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Chapter 17 Measurability of multifunctions with the (J) property

GRAŻYNA KWIECIŃSKA

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In various problems, one encounters measurability of multifunctions (called also set-valued functions) of two variables. Obviously, each multifunction of two variables $x \in X$ and $y \in Y$ may be treated as a multifunction of the single variable $(x, y) \in X \times Y$. The essential difference is the possibility of formulating hypotheses concerning the multifunction in terms of its sectionwise properties. In this case, we can speak about product (sometimes called joint) measurability and superpositional measurability (sup-measurability for short), i.e., roughly speaking, measurability with respect to a product σ -algebra and measurability of Carathéodory type superposition F(x, G(x)), respectively, where F and Gare multifunctions.

The difference between sup-measurability and joint measurability is essential. In general, neither of the inclusions between the class of joint measurable multifunctions and the class of sup-measurable multifunctions is true. It is easy to define a joint Lebesgue measurable real function which is not supmeasurable [15]. On the other hand Z. Grande and J. S. Lipiński have given an example of a sup-measurable real function which is not measurable as a function of two variables [8]. In the single valued version, the problem of product measurability and supmeasurability has been studied very extensively (an overview of some papers in this field can be found in [7]). An important contribution to this field, among others, has made J. S. Lipiński. Far less is known, however, in the multivalued case.

There are various sufficient conditions on sections of f ensuring that f is product measurable (e.g. [2], [3] and [5]–[7]). The most important one (given by H. D. Ursell [14]) is measurability of f in the first and its continuity in the second variable. The measurability of f can be obtained from weaker assumptions. J. S. Lipiński [11] has shown that under an additional assumption one can obtain product measurability of f if it is a derivative in the second variable. In order to attain this result he introduced the (J) property of a real function of two real variables (intensively studied by Z. Grande in the case of real functions defined on more general spaces [7]). Our purpose is to consider this topic in the case of multifunctions.

Let *S* and *Z* be nonempty sets and let Φ be a mapping which associates to each point $s \in S$ a nonempty set $\Phi(s) \subset Z$. Such a mapping is called *a multifunction* from *S* to *Z* and we write $\Phi : S \rightsquigarrow Z$.

If $\Phi : S \rightsquigarrow Z$ is a multifunction, then for a set $A \subset Z$ two *inverse images* of *A* under Φ are defined as follows:

$$\Phi^+(A) = \{s \in S : \Phi(s) \subset A\} \text{ and } \Phi^-(A) = \{s \in S : \Phi(s) \cap A \neq \emptyset\}.$$

A function $f: S \to Z$ may be considered as a multifunction assigning to $s \in S$ the singleton $\{f(s)\}$. It is clear that in this case for a set $A \subset Z$ we have

$$f^+(A) = f^-(A) = f^{-1}(A).$$

Let us suppose that (Z,d) is a metric space. If $z_0 \in Z$ and $M \subset Z$, then in standard notation, $d(z_0,M) = \inf_{z \in M} \{d(z_0,z)\}.$

Let $\mathcal{P}(Z)$ be the power set of *Z* and let $\mathcal{P}_0(Z) = \mathcal{P}(Z) \setminus \{\emptyset\}$. We put

$$\mathcal{C}_b(Z) = \{A \in \mathcal{P}_0(Z) : A \text{ is closed and bounded}\},\$$

$$\mathcal{K}(Z) = \{A \in \mathcal{P}_0(Z) : A \text{ is compact}\}.$$

Let *h* be the Hausdorff metric in $C_b(Z)$ generated by the metric *d*, i.e. for $A, B \in C_b(Z)$

$$h(A,B) = \max(\sup_{z \in B} \{d(z,A)\}, \sup_{z \in A} \{d(z,B)\}).$$

There are several ways of defining convergence in $\mathcal{P}_0(Z)$ and in consequence its connections with continuity. Throughout the chapter, convergence in the space $\mathcal{C}_b(Z)$ will be convergence in the Hausdorff metric *h*. A sequence $(\Phi_n)_{n\in\mathbb{N}}$ of multifunctions $\Phi_n : S \rightsquigarrow Z$ with values in $\mathcal{C}_b(Z)$ is called *converging* to a multifunction $\Phi : S \rightsquigarrow Z$ if for each $s \in S$ the sequence $(\Phi_n(s))_{n\in\mathbb{N}}$ converges to $\Phi(s)$ with respect to the Hausdorff metric *h*. We will write $\Phi = h\text{-lim}_{n\to\infty} \Phi_n$.

It is clear that

(1) If
$$s \in S$$
 and $\Phi(s) = h - \lim_{n \to \infty} \Phi_n(s)$, then for each $z \in Z$
 $d(z, \Phi(s)) = \lim_{n \to \infty} d(z, \Phi_n(s))$.

Now let (X, \mathcal{A}) be a measurable space and (Z, \mathcal{T}) a topological space. We will say a multifunction $\Phi : X \rightsquigarrow Z$ is \mathcal{A} -measurable (weakly \mathcal{A} -measurable) if $\Phi^+(G) \in \mathcal{A}$ ($\Phi^-(G) \in \mathcal{A}$) for each $G \in \mathcal{T}$.

It is evident that in the case of a single valued function $f: X \to Z$, the notions of A-measurability of f and weak A-measurability of f coincide with the usual notion of measurability of f, i.e., $f^{-1}(G) \in A$ for each $G \in \mathcal{T}$.

Excellent source of information on measurability properties of multifunctions with values in a metric space is the paper of Castaing and Valadier [1]. We now mention those properties which will be useful later on.

Proposition 17.1. *If* (X, A) *is a measurable space,* (Z, d) *is a metric space and* $\Phi : X \rightsquigarrow Z$ *is a multifunction, then*

(i) A-measurability of Φ implies weak A-measurability of Φ .

(ii) If Φ is compact valued, then A-measurability of Φ and weak A-measurability of Φ are equivalent.

(iii) If the space (Z,d) is separable, then Φ is weakly A-measurable if and only if the function $g_z : X \to \mathbb{R}$ given by $g_z(x) = d(z, \Phi(x))$ is A-measurable for each $z \in Z$.

(iv) If Φ is compact valued, then A-measurability and weak A-measurability of Φ are equivalent to A-measurability of the function $\Phi : X \to (\mathcal{K}(Z), h)$.

Observe that, by (1) and Proposition 17.1 (iii), the following property is true.

(2) If (Z,d) is separable and a sequence $(\Phi_n)_{n\in\mathbb{N}}$ converges to Φ , then Φ is weakly \mathcal{A} -measurable whenever Φ_n is weakly \mathcal{A} -measurable for each $n \in \mathbb{N}$.

There are several ways of defining continuity of multifunctions. Since we well consider multifunctions with values in a metric space we mention only continuity with respect to the Hausdorff metric h.

Let (Y, ρ) be a metric space and let $\Phi : Y \rightsquigarrow Z$ be a multifunction with values in $C_b(Z)$. The statement that Φ is *h*-continuous will mean that Φ treated as a function from *Y* to the space $(C_b(Z), h)$ is continuous. From now on, let $(Z, || \cdot ||)$ be a reflexive Banach space with a metric *d* generated by the norm; θ will denote the origin of *Z*, $||K|| = h(K, \{\theta\})$ when $K \in C_b(Z)$; co(*K*) will denote the convex hull of *K*.

If $A \subset Z$ and $B \subset Z$ and $\alpha \in \mathbb{R}$ then, as usual,

$$A+B=\{a+b:a\in A\wedge b\in B\}$$
 and $\alpha A=\{\alpha a:a\in A\}.$

It is known that ([4], Lem. 2.2 (ii))

(3) If
$$A_i, B_i \in C_b(Z)$$
 for $i = 1, 2$, then
 $h(A_1 + A_2, B_1 + B_2) \le h(A_1, B_1) + h(A_2, B_2)$.

We put

$$\mathcal{C}_{bc}(Z) = \{A \in \mathcal{C}_b(Z) : A \text{ is convex}\}.$$

By reflexivity of $(Z, || \cdot ||)$, the space $C_{bc}(Z)$ with the addition defined above is a commutative semigroup which satisfies the cancellation law (see [13]). The assumption that $(Z, || \cdot ||)$ is reflexive is used to show that

- (4) $A + B \in \mathcal{C}_{bc}(Z)$ whenever $A, B \in \mathcal{C}_{bc}(Z)$ ([13], Th. 2).
- (5) If $A, B, C \in \mathcal{C}_{bc}(Z)$, then h(A, B) = h(A + C, B + C) ([13], Lem. 3).

The completeness of (Z,d) implies $(\mathcal{C}_b(Z),h)$ is complete. Therefore Price's inequality $h(\operatorname{co}(A),\operatorname{co}(B)) \leq h(A,B)$ ([12], (2.9), p.4) implies that

(6) If (Z,d) is complete, then a Cauchy sequence in $\mathcal{C}_{bc}(Z)$ must converge to an element of $\mathcal{C}_{bc}(Z)$.

From now on, unless otherwise stated, we assume that all considered multifunctions have values in $C_{bc}(Z)$.

Let $T \subset \mathbb{R}$ be an \mathcal{L} -measurable set and let $\Phi : T \rightsquigarrow Z$ be an \mathcal{L} -measurable multifunction. Suppose that Φ is *bounded*, i.e. there is a totally bounded set $K \subset Z$ such that $\Phi(t) \subset K$ for each $t \in T$.

We define an integral of Φ as follows (cf. [9], p. 218, in the case $Z = \mathbb{R}^k$). If Φ takes only a finite number of values $B_1, B_2, ..., B_n$, then we put

$$\int_E \Phi(t) dt = \sum_{i=1}^n \lambda(D_i) \cdot B_i,$$

where $E \subset T$ is a bounded \mathcal{L} -measurable set and $D_i = \{t \in E : \Phi(t) = B_i\}$ for i = 1, 2, ..., n. By (4),

(7) $\int_E \Phi(t) dt \in \mathcal{C}_{bc}(Z).$

It is easy to see that

(8) If $A, B \in \mathcal{L}$ are non-overlapping and $E = A \cup B$, then

$$\int_{E} \Phi(t) dt = \int_{A} \Phi(t) dt + \int_{B} \Phi(t) dt.$$

Let $\Psi: T \rightsquigarrow Z$ be an \mathcal{L} -measurable and bounded multifunction. Using (3) one obtains

(9)
$$h\left(\int_{E} \Phi(t) dt, \int_{E} \Psi(t) dt\right) \leq \int_{E} h(\Phi(t), \Psi(t)) dt$$

whenever Φ and Ψ take a finite number of values.

For a general case of an \mathcal{L} -measurable and bounded multifunction the definition of its integral is based on the following lemma ([10], Lem. 1).

Lemma 17.2. Let a totally bounded convex set $K \subset Z$ and a number $\delta > 0$ be given. Then there exists a finite family $\mathcal{F}_{\delta} \subset \mathcal{C}_{bc}(Z)$ such that if $D \in \mathcal{C}_{bc}(K)$, then there exists a smallest set $B \in \mathcal{F}_{\delta}$ such that $D \subset B \subset B(D, \delta)$.

Now, take *K* in the lemma to be the totally bounded convex set containing all the values of Φ . Suppose $t \in T$. Let \mathcal{F}_{δ} be the family corresponding to $\delta > 0$, and let $\Phi_{\delta}(t)$ be the smallest member of \mathcal{F}_{δ} containing $\Phi(t)$.

Then $h(\Phi(t), \Phi_{\delta}(t)) < \delta$ and $\Phi_{\delta} : T \rightsquigarrow Z$ takes only a finite number of values. Moreover, if $(\delta_n)_{n \in \mathbb{N}}$ is a sequence of positive real numbers and $\lim_{n\to\infty} \delta_n = 0$, then, by (7) and (9),

$$\left(\int_E \Phi_{\delta_n}(t)\,dt\right)_{n\in\mathbb{N}}$$

is a Cauchy sequence in $C_{bc}(Z)$. Thus, by (6), the limit $h - \lim_{\delta \to 0} \int_E \Phi_{\delta}(t) dt$ exists in $C_{bc}(Z)$ and we take this limit to be the integral of Φ on E, i.e.

$$\int_{E} \Phi(t) dt := h - \lim_{\delta \to 0} \int_{E} \Phi_{\delta}(t) dt \in \mathcal{C}_{bc}(Z).$$

Note that by a passage to a limit in (8) and (9) we see that

(10) The properties (8) and (9) are true for each *L*-measurable and bounded multifunction. In particular,

$$\left|\int_{E} \Phi(x) \, dx\right| \leq \int_{E} \left| |\Phi(x)| \right| \, dx.$$

From now on we make the assumption that $I \subset \mathbb{R}$ is an interval.

Lemma 17.3. Let I = [a,b]. If an \mathcal{L} -measurable multifunction $\Phi : I \rightsquigarrow Z$ is bounded and $0 < \delta < b - a$, then the multifunction $\Phi_{\delta} : I \rightsquigarrow Z$ given by

$$\Phi_{\delta}(x) = \begin{cases} \int_{x}^{x+\delta} \Phi(t) dt & \text{if } a \leq x < b - \delta, \\ \int_{b-\delta}^{b} \Phi(t) dt & \text{if } b - \delta \leq x \leq b, \end{cases}$$

is h-continuous.

Proof. Let $x_0 \in I$ be fixed. Let us suppose that $x_0 < b - \delta$ and $x_0 < x < b - \delta$. Then

$$h(\Phi_{\delta}(x_0), \Phi_{\delta}(x)) = h\left(\int_{x_0}^{x_0+\delta} \Phi(t) dt, \int_{x}^{x+\delta} \Phi(t) dt\right) =$$

= $h\left(\int_{x_0}^{x} \Phi(t) dt + \int_{x}^{x_0+\delta} \Phi(t) dt, \int_{x}^{x_0+\delta} \Phi(t) dt + \int_{x_0+\delta}^{x+\delta} \Phi(t) dt\right) =$
= $h\left(\int_{x_0}^{x} \Phi(t) dt, \int_{x_0+\delta}^{x+\delta} \Phi(t) dt\right)$, by (5).

Thus, by (10),

$$h(\Phi_{\delta}(x_0), \Phi_{\delta}(x)) = h\left(\int_{x_0}^x \Phi(t) dt, \int_{x_0+\delta}^{x+\delta} \Phi(t) dt\right) \le \\ \le \left| \left| \int_{x_0}^x \Phi(t) dt \right| \right| + \left| \left| \int_{x_0+\delta}^{x+\delta} \Phi(t) dt \right| \right| \to 0 \text{ as } x \to x_0.$$

If $x_0 - \delta < x < x_0$, then

$$h(\Phi_{\delta}(x_0), \Phi_{\delta}(x)) = h\left(\int_{x_0}^{x_0+\delta} \Phi(t) dt, \int_{x}^{x+\delta} \Phi(t) dt\right) =$$

= $h\left(\int_{x_0}^{x+\delta} \Phi(t) dt + \int_{x+\delta}^{x_0+\delta} \Phi(t) dt, \int_{x}^{x_0} \Phi(t) dt + \int_{x_0}^{x+\delta} \Phi(t) dt\right) =$
= $h\left(\int_{x+\delta}^{x_0+\delta} \Phi(t) dt, \int_{x}^{x_0} \Phi(t) dt\right) \to 0 \text{ as } x \to x_0.$

Now let us suppose that $x_0 \ge b - \delta$. Since Φ_{δ} is constant for $b - \delta \le x \le b$, it is enough to consider only the case $x_0 = b - \delta$ and $x_0 - \delta < x < x_0$. Then

$$h(\Phi_{\delta}(x_0), \Phi_{\delta}(x)) = h\left(\int_{x_0}^{b} \Phi(t) dt, \int_{x}^{x+\delta} \Phi(t) dt\right) =$$

= $h\left(\int_{x_0}^{x+\delta} \Phi(t) dt + \int_{x+\delta}^{b} \Phi(t) dt, \int_{x}^{x_0} \Phi(t) dt + \int_{x_0}^{x+\delta} \Phi(t) dt\right) =$
= $h\left(\int_{x+\delta}^{x_0+\delta} \Phi(t) dt, \int_{x}^{x_0} \Phi(t) dt\right) \to 0 \text{ as } x \to x_0,$

which proves Lemma 17.3.

Let Φ : $I \rightsquigarrow Z$ be an \mathcal{L} -measurable bounded multifunction and $x_0 \in I$.

Definition 17.4. The statement that Φ is a derivative at $x_0 \in I$ means, that

$$\Phi(x_0) = h - \lim_{x \to x_0} \frac{1}{x - x_0} \int_{x_0}^x \Phi(t) dt.$$

 Φ is a derivative if it is a derivative at each point $x \in I$.

Similarly to the case of real functions one can show:

Proposition 17.5. Let $x_0 \in I$. If a multifunction $\Phi : I \rightsquigarrow Z$ is h-continuous at x_0 , then Φ is a derivative at x_0 .

Now we present a different approach of defining integrability for multifunctions. It is based on the definition of Riemann integral. Moving from Hukuhara's idea (cf. [9] in the case $Z = \mathbb{R}^k$) we define *R*-integrability of multifunctions in a more general case.

Let $\Phi : I \rightsquigarrow Z$ be a bounded multifunction. Let $\Delta = \{a_0, a_1, ..., a_n\}$ be a partition of *I* and let $v(\Delta) = \max\{a_{i+1} - a_i\}$ be the diameter of the partition. Let \mathcal{P} denote the family of all pairs (Δ, τ) , where $\tau = (t_0, t_1, ..., t_{n-1})$ is a sequence of points such that $t_i \in [a_i, a_{i+1}]$ for i = 0, ..., n - 1. We put

$$C_{\boldsymbol{\Phi}}(\boldsymbol{\Delta}, \boldsymbol{\tau}) = \sum_{i=0}^{n-1} (a_{i+1} - a_i) \boldsymbol{\Phi}(t_i)$$

for $(\Delta, \tau) \in \mathcal{P}$. Note that (4) implies $C_{\Phi}(\Delta, \tau) \in \mathcal{C}_{bc}(Z)$.

We say that a multifunction $\Phi : I \rightsquigarrow Z$ is *R*-integrable (on *I*) if there exists $B \in C_{bc}(Z)$ such that

$$\forall_{\varepsilon>0} \exists_{\eta>0} \forall_{(\Delta,\tau)\in\mathcal{P}} [\nu(\Delta) < \eta \Rightarrow h(C_{\Phi}(\Delta,\tau),B) < \varepsilon],$$

and we define $(R) \int_{I} \Phi(t) dt$ to be the set *B*. Note that, by (3),

$$h(C_{\mathbf{\Phi}}(\Delta, \tau), C_{\mathbf{\Psi}}(\Delta, \tau)) \leq \sum_{i=0}^{n-1} (a_{i+1} - a_i) h(\mathbf{\Phi}(t_i), \mathbf{\Psi}(t_i))$$

whenever $\Psi: I \rightsquigarrow Z$ is a bounded multifunction.

Thus

$$h\left(\int_{I} \Phi(t) dt, \int_{I} \Psi(t) dt\right) \leq \int_{I} h(\Phi(t), \Psi(t)) dt \leq (b-a)\varepsilon,$$

provided that $h(\Phi(t_i), \Psi(t_i)) \leq \varepsilon$ for each $t \in I$.

Therefore, similarly to the case of real functions,

(11) If $\Phi: I \rightsquigarrow Z$ is *h*-continuous, then Φ is *R*-integrable.

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Proposition 17.6. If a multifunction $\Phi : I \rightsquigarrow Z$ is bounded and almost everywhere *h*-continuous, then Φ is *R*-integrable.

Proof. Let $K \in C_{bc}(Z)$ be such that $\Phi(t) \subset K$ for $t \in I$. Let D_{Φ} denote the set of discontinuity points of Φ . By assumption, $\lambda(D_{\Phi}) = 0$. Fix $\varepsilon > 0$. Let $(I_n)_{n \in \mathbb{N}}$ be a sequence of open intervals such that $D_{\Phi} \subset \bigcup_{n \in \mathbb{N}} I_n$ and $\sum_{n \in \mathbb{N}} \lambda(I_n) < \varepsilon$. Without loss of generality we can assume that $I_n \cap I_m = \emptyset$ for $n \neq m$. Let $I_n = (\alpha_n, \beta_n)$ for $n \in \mathbb{N}$ and $A_{\varepsilon} = [a, b] \setminus \bigcup_{n \in \mathbb{N}} I_n$. Then $\lambda(A_{\varepsilon}) > b - a - \varepsilon$. We define a multifunction $\Phi_{\varepsilon} : I \rightsquigarrow Z$ by

$$\Phi_{\varepsilon}(t) = \begin{cases} \Phi(t) & \text{if } t \in A_{\varepsilon}, \\ \frac{\beta_n - t}{\beta_n - \alpha_n} \Phi(\alpha_n) + \frac{t - \alpha_n}{\beta_n - \alpha_n} \Phi(\beta_n) & \text{if } t \in (\alpha_n, \beta_n) \cap I, n \in \mathbb{N}. \end{cases}$$

Note that $\Phi_{\varepsilon}(t) \in \mathcal{C}_{bc}(Z)$. Moreover, Φ_{ε} is *h*-continuous and, by (11), also *R*-integrable. Let $B \in \mathcal{C}_{bc}(Z)$ be such that $\int_{I} \Phi_{\varepsilon}(t) dt = B$. Let $(\Delta, \tau) \in \mathcal{P}$ and $\eta > 0$ be such that $\nu(\Delta) < \eta$ and $h(C_{\Phi_{\varepsilon}}(\Delta, \tau), B) < \varepsilon$.

Then

$$h(C_{\Phi}(\Delta,\tau),B) \le h(C_{\Phi}(\Delta,\tau),C_{\Phi_{\varepsilon}}(\Delta,\tau)) + h(C_{\Phi_{\varepsilon}}(\Delta,\tau),B) =$$
$$= h(\Sigma_{i=0}^{n-1}(a_{i+1}-a_i)\Phi(t_i),\Sigma_{i=0}^{n-1}(a_{i+1}-a_i)\Phi_{\varepsilon}(t_i)) + h(C_{\Phi_{\varepsilon}}(\Delta,\tau),B)$$

and then, by (3),

$$h(C_{\Phi}(\Delta,\tau),B) \leq \Sigma_{i=0}^{n-1}(a_{i+1}-a_i)h(\Phi(t_i)), \Phi_{\varepsilon}(t_i))) + h(C_{\Phi_{\varepsilon}}(\Delta,\tau),B).$$

For that reason

$$h(C_{\Phi}(\Delta, \tau), B) \leq 2\varepsilon ||K|| + \varepsilon,$$

since $\Phi(t_i) = \Phi_{\varepsilon}(t_i)$ for $t_i \in [a_{i-1}, a_i] \cap A_{\varepsilon}$ and $h(\Phi(t_i), \Phi_{\varepsilon}(t_i)) \le 2 ||K||$ for $t_i \in [a_{i-1}, a_i] \setminus A_{\varepsilon}$. This finishes the proof of Proposition 17.6.

Followig Hukuhara [9], one can prove that

(12) If a bounded \mathcal{L} -measurable multifunction $\Phi: I \rightsquigarrow Z$ is *R*-integrable, then $(R) \int_I \Phi(t) dt = \int_I \Phi(t) dt$.

Now we pass to the multifunctions of two variables.

If $S = X \times Y$, $F : X \times Y \rightsquigarrow Z$ is a multifunction and $(x_0, y_0) \in X \times Y$, then the multifunction $F_{x_0} : Y \rightsquigarrow Z$ defined by $F_{x_0}(y) = F(x_0, y)$ is called the x_0 -section of F, and the multifunction $F^{y_0} : X \rightsquigarrow Z$ defined by $F^{y_0}(x) = F(x, y_0)$ is called the y_0 -section of F.

It is well known that if (X, \mathcal{A}) is a measurable space, (Y, ρ) is a separable metric space and (Z, d) is a metric space, then a function $f : X \times Y \to Z$,

 \mathcal{A} -measurable in the first and continuous in the second variable is measurable with respect to the product of \mathcal{A} and the Borel σ -algebra of Y. Thus by Proposition 1 (iv) we have the following result (cf. [15], Th. 2)

Proposition 17.7. If (X, \mathcal{A}) is a measurable space, (Y, ρ) is a separable metric space and (Z, d) is a metric space, and if $F : X \times Y \rightsquigarrow Z$ is a compact valued multifunction such that each section F_x is h-continuous and each section F^y is \mathcal{A} -measurable, then F is $\mathcal{A} \otimes \mathcal{B}or(Y)$ -measurable.

The product measurability of multifunctions can be obtained from weaker assumptions. We introduce a concept of multifunctions with the (J) property, which may be considered as a multivalued counterpart of the (J) property given by J. S. Lipiński and we show that a multifunction with the (J) property which is a derivative in the second variable is product measurable and sup-measurable.

Let (X, \mathcal{A}, μ) be a measure space with μ σ -finite. Still let $(Z||\cdot||)$ be a reflexive Banach space with the metric *d* generated by the norm, and still we will consider multifunctions $F : X \times I \rightsquigarrow Z$ with values in $\mathcal{C}_{bc}(Z)$.

Let $\mathcal{A} \otimes \mathcal{B}or$ be the σ -algebra generated by the family of sets $A \times B$, where $A \in \mathcal{A}$ and $B \in \mathcal{B}or$.

Definition 17.8. A bounded multifunction $F : X \times I \rightsquigarrow Z$ has the (*J*) property if, for each $y \in I$, F^y is weakly \mathcal{A} -measurable, for each $x \in X$, F_x is weakly \mathcal{L} -measurable and for each interval $P \subset I$, the multifunction $\Phi_P : X \rightsquigarrow Z$ given by

(13)
$$\Phi_P(x) = \int_P F(x, y) \, dy$$

is weakly A-measurable.

A multifunction with the (J) property need not be product measurable.

Example 17.9. Suppose CH. Let $E \subset \mathbb{R}^2$ be Sierpiński's set such that $E \notin \mathcal{L}_2$ and each *x*-section of *E*, i.e. $E_x = \{y \in \mathbb{R} : (x, y) \in E\}$, and each *y*-section of *E*, i.e. $E^y = \{x \in \mathbb{R} : (x, y) \in E\}$, have at most two elements. Let $F : \mathbb{R}^2 \rightsquigarrow \mathbb{R}$ be given by

$$F(x,y) = \begin{cases} [0,1], \text{ if } (x,y) \notin E, \\ \{0\}, \text{ if } (x,y) \in E. \end{cases}$$

Then *F* is not \mathcal{L}_2 -measurable, but *F* has the (J) property.

Proposition 17.10. If (Z,d) is separable and $F: X \times I \rightsquigarrow Z$ is a bounded multifunction such that each section F_x is *R*-integrable and each section F^y is weakly A-measurable, then *F* has the (J) property. *Proof.* Let $P = [c,d] \subset I$ be fixed. We only need to show that the multifunction Φ_P , given by (13), is weakly \mathcal{A} -measurable. Let $y_i = c + i\frac{d-c}{n}$ for i = 0, 1, 2, ..., n and $n \in \mathbb{N}$. If $x \in X$, then, by *R*-integrability of F_x , we have

$$(R) \int_{P} F(x, y) \, dy = h - \lim_{n \to \infty} \sum_{i=1}^{n} \frac{1}{n} F_{x}(y_{i}) = h - \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} F^{y_{i}}(x),$$

and then, applying (12), we have

$$\Phi_P(x) = h - \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^n F^{y_i}(x).$$

Let $n \in \mathbb{N}$ be fixed and let as define the multifunction $\Phi_n : X \rightsquigarrow Z$ by

$$\Phi_n(x) = \sum_{i=1}^n F^{y_i}(x).$$

Then $\Phi_n(x) \in C_{bc}(Z)$ for $x \in X$ (see (4)). Since the multifunction F^{y_i} is weakly \mathcal{A} -measurable for i = 0, 1, ..., n, the multifunction Φ_n is weakly \mathcal{A} -measurable, by Theorem III.40 in [1]. Thus Φ_P is weakly \mathcal{A} -measurable, by (2).

Theorem 17.11. Suppose that (Z,d) is separable. If a bounded multifunction $F: X \times I \rightsquigarrow Z$ has the (J) property and for each $x \in X$, F_x is a derivative, i.e.,

$$F_x(y) = h - \lim_{\Delta y \to 0} \frac{1}{\Delta y} \int_y^{y + \Delta y} F_x(t) dt \text{ for } y \in I,$$

then *F* is measurable with respect to the $\mu \times \lambda$ -completion of $\mathcal{A} \otimes \mathcal{B}$ or.

Proof. Let $n \in \mathbb{N}$ be fixed and let $\Delta = \{y_{0,n}, y_{1,n}, \dots, y_{n,n}\}$ be a partition of *I* into *n* equal intervals. Let us put

$$F_n(x,y) = \begin{cases} \frac{1}{y_{i,n} - y_{i-1,n}} \int_{y_{i-1,n}}^{y_{i,n}} F(x,y) \, dy & \text{if } x \in X \text{ and } y \in (y_{i-1,n}, y_{i,n}), \\ \{\theta\} & \text{if } x \in X \text{ and } y = y_{i,n}, i = 0, 1, \dots, n. \end{cases}$$

Next, let $\Phi_{i,n}$: $X \rightsquigarrow Z$, for i = 1, 2, ..., n, be a multifunction given by

$$\Phi_{i,n}(x) = \int_{y_{i-1,n}}^{y_{i,n}} F(x,y) \, dy.$$

By the (J) property of F, we see that

(14) $\Phi_{i,n}$ is weakly \mathcal{A} -measurable for each i = 1, 2, ..., n. Define $\Phi_n : X \times \bigcup_{i=1}^n (y_{i-1,n}, y_{i,n}) \rightsquigarrow Z$ by 17. Measurability of multifunctions with the (J) property

$$\Phi_n(x,y) = \Phi_{i,n}(x) \quad \text{for} \quad y \in (y_{i-1,n}, y_{i,n}).$$

If V is an open subset of Z, then, by (14), we have

$$\Phi_n^-(V) = \bigcup_{i=1}^n \Phi_{i,n}^-(V) \times (y_{i-1,n}, y_{i,n}) \in \mathcal{A} \otimes \mathcal{B}or.$$

Therefore F_n is weakly $\mathcal{A} \otimes \mathcal{B}or$ -measurable and by (2) we only need to show that

(15) $h-\lim_{n\to\infty} F_n(x,y) = F(x,y)$ for every $x \in X$ and for almost every $y \in I$. Fix $(x_0, y_0) \in X \times I$ such that $y_0 \neq y_{i,n}$ for $n \in \mathbb{N}$ and i = 1, 2, ..., n, and choose a sequence $(y_{i_n,n})$ such that $y_{i_n-1} < y_0 < y_{i_n}$. Since F_{x_0} is a derivative at y_0 , it follows that

$$F(x_0, y_0) = h - \lim_{\Delta y \to 0} \frac{1}{\Delta y} \int_{y_0}^{y_0 + \Delta y} F(x_0, y) \, dy.$$

Assume that

$$A_n = \frac{1}{y_0 - y_{i_n - 1, n}} \int_{y_{i_n - 1, n}}^{y_0} F(x_0, y) \, dy \quad \text{and} \quad B_n = \frac{1}{y_{i_n, n} - y_0} \int_{y_0}^{y_{i_n, n}} F(x_0, y) \, dy.$$

Then

(16)
$$\lim_{n\to\infty} h(A_n, F_0) = 0$$
 and $\lim_{n\to\infty} h(B_n, F_0) = 0$,
where $F_0 = F(x_0, y_0)$.

Let us put $z_n = h(F_n(x_0, y_0), F_0)$. Note that

$$z_{n} = h\left(\frac{1}{y_{i_{n},n} - y_{i_{n}-1,n}} \int_{y_{i_{n}-1,n}}^{y_{i_{n},n}} F(x_{0}, y) \, dy, \frac{1}{y_{i_{n},n} - y_{i_{n}-1,n}} \int_{y_{i_{n}-1,n}}^{y_{i_{n},n}} F_{0} \, dy\right) =$$
$$= \frac{1}{y_{i_{n},n} - y_{i_{n}-1,n}} h\left(\int_{y_{i_{n}-1,n}}^{y_{i_{n},n}} F(x_{0}, y) \, dy, \int_{y_{i_{n}-1,n}}^{y_{i_{n},n}} F_{0} \, dy\right).$$

By (10), we have

$$\int_{y_{i_n-1,n}}^{y_{i_n,n}} F(x_0, y) \, dy = \int_{y_{i_n-1,n}}^{y_0} F(x_0, y) \, dy + \int_{y_0}^{y_{i_n,n}} F(x_0, y) \, dy$$

and

$$\int_{y_{i_n-1,n}}^{y_{i_n,n}} F_0 \, dy = \int_{y_{i_n-1,n}}^{y_0} F_0 \, dy + \int_{y_0}^{y_{i_n,n}} F_0 \, dy.$$

Next, (3) shows that

$$h\left(\int_{y_{i_n-1,n}}^{y_0} F(x_0, y) \, dy + \int_{y_0}^{y_{i_n,n}} F(x_0, y) \, dy, \int_{y_{i_n-1,n}}^{y_0} F_0 \, dy + \int_{y_0}^{y_{i_n,n}} F_0 \, dy\right) \le \\ \le h\left(\int_{y_{i_n-1,n}}^{y_0} F(x_0, y) \, dy, \int_{y_{i_n-1,n}}^{y_0} F_0 \, dy\right) + h\left(\int_{y_0}^{y_{i_n,n}} F(x_0, y) \, dy, \int_{y_0}^{y_{i_n,n}} F_0 \, dy\right).$$

Moreover

$$\frac{1}{y_{i_n,n} - y_{i_n-1,n}} < \frac{1}{y_0 - y_{i_n-1,n}} \quad \text{and} \quad \frac{1}{y_{i_n,n} - y_{i_n-1,n}} < \frac{1}{y_{i_n,n} - y_0}$$

Therefore,

$$z_n < \frac{1}{y_0 - y_{i_n - 1, n}} h\left(\int_{y_{i_n - 1, n}}^{y_0} F(x_0, y) \, dy, \int_{y_{i_n - 1, n}}^{y_0} F_0 \, dy\right) + \frac{1}{y_{i_n, n} - y_0} h\left(\int_{y_0}^{y_{i_n, n}} F(x_0, y) \, dy, \int_{y_0}^{y_{i_n, n}} F_0 \, dy\right),$$

and finally

$$h(F_n(x_0, y_0), F_0) < h(A_n, F_0) + h(B_n, F_0).$$

Thus, by (16), (15) is true, which finishes the proof of Theorem 17.11. \Box

Note that product measurability of a multifunction $F : X \times I \rightsquigarrow Z$ with compact values such that each section F_x is *h*-continuous and each section F^y is \mathcal{L} -measurable follows from Theorem 17.11 as a consequence of (11) and Proposition 17.10.

The remainder of this chapter will be devoted to sup-measurability.

Let (X, \mathcal{A}) be a measurable space and let $(Y, \mathcal{T}(Y))$ and $(Z, \mathcal{T}(Z))$ be topological spaces. If $F : X \times Y \rightsquigarrow Z$ is such that the superposition of the Carathéodory type

$$H(x) = F(x, G(x)) = \bigcup_{y \in G(x)} F(x, y)$$

is A-measurable (resp. weakly A-measurable) for every closed valued A-measurable multifunction $G: X \rightsquigarrow Y$, then F is called A-sup-measurable (resp. weakly A-sup-measurable).

The following theorem is known (see [17], Theorem 1).

Theorem 17.12. Let (X, \mathcal{A}, μ) be a measure space with $\mu \sigma$ -finite. Let Y be a Polish space and let $(Z, \mathcal{T}(Z))$ be a topological space. If $F : X \times Y \rightsquigarrow Z$ is an $\mathcal{A}_{\mu} \otimes \mathcal{B}or(Y)$ -measurable multifunction, then it is \mathcal{A}_{μ} -sup-measurable (where \mathcal{A}_{μ} denotes a μ completion of \mathcal{A}).

From the above theorem it follows that each $\mathcal{A} \otimes \mathcal{B}or(Y)$ -measurable multifunction is \mathcal{A}_{μ} -sup-measurable, whenever the measure μ is σ -finite and Y is a Polish space. The following example shows that for more general σ -algebra in $X \times Y$ than the product $\mathcal{A}_{\mu} \otimes \mathcal{B}or(Y)$, this property may not be true.

Example 17.13. Let $X = Y = \mathbb{R}$ and let $E \notin \mathcal{L}$. If $F : \mathbb{R}^2 \rightsquigarrow \mathbb{R}$ is given by

$$F(x,y) = \begin{cases} [0,2] \text{ if } x \neq y\\ [0,1] \text{ if } x = y \land x \in E,\\ \{0\} \text{ if } x = y \land x \notin E, \end{cases}$$

then *F* is \mathcal{L}_2 -measurable. But $H(x) = F(x, \{x\})$ is not \mathcal{L} -measurable, i.e., *F* is not \mathcal{L} -sup-measurable.

As a straightforward consequence of Theorem 17.12 and Proposition 17.7 we have the following corollary (cf. [16]).

Corollary 17.14. If (X, \mathcal{A}, μ) is a measure space with $\mu \sigma$ -finite, Y is a Polish space, (Z,d) is a separable metric space and $F : X \times Y \rightsquigarrow Z$ is a compact valued multifunction such that each section F_x is h-continuous and each section F^y is \mathcal{A} -measurable, then F is \mathcal{A}_{μ} -sup-measurable.

Proposition 17.15. Let (X, \mathcal{A}) be a measurable space and let (Z, d) be separable. If $F_n : X \times Y \rightsquigarrow Z$ is \mathcal{A} -sup-measurable for each $n \in \mathbb{N}$ and the multifunction $F = h \text{-lim}_{n \to \infty} F_n$, then F is weakly \mathcal{A} -sup-measurable.

Proof. Let $z \in Z$. By (1), $\lim_{n\to\infty} d(z, F_n(x, y)) = d(z, F(x, y))$ for each $(x, y) \in X \times Y$. Let $G : X \rightsquigarrow Y$ be an \mathcal{A} -measurable multifunction with closed values. Let $x \in X$ and $H_n(x) = F_n(x, G(x))$ for each $n \in \mathbb{N}$, and let H(x) = F(x, G(x)). It is clear that $\lim_{n\to\infty} d(z, H_n(x)) = d(z, H(x))$. Fix $n \in \mathbb{N}$. Note that F_n being \mathcal{A} -sup-measurable implies F_n is weakly \mathcal{A} -sup-measurable. Hence H_n is weakly \mathcal{A} -measurable. Therefore, by Proposition 1 (iii), the real function $x \to d(z, H_n(x))$ is \mathcal{A} -measurable. Thus the real function $x \to d(z, H(x))$ is \mathcal{A} -measurable and, again by Proposition 1 (iii), H is weakly \mathcal{A} -measurable. \Box

Now we will consider the sup-measurability of multifunctions with the (J) property. Note that a multifunction with the (J) property may not be sup-measurable.

Example 17.16. Let $F : [0,1]^2 \rightsquigarrow \mathbb{R}$ be the multifunction given by

$$F(x,y) = \begin{cases} [1,2] & \text{if } x \in A \text{ and } y \le x, \\ [1,2] & \text{if } x \in \mathbb{R} \setminus A \text{ and } y < x, \\ \{0\} & \text{in other cases.} \end{cases}$$

where $A \subset [0,1]$ and $A \notin \mathcal{L}$. Then each section F_x is *h*-continuous with the exception of one point. Furthermore each section F^y is \mathcal{L} -measurable. Therefore, by Proposition 17.10, *F* has the (J) property. But *F* is not \mathcal{L} -sup-measurable, since the multifunction $H(x) = F(x, \{x\})$ is not \mathcal{L} -measurable.

Theorem 17.17. Let $(Z, || \cdot ||)$ be a separable Banach space and I = [a, b]. If a multifunction $F : X \times I \rightsquigarrow Z$ with compact convex values has the (J) property and each section F_x is a derivative, then F is A_{μ} -sup-measurable.

Proof. Let $(x, y) \in X \times I$. Since F_x is a derivative at y,

(17)
$$F(x,y) = h - \lim_{\Delta y \to 0} \frac{1}{\Delta y} \int_{y}^{y + \Delta y} F(x,t) dt.$$

For every $n \in \mathbb{N}$ we define $F_n : X \times I \rightsquigarrow Z$ by

$$F_n(x,y) = \begin{cases} n \int_y^{y+\frac{1}{n}} F(x,t) dt & \text{, if } a \le y < b - \frac{1}{n}, \\ n \int_{b-\frac{1}{n}}^{b} F(x,t) dt & \text{, if } b - \frac{1}{n} \le y \le b. \end{cases}$$

Then $h-\lim_{n\to\infty} F_n(x,y) = F(x,y)$ for $(x,y) \in X \times Y$, by (17). For fixed $n \in \mathbb{N}$, each section $(F_n)_x$ is *h*-continuous, by Lemma 17.3. Since *F* has the (J) property, $(F_n)^y$ is \mathcal{A} -measurable for every $y \in I$ and, by Corollary 1, F_n is \mathcal{A}_{μ} -sup-measurable. Thus, by Proposition 17.15, *F* is weakly \mathcal{A}_{μ} -sup-measurable, and hence also \mathcal{A}_{μ} -sup-measurable, since *F* is compact valued.

Observe that, by Proposition 17.10 and Theorem 17.17, we have the following corollary.

Corollary 17.18. If $(Z, || \cdot ||)$ is a separable Banach space and $F : X \times I \rightsquigarrow Z$ is a multifunction with compact convex values such that each section F_x is an *R*-integrable derivative and each section F^y is A-measurable, then F is A_{μ} -sup-measurable.

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GRAŻYNA KWIECIŃSKA

Institute of Mathematics, Pomeranian Academy

ul. Arciszewskiego 22 d, 76-200 Słupsk, Poland

E-mail: grazyna.kwiecinska@apsl.edu.pl