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# A NOTE ON DIVERGENCE-FREE POLYNOMIAL DERIVATIONS IN POSITIVE CHARACTERISTIC

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Abstract. In this paper we discuss an explicit form of divergence-free polynomial derivations in positive characteristic. It involves Jacobian derivations.

## 1. Introduction

Throughout this paper by a ring we mean a commutative ring with unity.

Let $K$ be a ring. Recall that if $d$ is a $K$-derivation of the polynomial algebra $K[x_1,\ldots,x_n]$ of the form $d = g_1\dfrac{\partial}{\partial x_1} + \ldots + g_n\dfrac{\partial}{\partial x_n}$, where $g_1,\ldots,g_n \in K[x_1,\ldots,x_n]$, then a polynomial

$$d^* = \frac{\partial g_1}{\partial x_1} + \ldots + \frac{\partial g_n}{\partial x_n}$$

is called the divergence of $d$. The derivation $d$ is called divergence-free if $d^* = 0$. See [2] and [3] for information on the properties of the divergence.

Given a polynomial $f \in K[x_1,\ldots,x_n]$, we denote by $d^f_{ij}$ a Jacobian derivation of the form

$$d^f_{ij}(g) = \begin{vmatrix} \dfrac{\partial f}{\partial x_i} & \dfrac{\partial f}{\partial x_j} \\ \dfrac{\partial g}{\partial x_i} & \dfrac{\partial g}{\partial x_j} \end{vmatrix}$$

for $g \in K[x_1,\ldots,x_n]$. Of course, $(d^f_{ij})^* = 0$.

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In the case of positive characteristic, a divergence-free derivation in two variables is closely related to a single Jacobian derivation, see [1], Theorem 4.5. In the general case of $n$ variables a divergence-free derivation is related to a sum of Jacobian ones (Theorem 3.1 below). This fact is well known (compare [4]). The aim of this paper is to give an elementary proof.

Note that the sum in Theorem 3.1 (ii) is different than the one in Theorem 8.3 of Nowicki's article [2]. There arises a natural question if, under the assumptions of Theorem 3.1, the derivation $d$ can be presented in the form $d = \sum_{i=1}^{n-1} d_{i,i+1}^{f_i} + \delta'$, where $f_1, \dots, f_{n-1} \in K[x_1, \dots, x_n]$, and $\delta'$ is a $K$-derivation satisfying

$$\frac{\partial}{\partial x_1}(\delta'(x_1)) = \ldots = \frac{\partial}{\partial x_n}(\delta'(x_n)) = 0.$$

## 2. Preliminaries

Let $K$ be a ring. Let $r$ be a positive integer and $i \in \{1, \dots, n\}$. For every polynomial $g \in K[x_1, \dots, x_n]$ we denote by $g_{i,0}$, $g_{i,1}$, $\dots$, $g_{i,r-1}$ the uniquely determined polynomials belonging to $K[x_1, \dots, x_{i-1}, x_i^r, x_{i+1}, \dots, x_n]$ satisfying the condition

$$g = g_{i,0} + g_{i,1}x_i + \ldots + g_{i,r-1}x_i^{r-1}.$$

The following easy observations will be used in the rest of the paper.

**Lemma 2.1.** *Every polynomial $g \in K[x_1, \dots, x_n]$ can be uniquely presented in the form*

$$g = u + wx_i^{r-1},$$

*where $u \in K[x_1, \dots, x_n]$, $u_{i,r-1} = 0$ and $w \in K[x_1, \dots, x_{i-1}, x_i^r, x_{i+1}, \dots, x_n]$. In particular, if $g = 0$, then $u = 0$ and $w = 0$.*

**Lemma 2.2.** *If $g \in K[x_1, \dots, x_{i-1}, x_i^r, x_{i+1}, \dots, x_n]$ and $j \in \{1, \dots, n\}$, $j \neq i$, then*

$$\frac{\partial g}{\partial x_j} \in K[x_1, \dots, x_{i-1}, x_i^r, x_{i+1}, \dots, x_n].$$

In Lemmas 2.3 and 2.4 we assume that $K$ is a ring of prime characteristic $p$.

**Lemma 2.3.** *Consider a polynomial $g \in K[x_1, \dots, x_n]$. The polynomial $g$ can be presented in the form*

$$g = \frac{\partial v}{\partial x_i}$$

*for some $v \in K[x_1, \dots, x_n]$ if and only if $g_{i,p-1} = 0$.*

**Lemma 2.4.** *Consider a polynomial $g \in K[x_1, \dots, x_n]$. The polynomial $g$ belongs to $K[x_1, \dots, x_{i-1}, x_i^p, x_{i+1}, \dots, x_n]$ if and only if*

$$\frac{\partial g}{\partial x_i} = 0.$$

## 3. Poincaré-type Lemma

Let $K$ be a ring of prime characteristic $p$. Consider the algebra of polynomials $K[x_1,\ldots,x_n]$. For $i=1,\ldots,n$ put $z_i = \prod_{\substack{j=1,\ldots,n\\ j\neq i}} x_j$.

**Theorem 3.1** (Poincaré-type Lemma). *Let $d$ be a $K$-derivation of $K[x_1,\ldots,x_n]$, where $n \geqslant 2$. The following conditions are equivalent:*

(i) $d^* = 0$,

(ii) $d = \sum_{1\leqslant i<j\leqslant n} d_{ij}^{f_{ij}} + \delta$, *where $f_{ij} \in K[x_1,\ldots,x_n]$ for $i<j$ and*

$$\delta = h_1 z_1^{p-1}\frac{\partial}{\partial x_1} + h_2 z_2^{p-1}\frac{\partial}{\partial x_2} + \ldots + h_n z_n^{p-1}\frac{\partial}{\partial x_n}$$

*for some $h_1,\ldots,h_n \in K[x_1^p,\ldots,x_n^p]$.*

Note a version of the above theorem for $n=1$.

**Lemma 3.2.** *If $d$ is a $K$-derivation of $K[x]$, then the following conditions are equivalent:*

(i) $d^* = 0$,

(ii) $d = h\dfrac{\partial}{\partial x}$, *where $h \in K[x^p]$.*

*Remark* 3.3. It is convenient to express Theorem 3.1 in terms of polynomials $g_1,\ldots,g_n \in K[x_1,\ldots,x_n]$, where $d = g_1\frac{\partial}{\partial x_1} + \ldots + g_n\frac{\partial}{\partial x_n}$.

The following conditions are equivalent:

(i) $\dfrac{\partial g_1}{\partial x_1} + \ldots + \dfrac{\partial g_n}{\partial x_n} = 0$,

(ii) there exist $f_{ij} \in K[x_1,\ldots,x_n]$ for $i<j$, and $h_1,\ldots,h_n \in K[x_1^p,\ldots,x_n^p]$, such that

$$\begin{aligned}
g_1 &= -\frac{\partial f_{12}}{\partial x_2} - \ldots - \frac{\partial f_{1n}}{\partial x_n} + h_1 z_1^{p-1},\\
g_2 &= \frac{\partial f_{12}}{\partial x_1} - \frac{\partial f_{23}}{\partial x_3} - \ldots - \frac{\partial f_{2n}}{\partial x_n} + h_2 z_2^{p-1},\\
&\vdots\\
g_{n-1} &= \frac{\partial f_{1,n-1}}{\partial x_1} + \ldots + \frac{\partial f_{n-2,n-1}}{\partial x_{n-2}} - \frac{\partial f_{n-1,n}}{\partial x_n} + h_{n-1} z_{n-1}^{p-1},\\
g_n &= \frac{\partial f_{1n}}{\partial x_1} + \ldots + \frac{\partial f_{n-1,n}}{\partial x_{n-1}} + h_n z_n^{p-1}.
\end{aligned}$$

## 4. Proof

*Proof.* Obviously, (ii) implies (i).

(i) $\Rightarrow$ (ii) We proceed by induction.

Let $n \geqslant 1$. Consider an arbitrary $K$-derivation of $K[x_1, \ldots, x_n, x_{n+1}]$

$$d = g_1 \frac{\partial}{\partial x_1} + \ldots + g_n \frac{\partial}{\partial x_n} + g_{n+1} \frac{\partial}{\partial x_{n+1}},$$

where $g_1, \ldots, g_{n+1} \in K[x_1, \ldots, x_{n+1}]$.

For each $i \in \{2, \ldots, n, n+1\}$ we present $g_i$ in the form $g_i = u_i + w_i x_1^{p-1}$, where $u_i \in K[x_1, \ldots, x_{n+1}]$, $(u_i)_{1,p-1} = 0$, $w_i \in K[x_1^p, x_2, \ldots, x_{n+1}]$ (Lemma 2.1). Moreover, we have $u_{n+1} = \frac{\partial v_1}{\partial x_1}$ for some $v_1 \in K[x_1, \ldots, x_{n+1}]$ (Lemma 2.3), so $g_{n+1} = \frac{\partial v_1}{\partial x_1} + w_{n+1} x_1^{p-1}$.

Now we compute the divergence:

$$\begin{aligned} d^* &= \frac{\partial g_1}{\partial x_1} + \ldots + \frac{\partial g_n}{\partial x_n} + \frac{\partial g_{n+1}}{\partial x_{n+1}} \\ &= \frac{\partial g_1}{\partial x_1} + \frac{\partial}{\partial x_2}\big(u_2 + w_2 x_1^{p-1}\big) + \ldots + \frac{\partial}{\partial x_n}\big(u_n + w_n x_1^{p-1}\big) + \\ &\quad \frac{\partial}{\partial x_{n+1}}\big(\frac{\partial v_1}{\partial x_1} + w_{n+1} x_1^{p-1}\big) \\ &= \frac{\partial g_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \ldots + \frac{\partial u_n}{\partial x_n} + \frac{\partial^2 v_1}{\partial x_1 \partial x_{n+1}} + \\ &\quad \frac{\partial w_2}{\partial x_2} x_1^{p-1} + \ldots + \frac{\partial w_n}{\partial x_n} x_1^{p-1} + \frac{\partial w_{n+1}}{\partial x_{n+1}} x_1^{p-1}. \end{aligned}$$

We obtain:

$$d^* = F + G x_1^{p-1},$$

where

$$F = \frac{\partial}{\partial x_1}\big(g_1 + \frac{\partial v_1}{\partial x_{n+1}}\big) + \frac{\partial u_2}{\partial x_2} + \ldots + \frac{\partial u_n}{\partial x_n},$$

$F \in K[x_1, \ldots, x_{n+1}]$, $F_{1,p-1} = 0$, and

$$G = \frac{\partial w_2}{\partial x_2} + \ldots + \frac{\partial w_{n+1}}{\partial x_{n+1}},$$

$G \in K[x_1^p, x_2, \ldots, x_{n+1}]$.

By Lemma 2.1, if $d^* = 0$, then $F = 0$ and $G = 0$.

Now, let $n = 1$ and assume that $d^* = 0$. We have $g_2 = \dfrac{\partial v_1}{\partial x_1} + w_2 x_1^{p-1}$, where $v_1 \in K[x_1, x_2]$, $w_2 \in K[x_1^p, x_2]$. Moreover, $\dfrac{\partial}{\partial x_1}\big(g_1 + \dfrac{\partial v_1}{\partial x_2}\big) = F = 0$ and $\dfrac{\partial w_2}{\partial x_2} = G = 0$. Hence, $g_1 + \dfrac{\partial v_1}{\partial x_2} \in K[x_1^p, x_2]$ and $w_2 \in K[x_1^p, x_2^p]$. Then $g_1 + \dfrac{\partial v_1}{\partial x_2} = u_1 + w_1 x_2^{p-1}$ for some $u_1 \in K[x_1^p, x_2]$, $(u_1)_{2,p-1} = 0$, $w_1 \in K[x_1^p, x_2^p]$ (Lemma 2.1), so $u_1 = \dfrac{\partial v_2}{\partial x_2}$ for some $v_2 \in K[x_1^p, x_2]$ (Lemma 2.3). Finally,

$$g_1 = -\frac{\partial(v_1 - v_2)}{\partial x_2} + w_1 x_2^{p-1}, \quad g_2 = \frac{\partial(v_1 - v_2)}{\partial x_1} + w_2 x_1^{p-1},$$

where $w_1, w_2 \in K[x_1^p, x_2^p]$. We have shown that for $n = 2$ condition (i) implies (ii).

Assume the statement for some $n \geqslant 2$, assume that $d^* = 0$.

By the inductive assumption for $K_1[x_1, \ldots, x_n]$, where $K_1 = K[x_{n+1}]$, since $F = 0$, we have:

$$\begin{aligned}
g_1 + \frac{\partial v_1}{\partial x_{n+1}} &= -\frac{\partial f_{12}}{\partial x_2} - \ldots - \frac{\partial f_{1n}}{\partial x_n} + h_1 z_1^{p-1},\\
u_2 &= \frac{\partial f_{12}}{\partial x_1} - \frac{\partial f_{23}}{\partial x_3} - \ldots - \frac{\partial f_{2n}}{\partial x_n} + h_2 z_2^{p-1},\\
&\vdots\\
u_{n-1} &= \frac{\partial f_{1,n-1}}{\partial x_1} + \ldots + \frac{\partial f_{n-2,n-1}}{\partial x_{n-2}} - \frac{\partial f_{n-1,n}}{\partial x_n} + h_{n-1} z_{n-1}^{p-1},\\
u_n &= \frac{\partial f_{1n}}{\partial x_1} + \ldots + \frac{\partial f_{n-1,n}}{\partial x_{n-1}} + h_n z_n^{p-1},
\end{aligned}$$

where $f_{ij} \in K_1[x_1, \ldots, x_n]$, $h_i \in K_1[x_1^p, \ldots, x_n^p]$ and $z_i = \prod\limits_{\substack{j=1,\ldots,n\\ j\neq i}} x_j$.

Moreover, by the inductive assumption for $K_2[x_2, \ldots, x_{n+1}]$, where $K_2 = K[x_1^p]$, since $G = 0$, we have:

$$\begin{aligned}
w_2 &= -\frac{\partial s_{23}}{\partial x_3} - \ldots - \frac{\partial s_{2,n+1}}{\partial x_{n+1}} + a_2 y_2^{p-1},\\
w_3 &= \frac{\partial s_{23}}{\partial x_2} - \frac{\partial s_{34}}{\partial x_4} - \ldots - \frac{\partial s_{3,n+1}}{\partial x_{n+1}} + a_3 y_3^{p-1},\\
&\vdots\\
w_n &= \frac{\partial s_{2,n}}{\partial x_2} + \ldots + \frac{\partial s_{n-1,n}}{\partial x_{n-1}} - \frac{\partial s_{n,n+1}}{\partial x_{n+1}} + a_n y_n^{p-1},\\
w_{n+1} &= \frac{\partial s_{2,n+1}}{\partial x_2} + \ldots + \frac{\partial s_{n,n+1}}{\partial x_n} + a_{n+1} y_{n+1}^{p-1},
\end{aligned}$$

where $s_{ij} \in K_2[x_2, \dots, x_{n+1}]$, $a_2, \dots, a_{n+1} \in K_2[x_2^p, \dots, x_{n+1}^p]$ and $y_i = \prod_{\substack{j=2,\dots,n+1 \\ j\neq i}} x_j$ for $i = 2, \dots, n+1$.

Now, we present each $h_i$, for $i = 1, \dots, n$, in the form $h_i = b_i + c_i x_{n+1}^{p-1}$, where $b_i \in K[x_1^p, \dots, x_n^p, x_{n+1}]$, $(b_i)_{n+1,p-1} = 0$, $c_i \in K[x_1^p, \dots, x_n^p, x_{n+1}^p]$ (Lemma 2.1). Then $b_i = \frac{\partial q_i}{\partial x_{n+1}}$ for some $q_i \in K[x_1^p, \dots, x_n^p, x_{n+1}]$, so $h_i = \frac{\partial q_i}{\partial x_{n+1}} + c_i x_{n+1}^{p-1}$ and $\frac{\partial q_i}{\partial x_j} = 0$ for $j = 1, \dots, n$. Denote: $t_i = \prod_{\substack{j=1,\dots,n+1 \\ j\neq i}} x_j$ for $i = 1, \dots, n+1$. We obtain

$$
\begin{aligned}
g_1 &= -\frac{\partial f_{12}}{\partial x_2} - \dots - \frac{\partial f_{1n}}{\partial x_n} - \frac{\partial (v_1 - q_1 x_2^{p-1} \dots x_n^{p-1})}{\partial x_{n+1}} + c_1 t_1^{p-1}, \\
g_2 &= \frac{\partial f_{12}}{\partial x_1} - \frac{\partial (f_{23} + s_{23} x_1^{p-1})}{\partial x_3} - \dots - \frac{\partial (f_{2n} + s_{2n} x_1^{p-1})}{\partial x_n} \\
&\quad - \frac{\partial (s_{2,n+1} x_1^{p-1} - q_2 x_1^{p-1} x_3^{p-1} \dots x_n^{p-1})}{\partial x_{n+1}} + (c_2 + a_2) t_2^{p-1}, \\
&\vdots \\
g_n &= \frac{\partial f_{1n}}{\partial x_1} + \frac{\partial (f_{2n} + s_{2n} x_1^{p-1})}{\partial x_2} + \dots + \frac{\partial (f_{n-1,n} + s_{n-1,n} x_1^{p-1})}{\partial x_{n-1}} \\
&\quad - \frac{\partial (s_{n,n+1} x_1^{p-1} - q_n x_1^{p-1} x_2^{p-1} \dots x_{n-1}^{p-1})}{\partial x_{n+1}} + (c_n + a_n) t_n^{p-1}, \\
g_{n+1} &= \frac{\partial (v_1 - q_1 x_2^{p-1} \dots x_n^{p-1})}{\partial x_1} + \frac{\partial (s_{2,n+1} x_1^{p-1} - q_2 x_1^{p-1} x_3^{p-1} \dots x_n^{p-1})}{\partial x_2} + \dots \\
&\quad + \frac{\partial (s_{n,n+1} x_1^{p-1} - q_n x_1^{p-1} x_2^{p-1} \dots x_{n-1}^{p-1})}{\partial x_n} + a_{n+1} t_{n+1}^{p-1}.
\end{aligned}
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□

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