - m_1 \in \{1, n\}.$$

Therefore, there are n 2-level sets in the family

$$B_n(i, i+1), \ i \in J_{n-1}; \ B_n(0, n) = B_n. \tag{5.103}$$

Vertex adjacency criterion for Δ_{n,m_1,m_2}

Corollary 5.9 is reformulated as the following.

Proposition 5.7. *in three ways: a) by a single $0 \leftrightarrow 1$-transposition (further $0 \leftrightarrow 1$-vertices); b) by a single $0 \to 1$-substitution (further $0 \to 1$-vertices); c) by a single $1 \to 0$-substitution (further $1 \leftrightarrow 0$-vertices).*

Namely:

- *if $x \in E^l$, then adjacent to it are its $0 \to 1$- and $0 \leftrightarrow 1$-vertices,*

- *if $x \in E^u$, then adjacent to x are its $1 \to 0$- and $0 \leftrightarrow 1$-vertices,*
- *$x \in E^m$, then its adjacent are $0 \to 1$- and $1 \to 0$-vertices,*

and only they are.

Here,

$$E^l = \begin{cases} B_n(m_1), & \text{if } m_1 \geq 1, \\ \emptyset, & \text{otherwise}, \end{cases} \qquad E^u = \begin{cases} B_n(m_2), & \text{if } m_2 \leq n-1, \\ \emptyset, & \text{otherwise}, \end{cases} \tag{5.104}$$

$$E^m = B_n(m_1, m_2) \setminus \{E^l, E^u\}.$$

Vertex regularity degree of Δ_{n,m_1,m_2}

Given (5.94), Proposition 5.3 becomes

Proposition 5.8. *Each of the sets* (5.104) *unites the vertices of Δ_{n,m_1,m_2} with the same degree of regularity:*

$$\begin{aligned} &\forall x \in E^l \ \mathcal{R}(x) = \mathcal{R}^l = (n - m_1) \cdot (m_1 + 1); \\ &\forall x \in E^u \ \mathcal{R}(x) = \mathcal{R}^u = (n - m_2 + 1) \cdot m_2; \\ &\forall x \in E^m \ \mathcal{R}(x) = \mathcal{R}^m = n. \end{aligned} \tag{5.105}$$

Simple polytopes among Δ_{n,m_1,m_2}

The following corollary follows from Proposition 5.5.

Corollary 5.10. *Among the $(0-1)$-partial permutohedra, there are six simple polytopes:*

- *$\Delta_{n,0,n}$ is the unit hypercube;*
- *$\Delta_{n,0,1}$ is the unit n-simplex;*
- *$\Delta_{n,n-1,n}$ is the n-simplex;*
- *$\Delta_{n,0,n-1}$ is the unit hypercube with the point $\mathbf{e}$ truncated along the adjacent vertices;*
- *$\Delta_{n,1,n}$ is the unit hypercube with the point $\mathbf{0}$ truncated along adjacent vertices;*
- *$\Delta_{n,1,n-1}$ is the unit hypercube with the points $\mathbf{0}, \mathbf{e}$ truncated along adjacent vertices.*

Polytopes Δ_{n,m_1,m_2} with regular vertices

In class Δ_{n,m_1,m_2}, we can single out the polytopes with regular vertices, i.e. those that satisfy the condition:

$$\exists\, \mathcal{R} : \forall\, x \in\ B_n(m_1, m_2)\ \mathcal{R}(x) = \mathcal{R}. \tag{5.106}$$

Six of them are listed in Corollary 5.10, but there is one more class of polytopes possessing this property.

It is easy to see from the formula (5.105) that if $B_n(m_1, m_2)$ contains an intermediate-level set, then for satisfying the condition (5.106), the polytope Δ_{n,m_1,m_2} must be simple:

$$\text{If done (5.105) and } E^m \neq \emptyset \Rightarrow \Delta_{n,m_1,m_2} \text{ is simple.}$$

Accordingly, all the polytopes meeting this requirement are listed in Corollary 5.10.

If there are no intermediate-level sets, this means $B_n(m_1, m_2)$ includes a lower- and an upper-level sets only. Taking into account (5.105), we get

$$\text{If (5.105)holds and } E^m = \emptyset \Rightarrow \Delta_{n,m_1,m_2} = E^l \cup E^u \Rightarrow$$

$$\Rightarrow m_1 = m_2 - 1,\ (n - m_1) \cdot (m_1 + 1) = (n - m_2 + 1) \cdot m_2. \tag{5.107}$$

Substituting the first condition from (5.107) into the second leads an identity, i.e. all polytopes corresponding to the EBPPC without intermediate-level sets will have regular vertices only.

This is generally a family of $n - 1$ polytopes:

$$\Delta_{n,j-1,j},\ j \in J_n \backslash \{1\}. \tag{5.108}$$

Based on (5.97), for them, $\eta = j + (n - j + 1) = n + 1$, i.e. the family (5.108) consists of the $(0-1)$-partial permutohedra combinatorically equivalent to the binary $n+1$-permutohedra:

$$\Delta_{n,j-1,j} \cong \Delta_{n+1,j},\ j \in J_n \backslash \{1\}.$$

This is another way to prove that the family (5.108) forms a class of polytopes with regular vertices because, as was shown in Section 4.5, all vertices of the polytopes are regular.

To summarize, considering that among the six polytopes listed in Corollary 5.10, two are from the family (5.108).

Proposition 5.9. *The polytopes of the family* (5.108) *along with* $\Delta_{n,0,n}$, $\Delta_{n,0,n-1}$, $\Delta_{n,1,n}$, $\Delta_{n,1,n-1}$ *and only polytopes with the same degree of regularity of vertices.*

Centrally symmetric B_{n,m_1,m_2} and Δ_{n,m_1,m_2}

We formulate this symmetry conditions as a consequence of Proposition 5.4.

Corollary 5.11. *The set* B_{n,m_1,m_2} *and polytope* Δ_{n,m_1,m_2} *are centrally symmetric if and only if* m_1, m_2 *satisfy the condition:*

$$m_1 + m_2 = n. \tag{5.109}$$

If (5.109) *holds, the point* $\mathbf{a} = \frac{\mathbf{1}}{\mathbf{2}}$ *is the center of their symmetry.*

The formula (5.109) is the result of substituting (5.97) into (5.81).

Remark 5.9. Taking into account that, by assumption, $m_1 < m_2$, one can say that, for a fixed n, there are $[\frac{n+1}{2}]$ symmetric EBPPCs:

$$B_{n,j,n-j}, \Delta_{n,j,n-j},\ j \in J^0_{[\frac{n-1}{2}]}.$$

5.9 The EUBPPC B_n

In Sections 5.5 and 5.8, properties of both vertex-located EPMPCs $E^n_{n+1,k}(G)$ and $E^n_{\eta 2}(G)$ were outlined.

According to (3.28) and (5.65), in the subclass of the ESPPCs, two extreme cases can be distinguished in η:

- the minimum possible η, $\eta = n + 1$, corresponds to the set $E^n_{n+1,2}(G)$ belonging simultaneously to Cases 5.1.1 and 5.1.2 (see Section 5.6);

- the maximum possible η, $\eta = 2n$, corresponds to the EUSPPC $\overline{E}_2^n(G)$ induced by two-element generating sets. Its peculiarity is that it simultaneously belongs to the classes $E_{\eta 2}^n(G)$ and $\overline{E}_k^n(G)$. This determines the specifics of this set and its convex hull called the special partial multipermutohedron with unbounded repetitions $\overline{\Pi}_2^n(G)$.

Consider the properties of $\overline{E}_2^n(G)$, $\overline{\Pi}_2^n(G)$, starting with $(0-1)$-case, and then we generalize to the entire class.

The main properties of B_n, $PB_n - convB_n$ were listed in Section 5.8 within consideration of the class EBPPC because B_n is its special case of the set $B_n(m_1, m_2)$, corresponding to a pair of parameters:

$$m_1 = 0, \ m_2 = n. \tag{5.110}$$

Thus,

$$B_n = B_{n,0,1}, \ PB_n = \Delta_{n,0,n}, \tag{5.111}$$

whence, in particular, it follows that PB_n is the unit n-cube.

Let us list some properties of the B_n set and the PB_n polytope based on the properties given in Sections 5.3 and 5.8. In this case, we will perform substitutions (5.110) and

$$k = 2, \ e_1 = 0, \ e_2 = 1.$$

As will be shown, the known properties of the unit hypercube PB_n directly following from the properties of Δ_{n,m_1,m_2} and $\overline{\Pi}_k^n(G)$.

The cardinality of B_n

$$|B_n| = 2^n. \tag{5.112}$$

This formula can be obtained in two ways. On the one hand, in this case, taking into account that $\mathcal{A} = \{0, 1\}$ is the generating set, the formula (5.48) becomes $B_n = \{0, 1\}^n$, while the formula (5.49) becomes (5.112).

On the other hand, the formula (5.99) is converted to $|B_n| = \sum\limits_{m=0}^{n} C_n^m = 2^n$ and gives

the same result.

The spherical locality of B_n

Like all sets of class $B_n\,(m_1, m_2)$, the set of binary vectors is spherically-located. Moreover, the hypersphere circumscribed about it is uniquely defined and has parameters (5.95).

Vertex locality of B_n

$$B_n = vert\, PB_n$$

like any set of class $B_n\,(m_1, m_2)$.

Polyhedral-sphericity of B_n

$$B_n = PB_n \cap S_{\frac{\sqrt{n}}{2}}\left(\frac{\mathbf{1}}{\mathbf{2}}\right);$$

The full-dimensionality of PB_n

$$\dim PB_n = n.$$

Indeed, the full-dimensionality is the commonality of polytopes of class $\Pi^n_{\eta k}(G)$.

Vertex adjacency criterion for PB_n

All vertices adjacent to an arbitrary point $x \in B_n$ are formed from it by a $0 \to 1$- or $1 \to 0$-replacement of one of its coordinates.

This property follows directly from the vertex adjacency criterion $\overline{\Pi}^n_k(G)$. It can also be formulated in terms of the Hamming distance, whereby the formula (5.53) becomes

$$\forall x, y \in B_n :\ x \leftrightarrow y \Leftrightarrow Hd(x, y) = 1.$$

The simplicity of PB_n

The simplicity of the polytope follows from (5.111) and Corollary 5.10. Also, it means that both the degree of the vertices of the polytope PB_n and its dimension are equal to n.

The irredundant H-representation of PB_n ($\Delta_{n,0,n}$.IHR)

The fact that the polytope PB_n is given by the irredundant system of $2n$ inequalities (4.90) follows, on the one hand, from the irredundant H-representation $\overline{\Pi}_k^n(G)$. Indeed, in this case, the formula (5.50) is converted to

$$PB_n = [0,1]^n,$$

and (5.51) becomes (4.90).

On the other hand, according to Proposition 5.6, for the case of (5.111), the irredundant representation of the polytope is given by the inequalities (4.90), denoted above by ($\Delta_{n,0,n}$.IHR). Now, it can now also be denoted (PB_n.IHR).

Symmetry of B_n and PB_n

For $(m_1, m_2) = (0,1)$, for the EBPPC and $(0-1)$-partial multipermutohedron, the symmetry condition (5.109) about the point $\frac{\mathbf{1}}{\mathbf{2}}$ holds. This means the set B_n and polytope PB_n are symmetric about this point. In addition, they will be symmetric about any hyperplane passing through the center of symmetry. In particular, B_n and PB_n have n symmetry hyperplanes parallel to the coordinate hyperplanes:

$$\Pi^i = \left\{ x \in \mathbb{R}^n : \, x_j = \frac{1}{2} \right\}. \tag{5.113}$$

Decomposition of B_n into EBPCs

$$B_n = \bigcup_{m=0}^{n} B_n\,(m).$$

This formula follows from (5.96) and defines a decomposition of B_n into a family of $n+1$

EBPCs, the first and last of which are singletons. The same formula yields the decomposition of B_n toward $\mathbf{e}$ and says that it is $n+1$-level toward this vector.

Remark 5.10. The symmetry of B_n about the point $\frac{\mathbf{1}}{\mathbf{2}}$ and hyperplanes (5.113) allows asserting that, in addition to $x^T\mathbf{e} = m,\ m \in J_n^0$, there are many other decompositions of B_n into $n+1$ parallel hyperplanes. These will be decompositions toward normal vectors whose vector of absolute values of its coordinates is proportional to $\mathbf{e}$.

Two-levelness of B_n

$$m(B_n) = 2.$$

This follows from the formula (5.103).

5.10 Combinatorically equivalent partial multipermutohedra

As seen from the H-representation ($\Pi^n_{\eta k}(G)$.IHR), the polytope $\Pi^n_{\eta k}(G)$ is completely determined by n first and n last elements of the multiset (1.2), and its combinatorial structure is fully determined by the multiplicities of these elements.

Let us introduce into consideration the corresponding n-element submultisets of G, as well as their underlying sets and primary specifications:

$$G^{\min} = \{g_i\}_{i\in J_n},\ G^{\max} = \{g_{\eta-n+i}\}_{i\in J_n};$$
$$k^{\min} = |S(G^{\min})|,\ \left[G^{\min}\right] = \left(\eta_i^{\min}\right)_{i\in J_{k^{\min}}};$$
$$k^{\max} = |S(G^{\max})|,\ [G^{\max}] = (\eta_i^{\max})_{i\in J_{k^{\max}}},$$

where

$$G^{\min} = \{(e_i^{\min})^{\eta_i^{\min}}\}_{i\in J_{k^{\min}}},$$
$$G^{\max} = \{(e_i^{\max})^{\eta_i^{\max}}\}_{i\in J_{k^{\max}}}.$$

The multisets $G^{\min}$ and $G^{\max}$ are called the <u>initial</u> and <u>final</u> <u>submultisets</u> of G.

It is easy to see that any two partial permutohedra whose vertices are n-dimensional e-configurations induced by multisets with the same initial and final submultisets are combinatorically equivalent.

Let us formulate this observation as a statement.

Proposition 5.10. *(Sufficient condition 1 for combinatorial equivalence of partial multipermutohedra). If for G and G',*

$$G^{\min} = G'^{\min},\ G^{\max} = G'^{\max}, \tag{5.114}$$

then $\forall n \leq \min\{\eta, \eta'\}$

$$\Pi^n_{\eta' k'}(G') \cong \Pi^n_{\eta k}(G) \tag{5.115}$$

Here, $\eta' = |G'|,\ k' = |S(G')|$, $G'^{\min} = \{g'_i\}_{i \in J_n},\ G'^{\max} = \left\{g'_{\eta'-n+i}\right\}_{i \in J_n}$.

Let us formulate other conditions under which the relation (5.115) is satisfied, and a pair of partial multipermutohedra are combinatorically equivalent. They are based on the analysis of the primary specifications of G and G' along with their initial and final submultisets.

We consider an arbitrary multiset G of the form (1.7) and form G' from it by eliminating a certain number of its intermediate elements:

$$G' = G \backslash G^{n+1,\eta-n},\ \text{where } G^{n+1,\eta-n} = \{g_{n+1}, ..., g_{\eta-n}\}. \tag{5.116}$$

In this case, two situations occur depending on G's cardinality:

- Case 5.10.1. $\eta < 2n$, then

$$G^{n+1,\eta-n} = \varnothing,\ G' = G,\ G' \subset G^{\min} \cup G^{\max},\ \eta' = \eta,\ k' = k;$$

- Case 5.10.2. If

$$\eta \geq 2n, \tag{5.117}$$

else

$$G' = G,\ G' = G^{\min} \cup G^{\max},\ \eta' = 2n,\ k' \leq k.$$

Overall, for G', it is true:

$$\eta' = \min\{\eta, 2n\}, \ k' \leq k. \tag{5.118}$$

Consider the n-dimensional EPPPC induced by a multiset (5.116) and the corresponding polytope $\Pi^n_{\eta' k'}(G')$. It follows from the construction of G' that the condition (5.114) is satisfied, and by Proposition 5.10, this polytope is combinatorically equivalent to the original $\Pi^n_{\eta k}(G)$. Moreover, $\Pi^n_{\eta' k'}(G')$ is induced by an η'-element multiset satisfying the condition (5.118). Thus, when studying the combinatorial structure of the partial multipermutohedra, we can restrict ourselves to considering multisets such that

$$n + 1 \leq \eta \leq 2n. \tag{5.119}$$

To exclude from consideration those multisets whose primary specifications have the inverse order to that given in G, we also assume that G satisfies the below condition similar to (4.104):

$$\exists\ i \in J^0_{k+1}: \ \eta_j = \eta_{k-j+1}, \ j \in J_{i-1}; \eta_i > \eta_{k-i+1}, \tag{5.120}$$

where $\eta_0 = 0, \ \eta_{k+1} = n + 1$.

Similarly, for G' we assume that

$$\exists\ i' \in J^0_{k'+1}: \ \eta'_j = \eta'_{k'-j+1}, \ j \in J_{i'-1}; \eta_{i'} > \eta_{k'-i'+1}, \tag{5.121}$$

where $[G'] = (\eta'_i)_{i \in J_{k'}}$ and $\eta'_0 = 0, \ \eta'_{k'+1} = n + 1$.

As indicated in Section 4.8, if (4.104) holds, then two multipermutohedra $\Pi_{nk}(G)$ and $\Pi_{n'k'}(G)$ are combinatorically equivalent if the primary specifications of their inducing multisets coincide. For these partial multipermutohedra, this condition is sufficient only.

Proposition 5.11. *(Sufficient condition 2 for combinatorial equivalence of polytopes of class $\Pi^n_{\eta k}(G)$).*

If the multisets G and G' have identical primary specifications, then the corresponding

n-partial multipermutohedra are combinatorically equivalent, i.e. if

$$G, G' : \ [G] = [G'] \,, \tag{5.122}$$

than $\eta = \eta', \ n = n' \ u \ \Pi^n_{\eta' k'}(G') = \Pi^n_{\eta k}(G') \cong \Pi^n_{\eta k}(G)$.

On the other hand, it is clear that under the conditions (5.120) and (5.121), multisets with distinct primary specifications of initial and final submultisets induce polytopes that are not combinatorically equivalent: if G and G' such that $([G^{\min}], [G^{\max}]) \neq (\left[G'^{\min}\right], [G'^{\max}])$, then $\Pi^n_{\eta' k'}(G')$ and $\Pi^n_{\eta k}(G)$ are not combinatorically equivalent.

Let us formulate one more condition of combinatorial equivalence.

Proposition 5.12. *(A necessary condition for the combinatorial equivalence of polytopes of class* $\Pi^n_{\eta k}(G)$*).*

If partial multipermutohedra are combinatorically equivalent, then the primary specifications of their initial and final submultisets are identical:

$$[G^{\min}] = [G'^{\min}], \ [G^{\max}] = [G'^{\max}]. \tag{5.123}$$

Remark 5.11. It will be shown in Section 5.11 that, on the one hand, there exist non-combinatorically equivalent polytopes in class $\Pi^n_{\eta k}(G)$ for which the condition (5.123) holds. On the other hand, combinatorically equivalent polytopes of these classes exist that do not satisfy the condition (5.123).

Here, we review only the option if the conditions (5.122) and (5.123) are equivalent and, therefore, they yield a criterion for combinatorial equivalence of two partial permutohedra. Namely, it concerns Case 5.10.2, where the inequality (5.117), given (5.119), is converted to equality:

$$\eta = 2n. \tag{5.124}$$

It is easy to see that if (5.124) holds and

$$g_n < g_{n+1}, \tag{5.125}$$

the conditions (5.122) and (5.123) are equivalent. This is because $G^{\min} \cap G^{\max} = \varnothing$. Respectively, the primary specifications $[G^{\min}]$ and $[G^{\max}]$ are independent.

Remark 5.12. The second possible situation under fulfillment of the condition (5.124) is

$$\exists\, e_i \in J_k :\ g_n = g_{n+1} = e_i, \tag{5.126}$$

when the removal of one of these elements from the inducing multiset does impact the combinatorial structure of the polytope, according to Proposition 5.10:

$$\Pi^n_{\eta k}(G) \cong \Pi^n_{\eta'-1,k'}(G \setminus \{e_i\}).$$

Removal of multiple elements equal to e_i can be continued until the initial and final submultisets remain unchanged. Respectively, the combinatorial structure of the polytope does not change.

5.11 Illustration of $E^n_{\eta k}(G)$ and $\Pi^n_{\eta k}(G)$ $(n = 2, 3)$

Example 5.1. The following can occur for $n = 2$ under the condition (5.119):

- two options corresponding to $\eta = n + 1 = 3$. They are represented by the formula (4.105) and already discussed in Section 5.7);
- three options corresponding to $\eta = 4$:

$$[G_3] = \left(2^2\right),\ [G_4] = \left(2, 1^2\right),\ [G_5] = (1, 2, 1)\,.$$

In the first two cases, two polytopes, the partial permutohedron $\Pi^2_3(G_1)$ and the partial multipermutohedron $\Pi^2_{32}(G_2) = \Delta_{2,0,1}$ are shown in Figures 4.1 and 4.2. In the third case, we deal with the set $\overline{E}^2_2(G_3)$ and polytope $\overline{\Pi}^2_2(G_3)$ shown in Figure 5.1. In the last two cases, $k = 3 \geq 2$ and $\eta = 4 = n + 1$, i.e. none of the vertex locality conditions is satisfied, and we deal with the non-vertex-located sets $E^2_{43}(G_4)$ and $E^2_{43}(G_5)$. The partial

multipermutohedron $\Pi^2_{43}(G_4)$ and corresponding set $E^2_{43}(G_4)$ are shown in Figure 5.2. Finally, in the last case, we choose, for example, $G_5 = \{1, 2^2, 3\}$. This multiset can be reduced because the condition (5.126) is satisfied, $g_2 = g_3 = e_2 = 2$. Applying Remark 5.12, we get $\Pi^2_{43}(G_5) \cong \Pi^2_3(G_5 \setminus \{2\}) = \Pi^2_3(J_3) = \Pi^2_3(G_1)$, i.e. again we ended with the polytope shown in Figure 4.1.

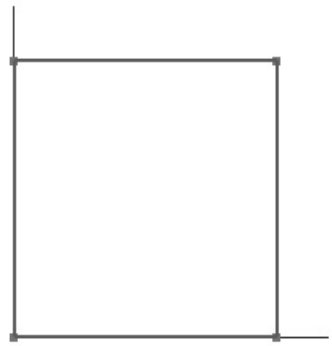

Figure 5.1: $\overline{\Pi}^2_2(G_3) = PB_2$

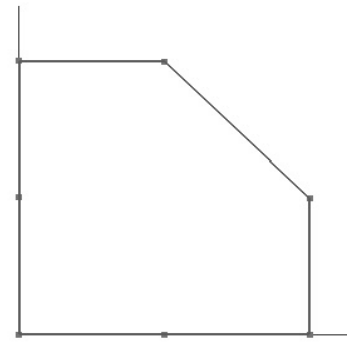

Figure 5.2: $\Pi^3_{42}(G_4)$

Example 5.2. The case of $n = 3$ corresponds to the three-dimensional partial multipermutohedra. All such sets $E^3_{\eta k}(G)$ can be divided into two groups:

1. Vertex-located three-dimensional EPMPCs (further referred to as Class 5.11.1);
2. Non-vertex three-dimensional EPMPCs (further referred to as Class 5.11.2).

   In turn, in the first group it can be singled out:

   (a) vertex-located EPMPCs not belonging to ESPPCs (see Section 5.7) (further referred to as Class 5.11.1.a) $(k > 2,\ \eta = n + 1 = 4)$;

   (b) the three-dimensional ESPPCs (further referred to as Class 5.11.1.b) $(k = 2)$.

Consider first Class 5.11.1.a. In addition to the EPMPC made from 3-dimensional e-configurations induced by the multisets $G_1 - G_5$ of the form (4.106), the following situation can occur:

$$[G_6] = (3, 2)\,, \quad [G_7] = \left(3^2\right). \tag{5.127}$$

By analogy with Example 4.3, we introduce the notation: $\forall\, i \in J_7$,

$$E^i = E^3_{\eta_i k_i}(G_i)\,,\ \eta_i = |G_i|\,,\ k_i = |S(G_i)|\,,\ P^i = conv\, E^i. \tag{5.128}$$

It is seen,

- $\eta_i = 4,\ i \in J_5;\ \eta_6 = 5,\ \eta_7 = 6;$
- $k_1 = 4,\ k_2 = k_3 = 3;\ k_i = 2,\ i = \overline{4,7}.$

Thus, Class 5.11.1.a includes

$$E^1 = E^3_4(G_1)\,,\ E^2 = E^3_{43}(G_2)\,,\ E^3 = E^3_{43}(G_3)\,,$$

shown in Figures 4.3-4.5 along with the polytopes $P^1 - P^3$.

The remaining sets (5.128) belong to Class 5.11.1.b. Two of them are

$$E^4 = E^3_{42}(G_5) = B_3(1,2)\,,\ E^5 = E^3_{42}(G_6) = B_3(0,1)\,.$$

Along with the corresponding $(0-1)$-partial permutohedra $\Delta_{3,1,2}$ and $\Delta_{3,0,1}$, they are shown in the Figures 4.6 and 4.7.

The following two figures, Figures 5.3 and 5.4), depict the EBPPCs induced by the below multisets given in the form of (5.127):

$$E^6 = E^3_{52}(G_6) = B_3(0,2)\,,\ E^7 = \overline{E}^3_2(G_7) = B_3,$$

and their corresponding polytopes.

In particular, Figure 5.3 depicts that $P^6 = \Delta_{3,0,2}$ has vertices of the degrees 3 and 4, i.e. this polytope has irregular vertices. The last set of this class is E^7, the set B_3 of three-dimensional $(0-1)$-vectors, which corresponds to the polytope that is the hypercube PB_3 shown in Figure 5.4.

Let us move on to Class 5.11.2. By (5.119), now, the search is limited to considering

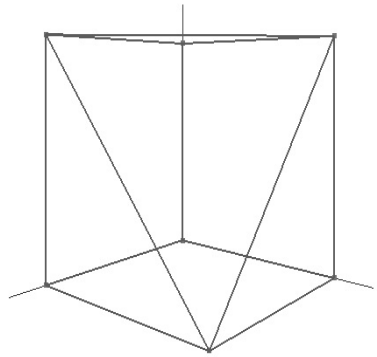

Figure 5.3: E^6 and P^6

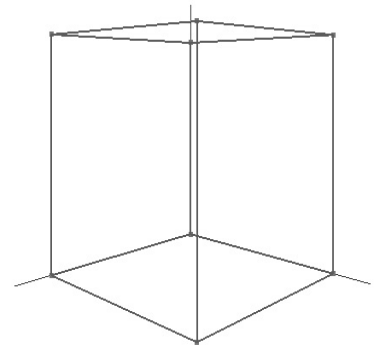

Figure 5.4: E^7 and P^7

G_i:

$$n + 2 = 5 \leq |G_i| \leq 2n = 6,\ 3 \leq |S(G_i)| \leq 6. \tag{5.129}$$

In Table 5.1, twelve possible primary specifications of multisets are listed that satisfy the conditions (5.120) and (5.129), with the numbering continued from $i = 8$. The main parameters of the multisets G_i are η_i, k_i and the value $\mathcal{R}^i$ of the regularity degree of P^i-vertices (or all possible degrees of the vertices) and the figure number with the image of E^i and P^i, $i = \overline{8, 19}$.

i	$[G_i]$	G_i	η_i	k_i	$\mathcal{R}_i$	Figure
8.	(1^5)	$G_8 = J_5$	5	5	3	5.5
9.	$(2, 1^3)$	$G_9 = \{0^2, 1, 2, 3\}$	5	4	3	5.6
10.	$(2, 1, 2)$	$G_{10} = \{0^2, 1, 2^2\}$	5	3	3	5.7
11.	$(3, 1^2)$	$G_{11} = \{0^3, 1, 2\}$	5	3	3	5.8
12.	$(1, 2, 1^2)$	$G_{12} = \{0, 1^2, 2, 3\}$	5	4	3,4	5.9
13.	$(2^2, 1)$	$G_{13} = \{0^2, 1^2, 2\}$	5	3	3,4	5.10
14.	$(1, 3, 1)$	$G_{14} = \{0, 1^3, 2\}$	5	3	4	5.11
15.	$(1, 2^2, 1)$	$G_{15} = \{0, 1^2, 2^2, 3\}$	6	4	3,4	5.12
16.	$(3, 2, 1)$	$G_{16} = \{0^3, 1^2, 2\}$	6	3	3,4	5.13
17.	$(2, 1, 2, 1)$	$G_{17} = \{0^2, 1, 2^2, 3\}$	6	4	3,4	5.14
18.	(2^3)	$G_{18} = \{0^2, 1^2, 2^2\}$	6	3	3	5.15
19.	$(3, 1, 2)$	$G_{19} = \{0^3, 1, 2^2\}$	6	3	3	5.16

Table 5.1: Class 5.11.2: parameters of the sets E^i and polytopes P^i, $i = \overline{8, 19}$

Below are images of the polytopes and sets of this class.

Let us analyze and compare the results with the ones obtained above. It is easy to see that some of the polytopes of Class 5.11.2 are combinatorically equivalent to those of

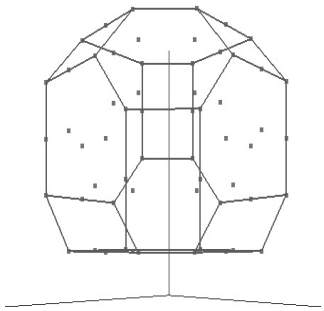

Figure 5.5: E^8 and P^8

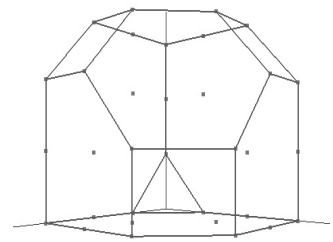

Figure 5.6: E^9 and P^9

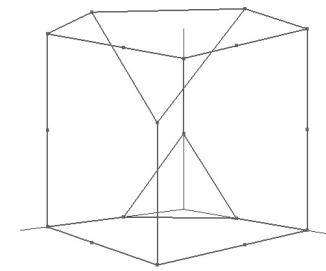

Figure 5.7: E^{10} and P^{10}

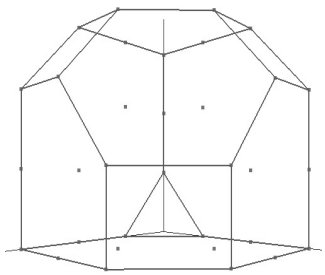

Figure 5.8: E^{11} and P^{11}

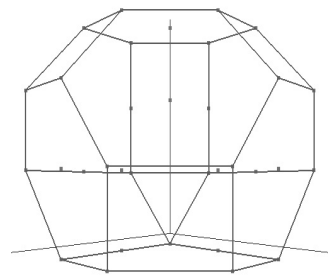

Figure 5.9: E^{12} and P^{12}

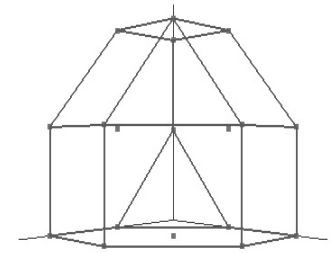

Figure 5.10: E^{13} and P^{13}

Class 5.11.1. Namely,

1. The polytopes of both classes are combinatorically equivalent polytopes $P^1 \cong P^8$. Moreover, sufficient conditions 1 and 2 do not detect this.

2. $P^3 \cong P^{14}$. Since $G_3 = \{0, 1^2, 2\}$, $G_{14} = \{0, 1^3, 2\}$, here, the sufficient condition 1 of combinatorial equivalence is satisfied, and applying Remark 5.12, we obtain

$$P^{14} = \Pi^3_{53}(G_{14}) \cong \Pi^3_{43}(G_{14} \setminus \{1\}) = \Pi^3_{43}(G_3) = P^3;$$

3. $P^{10} \cong P^{18}$. Here, the combinatorial equivalence is established similarly to the previous case, taking into account that

$$P^{18} = \Pi^3_{63}(G_{18}) \cong \Pi^3_{53}(G_{18} \setminus \{1\}) = \Pi^3_{53}(G_{10}) = P^{10}.$$

4. $P^{13} \cong P^{17}$. For the inducing multisets $G_{13} = \{0^2, 1^2, 2\}$, $G_{17} = \{0^2, 1, 2^2, 3\}$ $[G^{\min}] =$

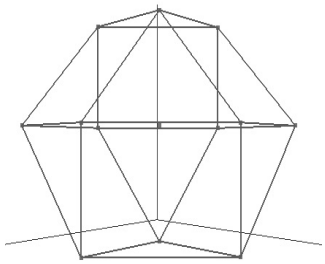

Figure 5.11: E^{14} and P^{14}

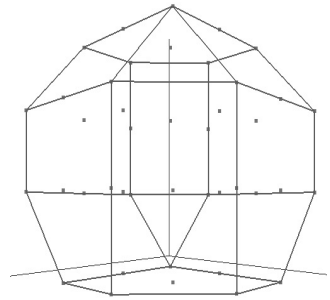

Figure 5.12: E^{15} and P^{15}

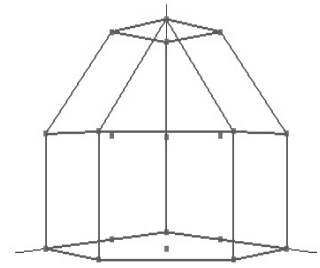

Figure 5.13: E^{16} and P^{16}

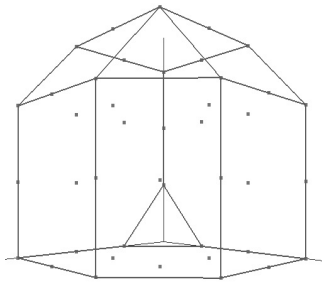

Figure 5.14: E^{17} and P^{17}

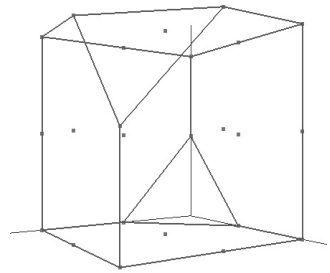

Figure 5.15: E^{18} and P^{18}

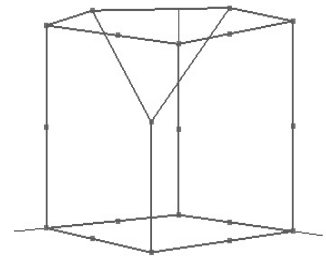

Figure 5.16: E^{19} and P^{19}

$[G^{\max}] = (2, 1)$, i.e. the necessary condition of combinatorial equivalence holds, while both the sufficient conditions are violated.

Finally, we check the condition (5.123) for the multisets:

$$G = G_3, \ G' = G_{15}.$$

Given that $G_3 = \{0, 1^2, 2\}$, $G_{15} = \{0, 1^2, 2^2, 3\}$, we have

$$\left[G^{\min}\right] = \left[G'^{\min}\right] = (1, 2)\,, \ \ [G^{\max}] = \left[G'^{\max}\right] = (2, 1)\,,$$

therefore, the necessary condition (5.123) is satisfied, while P^3 and P^{15} are not combinatorically equivalent. Note that the multiset G_{15} satisfies the conditions (5.124) and (5.125). Respectively, its reduction changes the combinatorial structure of the polytope, according to Remark 5.12. This can be seen in the example of P^3 and P^{15}.

Thus, seven different types of polytopes were found in class $\Pi^2_{\eta k}(G)$. For dimension three, taking into account the combinatorically equivalent polytopes among those listed, it is established that there are fifteen non-combinatorically equivalent types of $\Pi^3_{\eta k}(G)$.

Conclusion

The main result of the presented monographic research is the study of combinatorial point configurations formed from mapping a set of combinatorial configurations into Euclidean space. The identification of combinatorial point configurations into a special class became possible thanks to an integrated approach. On the one hand, the approach is based on the peculiarities of finite point configurations in Euclidean space. On the other hand, it utilizes properties of combinatorial configurations. Therefore, the monograph pays special attention to deriving and systematizing these properties.

The concept of combinatorial point configuration is introduced as mapping a finite abstract set of a certain structure into Euclidean space. Algebro-topological and topological-metric properties of combinatorial point configurations and corresponding combinatorial polytopes are derived. A bijective mapping of finite point configurations with specific sets of combinatorial configuration sets is established. The class of so-called Euclidean combinatorial sets is further studied using specifics of their immersion into Euclidean space.

A general approach is proposed to decompose finite point configurations into hyperplanes and partition them into pairwise disjoint subsets. A typology of surface-located sets is proposed based on the properties of strictly convex surfaces, including classes of ellipsoidally, spherically and super spherically located sets. Classes of vertex-located and polyhedral-surface sets are singled out, and their properties are explored. Approaches to decomposing finite point configurations into vertex-located subsets are offered. Polyhedral combinatorics have been further developed in the study of convex hulls of finite-point configurations. Analytical forms for describing the corresponding combinatorial polytopes, including multi-level ones, are

found. Approaches to the functional-analytic representation of various classes of finite point configurations are proposed and theoretically substantiated.

The main attention in the monographs is paid to the study of permutation point configurations, multipermutation point configurations, partial permutation point configurations, and partial multipermutation point configurations. The configuration found multiple applications as a search domain in operations research, optimization and polyhedral combinatorics. First of all, we refer to problems of combinatorial and discrete optimization [25, 32, 37, 38, 70, 72, 73, 110]. If a search domain of such problems is a combinatorial point configuration, we deal with the so-called Euclidean combinatorial optimization problem is identified. The fundamental principles of Euclidean combinatorial optimization are discussed in detail in [18, 39, 40, 48, 49, 79, 83–85, 90, 92–96, 98, 104].

Indeed, the feasible domain of discrete optimization problems is a set of isolated points in Euclidean space, that is, a finite point configuration. This means every finite point configuration is combinatorial since the coordinates of each its point form an ordered sequence of real numbers taken from a finite set. Therefore, when formalizing the feasible domain, one can use the functional-analytic representations of finite point configurations offered in Chapters 1 and 2.

Most techniques for solving discrete optimization problems utilize decompositions of discrete sets into subsets, particularly partitions. Among these methods are branches and bounds, sequential analysis of variants, cutting-plane methods, etc. General approaches to decomposing finite point configurations proposed in this monograph can be used when implementing these methods. The contributions concerning discrete sets' decompositions enable the extension of the class of combinatorial optimization problems solvable through the decomposition of search domains.

Let Π be a finite set whose elements are combinatorial configurations and on which the functional

$$\xi : \Pi \to \mathbb{R}^1$$

is given.

The objective is to find

$$\pi^* = \arg \min_{\pi \in P \subseteq \Pi} \xi(\pi),$$

where $P \subseteq \Pi$ is the feasible domain.

We establish the bijection mapping $\varphi : \Pi \to E$ between the combinatorial configurations π constituting the set Π and a point $x = (x_1, x_2, ..., x_n)$ of a certain set $E \subset \mathbb{R}^n$ by setting

$$x = \varphi(\pi), \quad \pi = \varphi^{-1}(x).$$

Let the function $f : E \to \mathbb{R}^1$ be such that $f(x) = \xi(\varphi^{-1}(x))$ for all $x \in E$. Then the original problem can be equivalently formulated as finding

$$x^* = \arg \min_{x \in X \subseteq E} f(x),$$

where $X = \varphi(P)$ is an image of P in $\mathbb{R}^n$.

Note that when formalizing the mapping $\varphi : \Pi \to E \subset \mathbb{R}^n$, one can use the basic schemes for the formation of various classes of combinatorial point configurations described in Chapters 3-5.

Let $E\,(A) \subset \mathbb{R}^n$ be a set of points of the form $x = (x_1, x_2, ..., x_n)$, where $x_i \in A$, $i \in J_n$. Then each point $x \in E$ is a combinatorial configuration under the mapping $\varphi : \Pi \to E \subset \mathbb{R}^n$. The condition $X = \phi(P)$ specifies the corresponding class of combinatorial point configurations.

The above reasoning allows considering the optimization problem over the set Π of combinatorial configurations under the mapping $\varphi : \Pi \to E \subset \mathbb{R}^n$ as a discrete optimization problem on the combinatorial point configuration, i.e. on a finite set of vectors in Euclidean space.

A specific form of constraints on mappings φ for some classes of combinatorial point configurations is proposed in Chapters 4 and 5 of the monograph. To describe the set $X \subset E$, general approaches to the functional-analytic representation of combinatorial point configurations proposed in Chapters 1, 2 are applicable. Moreover, from the point of view of

such a description, of particular interest are vertex-located and polyhedral-spherical combinatorial point configurations [39–42, 45, 48, 49, 51, 52, 86]. Such sets laid the foundations for the theory of convex extensions of functions defined on the vertices of convex polytopes [83, 84, 90, 91]. The current state of the theory of convex extensions and continuous functional representations of combinatorial point configurations are presented in the papers [18, 39–42, 44, 48, 50, 98, 98, 105, 105]. The properties of such sets underlie methods of polyhedral-spherical optimization, the main ideas of which are described in [86, 90]. and further developed in [30, 43, 46, 47, 53–62, 77, 78, 87, 89, 98–104].

For vertex-located combinatorial point configurations, the theory of convex extensions of functions is developed [83, 84, 90, 91], and the current state is covered in the publications [18, 40, 48, 49, 98, 105]. General approaches to continuous functional representation of various classes of combinatorial point configurations are presented in [39–41, 48–50].

In summary, the monograph offers an in-depth exploration of trends and processes that are effectively modeled through diverse combinatorial point configurations. The unique properties inherent in these configurations present valuable opportunities for innovating new optimization methods across a broad spectrum of application domains.

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The monograph is dedicated to exploring combinatorial point configurations derived from mapping a set of combinatorial configurations into Euclidean space. Various methods for this mapping, along with the typology and properties of the resultant configurations, are presented. In addition, the study revolves around combinatorial polytopes defined as convex hulls of combinatorial point configurations. The primary focus lies in examining multipermutation and partial multipermutation point configurations alongside their associated combinatorial polytopes known as multipermutohedra and partial multipermutohedra. Our theoretical contributions are substantiated through the proof of theorems and supporting auxiliary statements. Examples and illustrations are included to enhance the comprehension of the material.

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