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ABSTRACT OF THE DOCTORAL THESIS

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Wigner function on topologically non-trivial manifolds

The aim of this work is to investigate the possibilities of constructing Wigner quasiprobability distributions for quantum mechanics on manifolds with non-trivial topology and to study the properties of the obtained functions. This issue is placed in the broader context of phase space formulation of quantum mechanics. The study of Wigner functions involves the use of the tools of quantum mechanics in phase space, such as coherent states, the Segal-Bargmann transform and the Husimi function, so the objects mentioned were also of interest in this work.

At first, the basic tools of quantum mechanics in phase space are discussed in the case where the configuration space of the system and its phase space are topologically equivalent manifolds of Euclidean space.

Two cases with non-trivial topology are then addressed: the circle and the sphere. These are manifolds of clear physical importance, primarily as manifolds indexing angles of orientation on a plane and in space, respectively. They are also manifolds with significantly different topologies. The main difference, however, concerns not them *per se* but their tangent bundles, which serve as phase spaces. The tangent bundle to the circle is a trivial bundle, while the tangent bundle to the sphere is not, which has profound implications.

In particular, in the case of the circle, an analysis and comparison of known coherent state constructions and Wigner function constructions are presented. The investigated properties of the Wigner function for coherent states on the circle are presented. Furthermore, some aspects of the free evolution of coherent states on the circle are studied.

For a sphere, an analysis and comparison of selected coherent state constructions is also presented. The construction of a Wigner function for a sphere is introduced, and its basic properties are discussed. As far as the author is aware, the presented Wigner function proposition for the sphere is the only example published to date of a Wigner function defined on a phase space that is a non-trivial bundle. The basic properties of the Wigner function in coherent states on the sphere are investigated.

The methods of directional statistics have been used to describe the statistical aspects of quantum mechanics on non-trivial manifolds. The foundations of the statistics of variables belonging to manifolds other than $\mathbb{R}^n$ are discussed, and its methods are applied to the description of quantum states on the circle and the sphere. In particular, the uncertainties of angular position and angular momentum in coherent states on the circle were investigated using the definition of variance valid in directional statistics. The location of the intrinsic mean of the angular position in

coherent states on the circle evolving freely was also investigated and compared with the location of the extrinsic mean. The form of jumps of these averages has been compared.

A general method for constructing Wigner functions for quantum mechanics on non-trivial manifolds by means of the Segal-Bargmann transform of the wave function, calculation of the Husimi function, and the inverse (generalised) Gauss-Weierstrass transform has been proposed. It has been tested in topologically trivial cases and for a circle.

Properties of the Wigner function in coherent states on the circle and the construction of the Wigner function for the sphere were the subjects of published papers. The other results mentioned have not been published.