http://dx.doi.org/10.18778/7525-971-1.15

Chapter 15
On approximately continuous integrals (a survey)

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2010 Mathematics Subject Classification: 26A15, 26A24, 26A36, 26A39, 26A42, 26A45, 26A46, 42A10.
Key words and phrases: Approximately continuous function, approximate derivative, approximate Kurzweil–Henstock integral, Lusin-type integral, Burkill's AP-integral, variational measure, differential basis, trigonometric series, approximate symmetric integral.

The origin of the notion of approximate derivative and, in turn, approximate continuity is closely related with integration theory. The purpose of this chapter is to recollect the lines of development of the theory of generalized integration in connection with approximate continuity. Among the results surveyed, some recent results of the authors are mentioned.

15.1 Introduction. Denjoy–Khintchine integral

The story begins a century ago, when A. Denjoy [8] and A. Khintchine [27] considered transfinite sequences of expanding extensions of the Lebesgue integral and, as a result (*totalisation*), obtained two integrals: D_*- and D-integral, both having the property of recovering an everywhere differentiable function from its (ordinary) derivative. Nowadays these integrals are also referred to as *restricted* and *wide* Denjoy integrals (or Denjoy–*Perron* and Denjoy–

Khintchine integrals), respectively. N. N. Lusin [39] described the constructive integrals of Denjoy and Khintchine in terms of generalized primitives. While the restricted integral allowed description with ordinary derivatives, the wide one did not. We shall skip the former of Lusin's descriptions and recall the latter only.

Let $E \subset \mathbb{R}$. A function $F\colon E \to \mathbb{R}$ is said to be an *AC-function*, if for every $\varepsilon > 0$ there exists an $\eta > 0$ such that for any pairwise nonoverlapping intervals $[a_1, b_1], \ldots, [a_n, b_n]$, with both endpoints in E,

$$\sum_{i=1}^{n} (b_i - a_i) < \eta \quad \Rightarrow \quad \sum_{i=1}^{n} |F(b_i) - F(a_i)| < \varepsilon.$$

We call sequence of sets $\{E_i\}_{i=1}^{\infty}$ with $E = \bigcup_{i=1}^{\infty} E_i$ an *E-form*. If, moreover, all E_i are closed (measurable) we say the E-form is closed (measurable). A function $F\colon E \to \mathbb{R}$ is said to be an *ACG*-function if there exists an E-form $\{E_i\}_i$ such that for each i, $F \restriction E_i$ is an *AC*-function. If the E-form can be chosen closed, F is said to be $[ACG]$. In an analogous manner one can define *VB*, *VBG*, and $[VBG]$ properties.

As it was mentioned by Lusin, the class of indefinite *D*-integrals coincides with the class of all continuous *ACG*-functions F (on a given $[a,b]$). As not all such functions are almost everywhere differentiable, the ordinary derivative is no longer a tool for recovery of integrand. It remains possible however, by Lebesgue theorem, to differentiate F at almost every x *along* $E_i \subset [a,b]$, provided $F \restriction E_i$ is *AC* and $x \in E_i$ is an accumulation point of E_i. As long as F is, moreover, continuous, all sets E_i from the definition of *ACG* can be chosen closed, so measurable, thus for almost all x the derivation along E_i can be seen (Lebesgue Density Theorem) as derivation along a set with density point at x. This leads to the notion of approximate derivative: *we say* $F\colon [a,b] \to \mathbb{R}$ *is approximately differentiable at x if there exists a set* $E \subset [a,b]$ *with density* 1 *at x, such that the limit*

$$\lim_{t \to x, t \in E} \frac{F(t) - F(x)}{t - x} = F'_{\mathrm{ap}}(x),$$

called the approximate derivative of F at x, exists. One can show that this definition is unique, i.e., it does not depend on E. (The notion of approximate (*asymptotique*) derivative has been brought first, as above, by Khintchine [27] in connection with the constructive approach to *D*-integral.)

Theorem 15.1 (Lusin). *A function* $f\colon [a,b] \to \mathbb{R}$ *is D-integrable if and only if the following condition is satisfied:*

there exists a continuous ACG-function $F\colon [a,b] \to \mathbb{R}$
such that $F'_{\mathrm{ap}}(x) = f(x)$ *at almost every* $x \in [a,b]$.

The further development of integration related to approximate derivatives was usually motivated in connection with the above result.

15.2 Derivation bases

It will be convenient to consider definitions of some integrals using the concept of derivation basis. We introduce two bases: the approximate basis and the approximate symmetric basis. To do this we need an auxiliary concept of density gauge.

A *density gauge* Δ on a set $E \subset \mathbb{R}$ is defined to be an indexed collection $\Delta = \{\Delta_x : x \in E\}$ of subsets $\Delta_x \subset [0,\infty)$ with the property that for each $x \in E$ there is a measurable $D_x \subset \Delta_x$ with the right density one at x. Clearly, Δ can be identified with the plane set $\Delta = \{(x,h) : x \in E, h \in \Delta_x\} \subset \mathbb{R}^2$. While defining the approximate symmetric basis, the latter is moreover assumed to be measurable.

Having a fixed density gauge Δ on E we define the set β_Δ of all possible pairs $(\langle x, x \pm h\rangle, x)$ with $(x,h) \in \Delta$. For all possible Δ *on* $E = \mathbb{R}$ these sets β_Δ will appear as elements of the *approximate basis* $\mathcal{B}_{\mathrm{ap}}$. Similarly, the *approximate symmetric basis* $\mathcal{B}_{\mathrm{ap.s}}$ is defined as the collection of elements $\beta^{\mathrm{s}}_\Delta$ for all possible *measurable* (as plane sets) density gauges Δ on $\mathbb{R}$, where $\beta^{\mathrm{s}}_\Delta$ is the set of all pairs $([x-h, x+h], x)$ with $(x,h) \in \Delta$ for a fixed Δ.

If $\mathcal{B}$ denotes any of the bases $\mathcal{B}_{\mathrm{ap}}$ and $\mathcal{B}_{\mathrm{ap.s}}$, then given $\beta \in \mathcal{B}$ we say that $(I,x) \in \beta$ is a β*-fine tagged interval*. A finite collection of β-fine tagged intervals $\{(I_i, x_i)\}_{i=1}^n$ with $I_i \cap \operatorname{int} I_j = \varnothing$ for $i \neq j$, is called a β*-fine division*. It is said to be *tagged* in a set $E \subset \mathbb{R}$ if $x_i \in E$ for each $i = 1, \ldots, n$. It is said to be a β*-fine partition of* an $[a,b]$ if $\bigcup_{i=1}^n I_i = [a,b]$. For $\beta \in \mathcal{B}$ and a set $E \subset \mathbb{R}$ we define $\beta[E] = \{(I,x) \in \beta : x \in E\}$.

Let $F\colon [a,b] \to \mathbb{R}$ and the interval function ΔF be defined by $\Delta F(I) = F(d) - F(c)$ for all intervals $I = [c,d] \subset [a,b]$ such that $(I,x) \in \beta$ for some x and some β. We define the *upper* and *lower derivative of* F *at a point* x *with respect to the basis* $\mathcal{B}$ by setting

$$\overline{F}'_{\mathcal{B}}(x) = \inf_{\beta \in \mathcal{B}} \sup_{(I,x) \in \beta[\{x\}]} \frac{\Delta F(I)}{\lambda(I)}, \qquad \underline{F}'_{\mathcal{B}}(x) = \sup_{\beta \in \mathcal{B}} \inf_{(I,x) \in \beta[\{x\}]} \frac{\Delta F(I)}{\lambda(I)}.$$

If $\overline{F}'_{\mathcal{B}}(x) = \underline{F}'_{\mathcal{B}}(x)$, then the common value is called the *derivative of F at x with respect to the basis $\mathcal{B}$* and is denoted by $F'_{\mathcal{B}}(x)$.

Derivative with respect to the basis $\mathcal{B}_{\mathrm{ap}}$ coincides with the approximate derivative $F'_{\mathrm{ap}}(x)$ defined in previous section. Respectively, the derivative with respect to the basis $\mathcal{B}_{\mathrm{ap.s}}$ is called the *approximate symmetric* derivative and is denoted as $F'_{\mathrm{ap.s}}(x)$. A similar notation will be used for respective upper and lower derivatives.

We shall also use the notions of approximate continuity and approximate symmetric continuity which can be introduced in an obvious way.

15.3 Burkill's *AP*-integral

Most of approaches to defining integrals that solve the problem of recovering a function from its ordinary derivative, including constructive and descriptive definitions coming from the original papers by Denjoy, Lusin and Khintchine, Perron approach, the Riemann-type Kurzweil–Henstock definition and some others, were eventually proved to define the same class of integrable functions (see [50], [55]).

For integration of approximate derivative the situation turned out to be more complicated. Most of researchers' effort in this field was exerted into finding relations between approximate Perron-type integrals and the Denjoy–Khintchine integral and its approximately continuous generalizations.

The approximately continuous Perron integral (*AP*-integral) was introduced by John Burkill in [7].

Let $f\colon [a,b] \to \mathbb{R}$. An approximately continuous function $M\colon [a,b] \to \mathbb{R}$ is an *ap-major function* of f on $[a,b]$ if $M(a) = 0$ and $\underline{M}'_{\mathrm{ap}}(x) \geq f(x)$ for all $x \in [a,b]$. An approximately continuous function $m\colon [a,b] \to \mathbb{R}$ is an *ap-minor function* of f on $[a,b]$ if $m(a) = 0$ and $\overline{m}'_{\mathrm{ap}}(x) \leq f(x)$ for all $x \in [a,b]$.

Definition 15.2. A function $f\colon [a,b] \to \mathbb{R}$ is (*Burkill's*) *AP-integrable* on $[a,b]$ if f has at least one ap-major function and at least one ap-minor function on $[a,b]$ and $\inf_M M(b) = \sup_m m(b)$, where inf and sup range over all ap-major functions and ap-minor functions of f, respectively. This common value is the *AP-integral* of f on $[a,b]$ and is denoted by $(AP)\int_a^b f$.

It is straightforward that *AP*-integral recovers any approximately differentiable function from its (approximate, of course) derivative F ($M = m = F$ can be taken). A natural question is whether *D*-integral can serve for the same purpose. As a function having everywhere a finite approximate derivative need not be continuous, the answer is "no". But assuming the antiderivative is continuous, the recovery with D-integral is possible. It is a result of G. P. Tolstov [64] who proved that every approximately differentiable function has $[ACG]$ property.

As an opposite problem one can ask the question as to whether Burkill's AP-integral, solving the recovery problem, is powerful enough to encompass D-integral, or, in other words, whether every continuous ACG-function is an indefinite AP-integral. Again Tolstov [65] provided an example that refutes such a hypothesis. Let $C \subset [0,1]$ be the Cantor ternary set. Put

$$F(x) = \begin{cases} 0 & \text{at } x \in C, \\ \frac{1}{m} & \text{at } x = \operatorname{mid} I_i^{(m)}, \\ \text{linear} & \text{on both } \left[\min I_i^{(m)}, \operatorname{mid} I_i^{(m)}\right] \text{ and } \left[\operatorname{mid} I_i^{(m)}, \max I_i^{(m)}\right], \end{cases}$$

where $I_i^{(m)}$, $i = 1, \dots, 2^{m-1}$, is one of the closed intervals of mth rank contiguous to C in $[0,1]$. So defined F on $[0,1]$ is clearly continuous and ACG. In order to prove F cannot be an integral in Burkill's sense, Tolstov proved that any indefinite AP-integral should fulfil the following property: for each perfect $E \subset [0,1]$ and each $\varepsilon > 0$ there is a portion E' of E such that, if (a_n, b_n), $n \in \mathbb{N}$, are intervals contiguous to E', for each n there is a measurable $E_n^{\varepsilon} \subset (a_n, b_n)$ such that $\lambda(E_n^{\varepsilon}) \geq (1-\varepsilon)(b_n - a_n)$ and

$$\sum_{n \in \mathbb{N}} \big(|F(x_n) - F(a_n)| + |F(y_n) - F(b_n)| \big) < \infty$$

for any choice of $x_n, y_n \in E_n^{\varepsilon}$, $n \in \mathbb{N}$. The above function F clearly does not satisfy it. P. Bullen [5], inspired by the property, gave a descriptive definition of an integral claiming that it is equivalent to the AP-integral. But this claim has been refuted: K. Liao in [35] constructed a fairly simple example of continuous nearly everywhere differentiable function without the above property. So the Bullen integral turned out to be more restrictive than the AP one.

The comparison of D- and AP-integrals results in neutral: they are incomparable. This initiated the seek for a common counterpart for both these integrals, which would unify, in particular, integration of all approximate derivatives with D-integration.

The first (and simplest) quest in this direction was made as early as in 1934 by J. Ridder. He called a function $f\colon [a,b] \to \mathbb{R}$, *$\beta$-integrable* [45] if there exists an approximately continuous $[ACG]$-function $F\colon [a,b] \to \mathbb{R}$ with $f(x) = F'_{\text{ap}}(x)$ at almost every $x \in [a,b]$, $\int_a^b f := F(b) - F(a)$. As the class of approximately continuous $[ACG]$-functions has monotonicity property with respect to approximate derivation, this definition is proper (i.e., the integral is defined uniquely). For a continuous F, if $F \restriction E$ is *AC*, then $F \restriction \operatorname{cl} E$ is so; therefore β-integral is a straightforward extension of *D*-integral (via Lusin's result, Theorem 15.1). Ridder generalized also Burkill's *AP*-integral (D_4-integral of [44]) and proposed a proof that β-integral and D_4-integral are equivalent. The same definitions (respectively *AD*-integral in [29], *AP**-integral in [30]) one can find in later papers by Y. Kubota. Ridder and Kubota claimed to have proved that these two generalizations are equivalent. However, they used a similar fallacious argument in justification of this claim. Then there were a few incorrect attempts to repair Ridder and Kubota's proof (for details see [24], [48]). Eventually C.-M. Lee in [33] and D. N. Sarkhel in [48] proved that each indefinite *AP*-integral F on $[a,b]$ is a *Baire one star function*, i.e., there is an $[a,b]$-form $\{E_i\}_{i=1}^{\infty}$ such that for each i, $F \restriction E_i$ is continuous. Consequently, F is an $[ACG]$-function (a generalization of Tolstov's result [64]) and so *AP*-integral is included in Ridder's β-integral.

We say a linear space $\mathcal{L}$ of functions on $[a,b]$ has the *monotonicity property* with respect to approximate derivation if for $F \in \mathcal{L}$, $F'_{\mathrm{ap}}(x) \geq 0$ for a.e. $x \in [a,b]$ implies F is nondecreasing. Any such $\mathcal{L}$ induces a descriptive definition of integral: $(\mathcal{L}) \int_a^b f = F(b) - F(a)$ as long as $f(x) = F'_{\mathrm{ap}}(x)$ almost everywhere on $[a,b]$ and $F \in \mathcal{L}$. As examples of $\mathcal{L}$, classes of continuous *ACG*-functions and approximately continuous $[ACG]$-functions ($\mathcal{L}_1$), inducing respectively *D*-integral and β-integral, can be mentioned.

Before the β-integral has been finally established as a generalization of *AP*-integral [33], [48], some other, weaker than $[ACG]$, Lusin-type conditions defining $\mathcal{L}$ were considered in order to provide such generalization: $\mathcal{L}_2$ consisting of approximately continuous $[VBG]$-functions satisfying condition (N), $\mathcal{L}_3$ of approximately continuous *ACG*-functions, and $\mathcal{L}_4$ of approximately continuous *VBG*-functions satisfying (N). The integrals defined by classes $\mathcal{L}_2$ to $\mathcal{L}_4$ are known as T_{ap}-integral [49], *AD*-integral [28], and $AK_{(N)}$-integral [24], respectively. These integrals, at the cost of more sophisticated proof of monotonicity property, were easier to show to cover the *AP*-integral. The classes $\mathcal{L}_i$, $i = 1,2,3,4$, do not coincide, so do the corresponding classes of integrands (see [49], [58]). Chart 1, borrowed from [58], depicts interrelations between them.

For other descriptively defined approximately continuous integrals see e.g. [11], [21], [59].

$$\begin{array}{ccccccccc} & & & \subsetneq & \mathcal{L}_2 & \subsetneq & & & \\ \mathcal{L}_1 & \subsetneq & \mathcal{L}_2\cap\mathcal{L}_3 & & \not\subset \ \not\supset & & \mathcal{L}_2+\mathcal{L}_3 & \subsetneq & \mathcal{L}_4 \\ & & & \subsetneq & \mathcal{L}_3 & \subsetneq & & & \end{array}$$

Chart 1 Relations between $\mathcal{L}_i$-integrals.

15.4 Approximate Kurzweil–Henstock integral

We discuss in this section an approximately continuous variant of the renowned concept of Riemann-type integral due to J. Kurzweil and R. Henstock. To define a Riemann-type integral with respect to the basis $\mathcal{B}_{\mathrm{ap}}$ it is important to notice that this basis has the *partitioning property*: for each $\beta \in \mathcal{B}_{\mathrm{ap}}$ and each $[a,b]$ there exists a β-fine partition of $[a,b]$ (see [22], Lemma 3).

Definition 15.3. We say a function $f\colon [a,b] \to \mathbb{R}$ is *AH-integrable* if there is a number $\mathbf{I} = (AH)\int_a^b f$, the value of *AH*-integral, such that for each $\varepsilon > 0$ we can find $\beta \in \mathcal{B}_{\mathrm{ap}}$ with the property that

$$\left|\sum_{i=1}^{n} f(x_i)\lambda(I_i) - \mathbf{I}\right| < \varepsilon \tag{15.1}$$

holds provided $\{(I_i, x_i)\}_{i=1}^{n}$ is a β-fine partition of $[a,b]$.

The value $(AH)\int_a^b f$ is unique because $\mathcal{B}_{\mathrm{ap}}$ is filtering, i.e., $\beta_3 \subset \beta_1 \cap \beta_2$ for all $\beta_1, \beta_2 \in \mathcal{B}_{\mathrm{ap}}$ and some $\beta_3 \in \mathcal{B}_{\mathrm{ap}}$.

The above definition can be used in fact for any basis with the partitioning property giving a corresponding *$\mathcal{B}$-integral*. The basis $\mathcal{B}_{\mathrm{ap.s}}$ does not possess the partitioning property. So a Riemann-type integral with respect to this basis requires more sophisticated definition, which will be considered in Section 15.6.

For a wide class of bases, Riemann-type integral is equivalent to the appropriately defined Perron-type integral (see [41], [55]). This is also true for the basis $\mathcal{B}_{\mathrm{ap}}$. But the Perron-type integral constructed according to that general scheme is defined by major and minor functions which are not supposed to be approximately continuous (see [55]), unlike our ap-major and ap-minor functions (page 236). The question as to whether this integral, and so also *AH*-integral, is equivalent to *AP*-integral, is still open. We can say only that Burkill's *AP*-integral is covered by *AH*-integral.

We consider now some kind of descriptive characterizations of the *AH*-integral. For this we need a few additional definitions.

The *variation of* a function $F\colon \mathbb{R} \to \mathbb{R}$ *on a set* $E \subset \mathbb{R}$ *with respect to* an element $\beta \in \mathcal{B}$ is defined as

$$V(F,E,\beta) = \sup \sum_{j=1}^{n} |\Delta F(I_j)|,$$

where sup ranges over all divisions $\{(I_j,x_j)\}_{j=1}^{n} \subset \beta[E]$. The *variational measure of a set* $E \subset \mathbb{R}$*, generated by* F*, with respect to a basis* $\mathcal{B}$ is defined to be

$$\mathrm{V}_F^{\mathcal{B}}(E) = \inf_{\beta\in\mathcal{B}} V(F,E,\beta). \tag{15.2}$$

If $\mathcal{B}$ stands for the basis $\mathcal{B}_{\mathrm{ap}}$, we shall call (15.2) the *approximate variational measure of* E, and denote it as $\mathrm{V}_F^{\mathrm{ap}}(E)$. Similarly, in the case of $\mathcal{B}_{\mathrm{ap.s}}$, (15.2) is called the *approximate symmetric variational measure* of E and is denoted by $\mathrm{V}_F^{\mathrm{ap.s}}(E)$.

We say $\mathrm{V}_F^{\mathcal{B}}$ is σ*-finite* on a set $E \subset \mathbb{R}$ if there is an E-form $\{E_n\}_{n=1}^{\infty}$ such that $\mathrm{V}_F^{\mathcal{B}}(E_n) < \infty$ for each n. We say $\mathrm{V}_F^{\mathcal{B}}$ is *absolutely continuous* if $\mathrm{V}_F^{\mathcal{B}}(N) = 0$ for each nullset N.

The class of functions generating absolutely continuous variational measure is widely used in the Kurzweil–Henstock theory of integration. In the case of approximate basis some authors (see for example [37]) use the term *ASL functions* (after *approximate strong Lusin* condition) to name functions of this class.

Another class, which also plays an important role in this theory, is the class of functions of generalized absolute continuity with respect to a basis. A function F is said to be $AC_{\mathcal{B}}$ on a set E if for any $\varepsilon > 0$ there exist $\eta > 0$ and an element $\beta \in \mathcal{B}$ such that $\sum_{i=1}^{n} |F(b_i) - F(a_i)| < \varepsilon$ holds for any division $\{([a_i,b_i],x_i)\}_{i=1}^{n} \subset \beta[E]$ with $\sum_{i=1}^{n}(b_i - a_i) < \eta$. F is said to be $ACG_{\mathcal{B}}$ on E if there is an E-form $\{E_k\}_{k=1}^{\infty}$ such that F is $AC_{\mathcal{B}}$ on E_k for each k.

For *AH*-integral many properties, known also for more general classes of bases, hold. In particular (see [3], [4], [11], [22], [23], [66]), the *AH*-indefinite integral F of $f\colon [a,b] \to \mathbb{R}$ is approximately continuous at each point of $[a,b]$ and it is approximately differentiable a.e. with $F'_{\mathrm{ap}}(x) = f(x)$ a.e. on $[a,b]$. Two important general properties of indefinite $\mathcal{B}$-integrals run as follows:

P1) A function $F\colon [a,b] \to \mathbb{R}$, $F(a) = 0$, is the indefinite $\mathcal{B}$-integral of a function f on $[a,b]$ if and only if F generates absolutely continuous $\mathrm{V}_F^{\mathcal{B}}$ and F is $\mathcal{B}$-differentiable a.e. with $F'_{\mathcal{B}}(x) = f(x)$ a.e. on $[a,b]$.

P2) A function $F\colon [a,b] \to \mathbb{R}$, $F(a) = 0$, is the indefinite $\mathcal{B}$-integral of a function f on $[a,b]$ if and only if F is an $ACG_{\mathcal{B}}$-function $\mathcal{B}$-differentiable a.e. with $F'_{\mathcal{B}}(x) = f(x)$ a.e. on $[a,b]$.

In order to make the definitions of $\mathrm{V}_F^{\mathcal{B}}$ and $ACG_{\mathcal{B}}$ compatible for a function $F\colon [a,b] \to \mathbb{R}$, we accept that $F(x) = F(b)$ for $x > b$ and $F(a) = 0$ for $x < a$.

Property P1) was proved in a more general local system setting in [66]. Property P2) is known to be equivalent to P1) for a wide class of bases $\mathcal{B}$, in particular for $\mathcal{B}_{\mathrm{ap}}$ (see [14], Theorem 5.1).

The properties P1) and P2) are examples of so-called partial descriptive characterizations of the indefinite integral (see [42]). A certain drawback of those characterizations is that $\mathcal{B}$-differentiability a.e. of the functions in the class of primitives is included into characterization as an additional assumption. That is why they are called “partial" to distinguish them from a deeper result called “full descriptive characterization" in which the differentiability a.e. of all the functions in the class of primitives is implied by the main characteristic of the class.

A full descriptive characterization of the $\mathcal{B}_{\mathrm{ap}}$-integral is given in

Theorem 15.4. *The class of indefinite $\mathcal{B}_{\mathrm{ap}}$-integrals coincides with the class of all functions $F\colon [a,b] \to \mathbb{R}$, $F(a) = 0$, generating absolutely continuous $\mathrm{V}_F^{\mathrm{ap}}$.*

This result can be easily obtained from [14], Theorem 5.1. In [61] this theorem was formulated as a corollary of a more general result. Making use of V. Ene’s result ([13], Theorem 3), we pointed out in [61] that even a weaker assumption than absolute continuity of $\mathrm{V}_F^{\mathrm{ap}}$, namely finiteness of $\mathrm{V}_F^{\mathrm{ap}}$ on each nullset, implies F is almost everywhere approximately differentiable. Moreover, in the last statement the assumption of finiteness of $\mathrm{V}_F^{\mathrm{ap}}$ can be replaced by the assumption of σ-finiteness of $\mathrm{V}_F^{\mathrm{ap}}$ on nullsets. So we have (see [53])

Theorem 15.5. *Assume $\mathrm{V}_F^{\mathrm{ap}}$ is σ-finite on each nullset. Then $\mathrm{V}_F^{\mathrm{ap}}$ is σ-finite and F is approximately differentiable a.e.*

The proof is based on the important result (see [61]):

Theorem 15.6. *If $\mathrm{V}_F^{\mathrm{ap}}$ is σ-finite on a set $E \subset \mathbb{R}$ then F is VBG on E.*

Having *VBG* property we can use a Denjoy–Khintchine result ([50], chapter 7 (4.3) p.222): *If a VBG-function F on a measurable set E is measurable, then it is approximately differentiable at almost every $x \in E$.* So the problem of measurability is involved here. In [61], we used the following result due to Ene ([13], Theorem 3): A measurable $F\colon [a,b] \to \mathbb{R}$ is *VBG* if and only if it is so on each nullset. The measurability assumption in this theorem is essential (we showed in [53] that there is a function $F\colon [0,1] \to [0,1]$ which is not *VBG*, but it is so on each null subset of $[0,1]$). To obtain the required measurability in this context we proved in [53] that *if* $\mathrm{V}_F^{\mathrm{ap}}$ *is* σ*-finite on each nullset then* $F\colon \mathbb{R} \to \mathbb{R}$ *is measurable*.

15.5 Composite–approximate Riemann-type integration

The problem of defining the wide Denjoy integrals in terms of Riemann sums (thus generalizing the description of restricted Denjoy integral due to Kurzweil and Henstock) was raised by Henstock himself, yet in 1967, with a suggestion of solution, see [25]. Henstock suggested the use of what one may call a mixed derivation basis: integration 'along' a given $[a,b]$-form should lead to *ACG* property of indefinite integrals, while integration with respect to the full integration basis (at separate, i.e., not all, points, so not to destroy the feature of the first component) should guarantee continuity of respective integrals.

Henstock's suggestion was considered later on in connection with approximate counterparts of wide Denjoy integrals ($\mathcal{L}_1$- to $\mathcal{L}_4$-integrals, see Section 3), [56] and [34], as well as in generalized setting [16], [60]. In all the cases (including the description of wide Denjoy integral) the problem was eventually settled only in a work of the 3rd author [60]. In our presentation, consequently, we restrict ourselves to the case of approximately continuous primitives.

Given an E-form $\{E_i\}_i$, we write $\mathrm{Is}\{E_i\}_i$ for the set of all x such that for some i, $x \in E_i$ and x is isolated from either side of E_i.

Fix a density gauge Δ on $A \subset [a,b]$ and an $[a,b]$-form $\{E_i\}_i$. On each E_i define a gauge δ_i, i.e., $\delta_i\colon E_i \to (0,\infty)$. We call the sequence $\{\delta_i\}_i$ *related* to $\{E_i\}_i$. Consider a tagged interval $(\langle x,y\rangle,x)$. We say it is $\{\delta_i\}_i$*-fine* if for some i, $x,y \in E_i$ and it is δ_i-fine. We say $(\langle x,y\rangle,x)$ is $(\beta_\Delta,\{\delta_i\}_i)$*-fine* if it is either $\{\delta_i\}_i$-fine or β_Δ-fine (see page 235). We say a division is $\{\delta_i\}_i$-fine, or $(\beta_\Delta,\{\delta_i\}_i)$-fine if all its members are such. One can refer to pairs $(\beta_\Delta,\{\delta_i\}_i)$ as to *composite–approximate gauges* and (as for $\{\delta_i\}_i$) call them related to $\{E_i\}_i$.

A result of Ene [16], Lemma 4.2 (see also [56], Theorem 3.1) says that for every $\{\delta_i\}_i$, related to a *closed* $[a,b]$-form $\{E_i\}_i$, and any density gauge Δ on $\mathrm{Is}\{E_i\}_i$ there is a $(\beta_\Delta,\{\delta_i\}_i)$-fine partition of $[a,b]$. It is an extension of Henstock's result [25], Exercise 43.9 (for the case of all Δ_x being ordinary neighborhoods of respective x).

Theorem 15.7. *A function* $f\colon [a,b] \to \mathbb{R}$ *is (Ridder's)* β*-integrable if and only if the following property holds:*

(A1) *there is a closed* $[a,b]$*-form* $\{E_i\}_i$ *with the following property: for each* $\varepsilon > 0$ *there is a sequence* $\{\delta_i\}_i$ *related to* $\{E_i\}_i$ *and, for each countable superset* $A \supset \operatorname{Is}\{E_i\}_i$*, a density gauge* Δ *on* A *such that* (15.1) *is fulfilled by each* $(\beta_\Delta, \{\delta_i\}_i)$*-fine partition* $\{(I_i, x_i)\}_{i=1}^n$ *of* $[a,b]$, $\mathbf{I} = \int_a^b f$.

Theorem 15.7 has been claimed in [34] (with the property **(A1)** called *AH*-integration), but finally settled only in [60] (*LL*-integration, Definition 7' there).

Theorem 15.8. *A function* $f\colon [a,b] \to \mathbb{R}$ *is* T_{ap}*-integrable if and only if the following property holds:*

(A2) *to each* $\varepsilon > 0$ *we can find a closed* $[a,b]$*-form* $\{E_i\}_i$ *and a sequence of gauges* $\{\delta_i\}_i$ *related to* $\{E_i\}_i$ *such that for any countable superset* $A \supset \operatorname{Is}\{E_i\}_i$ *there is a density gauge* Δ *on* A *such that* (15.1) *holds for each* $(\beta_\Delta, \{\delta_i\}_i)$*-fine partition* $\{(I_i, x_i)\}_{i=1}^n$ *of* $[a,b]$, $\mathbf{I} = \int_a^b f$.

This result has been finally settled in [60] (see also [16]). The condition **(A2)** was called $[\mathcal{S}_1\mathcal{S}_2\mathcal{R}]$- or $[\mathcal{S}\mathcal{R}]$-integrability [16] and *E*-integrability in [60], and is a reformulation of a one from [56] (*AH*-integral there).

By the aforementioned Ene's partitioning result [16], Lemma 4.2, both conditions **(A1)** and **(A2)** are nontrivial; i.e., the number **I** is unique. For more detailed discussion on the above reformulations consult [60].

As the partitioning result is no longer true for non-closed $[a,b]$-forms, a Riemann-type description of Kubota's *AD*-integral of [28] (the one defined with $\mathcal{L}_3$) is an issue. One can provide instead, a variational characterization of this integral.

Theorem 15.9 ([60], § 4.3). *A function* $f\colon [a,b] \to \mathbb{R}$ *is AD-integrable [28], with a primitive* $F\colon [a,b] \to \mathbb{R}$*, if and only if the following property holds:*

(A3) *there is an* $[a,b]$*-form* $\{E_i\}_i$ *with the following property: for each* $\varepsilon > 0$ *there is a sequence* $\{\delta_i\}_i$ *related to* $\{E_i\}_i$ *and, for each countable superset* $A \supset \operatorname{Is}\{E_i\}_i$*, a density gauge* Δ *on* A *such that*

$$\left| \sum_{i=1}^{n} \bigl(f(x_i)\lambda(I_i) - \Delta F(I_i) \bigr) \right| < \varepsilon$$

is fulfilled by each $(\beta_\Delta, \{\delta_i\}_i)$*-fine division* $\{(I_i, x_i)\}_{i=1}^n$ *in* $[a,b]$.

Analogous variational descriptions of integrals defined with classes $\mathcal{L}_1$ and $\mathcal{L}_2$ easily follow from their Riemann-type characterizations (Theorems 15.7 and 15.8). It is interesting to note that in **(A3)** the $[a,b]$-form $\{E_i\}_i$ can be made dependent on ε, as in **(A2)**, nevertheless still this condition characterizes the same integral. Therefore, it is an open question how to define in analogous terms the $\mathcal{L}_4$-integral (Gordon's $AK_{(N)}$-integral).

For Perron-type definitions of integrals defined with $\mathcal{L}_1$ to $\mathcal{L}_4$ see [11], [12].

15.6 Approximate symmetric Kurzweil–Henstock integral

As we have already mentioned, the approximate symmetric basis $\mathcal{B}_{\mathrm{ap.s}}$ does not have the partitioning property. An extensive proof of D. Preiss and B. Thomson [43] (other proof was given in [18]), leads however to the following result which one may find as a weaker version of partitioning property: for any $\beta_\Delta^{\mathrm{s}} \in \mathcal{B}_{\mathrm{ap.s}}$ there exists a set $N \subset \mathbb{R}$ of measure zero such that for every interval $[c,d]$ with $c,d \in \mathbb{R} \setminus N$ there is a $\beta_\Delta^{\mathrm{s}}$-fine partition of $[c,d]$.

Definition 15.10 (periodic version, [43], § 11). A periodic function $f\colon \mathbb{R} \to \mathbb{R}$ is called *ASH*-integrable over a period $T > 0$, if there exists a number $\mathbf{I} \in \mathbb{R}$ with the property that for every $\varepsilon > 0$ there is an element $\beta_\Delta^{\mathrm{s}} \in \mathcal{B}_{\mathrm{ap.s}}$ such that for any $\beta_\Delta^{\mathrm{s}}$-fine partition $\{([x_i - h_i, x_i + h_i], x_i)\}_{i=1}^n$ of any interval of the form $[p, p+T]$, $p \in \mathbb{R}$, for which this partition exists, the inequality

$$\left| \sum_{i=1}^{n} 2f(x_i)h_i - \mathbf{I} \right| < \varepsilon$$

holds. The number $\mathbf{I}$ is called the *approximate symmetric Henstock–Kurzweil integral* (*ASH-integral*) of f over T and is denoted as $(ASH)\int_T f$.

Due to the weak partitioning property, the integrability condition is not empty: for any $\beta_\Delta^{\mathrm{s}}$ there is a $\beta_\Delta^{\mathrm{s}}$-fine partition of $[p, p+T]$ for almost all $p \in \mathbb{R}$.

Any function $f\colon [a,b) \to \mathbb{R}$ can be considered for *ASH*-integrability: it is enough to extend it periodically from $[a,b)$ onto $\mathbb{R}$ and find $\int_a^b f$ as the integral of the extension over the period $b-a$.

Definition 15.11 (variational version, $\mathcal{A}$-integral of [43], § 8). A function $f\colon \mathbb{R} \to \mathbb{R}$ is said to be *ASH*-integrable if there is a function F, an indefinite integral of f, defined almost everywhere on $\mathbb{R}$ with the property that for every $\varepsilon > 0$ there is an element $\beta_\Delta^{\mathrm{s}} \in \mathcal{B}_{\mathrm{ap.s}}$ such that for any $\beta_\Delta^{\mathrm{s}}$-fine division $\{([x_i - h_i, x_i + h_i], x_i)\}_{i=1}^n$ in $\mathbb{R}$,

$$\sum_{i=1}^{n} \left| 2f(x_i)h_i - F(x_i + h_i) + F(x_i - h_i) \right| < \varepsilon.$$

Such indefinite integral F is unique up to an additive constant and a nullset in $\mathbb{R}$. One can prove, see [43], § 11, that for a periodic function, integrability conditions from Definitions 15.10 and 15.11 are equivalent, moreover, the periodic intergral over T is $F(p+T)-F(p)$ for almost all $p \in \mathbb{R}$.

An appropriate Perron type definition (*ASP*-integral) and descriptive definitions similar to the ones considered in Section 15.4 for the basis $\mathcal{B}_{\text{ap}}$, can be given for the basis $\mathcal{B}_{\text{ap.s}}$. Note that in definition of *ASP*-integral on an interval $[a,b]$ the endpoints a,b should be treated in a special way. Namely, the approximate limit of the difference $M(b-h)-M(a+h)$, while $h \to 0$, should exist for all major functions, together with analogous assumption imposed on minor functions; for details see [43].

Approximate symmetric variation

The classical Perron integral was defined using continuous major and minor functions. A remarkable theorem of J. Marcinkiewicz asserts that the integrability of a measurable f can be deduced from the existence of a single pair of continuous major and minor functions. This theorem has been extended, in terms of respective continuity, to some more general Perron-type integrals, including *AP*-integral (see [6]). It was shown in [54] that the corresponding integral with respect to the basis $\mathcal{B}_{\text{ap.s}}$ does not have the Marcinkiewicz property. In fact, a stronger result is obtained in [54]:

Theorem 15.12. *There is a measurable function f that is not ASP-integrable on an interval $[a,b]$ and yet f has a pair of continuous (in the usual sense) major and minor (defined with respect to $\mathcal{B}_{\text{ap.s}}$) functions.*

The main tool used in construction of the above example is the approximate symmetric $\beta_{\Delta}^{\text{s}}$-variation and its continuity in the case it is generated by a continuous function. A similar method, based on continuity of β_{Δ}-variation, can be used to prove that given continuous indefinite *AH*-integral, the corresponding Perron integral can be defined with continuous ap-major and ap-minor functions [2]. In the general case, as we have already mentioned, the question as to whether we can drop the requirement of approximate continuity of ap-major and ap-minor functions in Definition 15.2 is open. A similar open problem can be formulated for *ASP*-integral.

It is known (see [20]) that *ASL* functions (see Section 15.4) satisfy Lusin's condition (N) (image of a nullset is a nullset). But it is not the case for functions generating absolutely continuous variational measure $\mathrm{V}_F^{\text{ap.s}}$.

Theorem 15.13 ([51]). *There exists a continuous function F on $[0,1]$ such that the approximate symmetric derivative $F'_{\text{ap.s}}(x)$ is finite everywhere on $(0,1)$, and $F'_{\text{ap.s}}(x) = 0$ on a perfect set S of measure zero. Furthermore, F maps S onto $[0,1]$ and thereby does not satisfy (N).*

It is easy to check that, with F of the above theorem, $\mathrm{V}_F^{\text{ap.s}}(S) = 0$. At the same time the construction of F in [51] implies that $\mathrm{V}_F^{\text{ap}}(S) > 0$. So the approximate and the approximate symmetric variational measure of the same function can disagree and we have

Theorem 15.14. *There exists a continuous function with non-zero V_F^{ap} on some set of measure zero, yet with $\mathrm{V}_F^{\text{ap.s}}$ equal to zero on the same set.*

Another consequence of this construction is the following assertion.

Theorem 15.15. *The ASH-integral and Denjoy–Khintchine integral can contradict each other on the class of exact approximate symmetric derivatives, that is, there exists a D-integrable function $f\colon [0,1] \to \mathbb{R}$ having a continuous on $[0,1]$ approximate symmetric antiderivative F with $F'_{\text{ap.s}}(x) = f(x)$ everywhere on $(0,1)$, but for which*

$$F(1) - F(0) = (ASH) \int_0^1 f \neq (D) \int_0^1 f.$$

An application of *ASH*-integral

Here we consider some application of approximate symmetric Kurzweil–Henstock integral to the theory of trigonometric series and trigonometric integrals.

One of the principle questions concerning trigonometric series

$$\sum_{n=-\infty}^{\infty} c_n e^{inx} \tag{15.3}$$

is the question of recovering the coefficients of every convergent trigonometric series from its sum. To answer this question the so-called Riemann and Lebesgue theories of trigonometric series were developed, the key objects of which are twice (the Riemann function) and once formally integrated (the Lebesgue function) series, respectively:

$$F(x) = c_0 \frac{x^2}{2} - \sum_{n\neq 0} \frac{c_n}{n^2} e^{inx}, \qquad l(x) = c_0 x + \sum_{n\neq 0} \frac{c_n}{in} e^{inx}.$$

Riemann proved that F is uniformly smooth if $c_n \to 0$ as $n \to \infty$, and $D^2F(x_0) = \sum_{n=-\infty}^{\infty} c_n e^{inx_0}$ if additionally (15.3) is convergent at x_0. Thus any integration process recovering a function F from its *second symmetric Riemann derivative* D^2F will solve the problem of recovering the coefficients.

As for the Lebesgue function, it is known (see [67]) that: if $c_n \to 0$ then the series defining $l(x)$ is almost everywhere convergent, and if (15.3) is convergent at x_0 to s, then $l'_{\mathrm{ap.s}}(x_0)$ exists and equals s. (Moreover, Zygmund jointly with Rajchman showed that if $c_n \to 0$, then l is everywhere approximately symmetrically continuous and approximately continuous at every point where it is finite.) Thus, analogously to the Riemann theory case, one should look for an integral recovering a function from its approximate symmetric derivative. The *ASH*-integral defined above, recovers a measurable function from its approximate symmetric derivative and thus handles the problem of recovery. More precisely,

Theorem 15.16 ([43]). *If a function $F\colon \mathbb{R} \to \mathbb{R}$ is measurable, approximately symmetrically continuous at each point of the line and has nearly everywhere approximate symmetric derivative f, then the function $f = F'_{\mathrm{ap.s}}$ is ASH-integrable with F being an indefinite integral of f.*

The proof follows a standard argument with the only peculiarity related to measurability of the density gauges used. The measurability assumption on F is essential here [43].

Theorem 15.17 ([43]). *If a trigonometric series* (15.3) *converges nearly everywhere to a finite function f, then functions $f(x)$ and $f(x)e^{-inx}$, $n \in \mathbb{Z}$, are ASH-integrable and*

$$c_n = \frac{1}{2\pi} \int_{2\pi} f(x)e^{-inx}\,dx,$$

where the integral is understood over the period 2π.

A generalized version of trigonometric series are trigonometric integrals; i.e., integrals of the form

$$\int_{-\infty}^{\infty} e^{i\lambda x} c(\lambda)\,d\lambda = \lim_{\omega\to\infty} (L) \int_{-\omega}^{\omega} e^{i\lambda x} c(\lambda)\,d\lambda.$$

The problem of recovery has a natural formulation for trigonometric integrals. It was shown in [57] that *ASH*-integral handles the recovery in this case as well. To this purpose an analogue of the Lebesgue theory was developed for trigonometric integrals. Consider the following assumption on a function c, called *condition* $\mathcal{N}_0$:

$$\lim_{u\to\pm\infty}\left\{\max_{0\leqslant h\leqslant 1}\left|\int_u^{u+h} c(\lambda)\,d\lambda\right|\right\}=0,$$

which is an analogue of the $c_n \to 0$ condition for trigonometric series. Note, that this condition necessarily holds for any c with trigonometric integral convergent on a set of positive measure.

Formal integration of the integral $\int_{-\infty}^{\infty} e^{i\lambda x}c(\lambda)\,d\lambda$ leads to the function

$$L(x)=\int\limits_{|\lambda|<1}\frac{e^{i\lambda x}-1}{i\lambda}c(\lambda)\,d\lambda+\int\limits_{|\lambda|\geq 1}\frac{e^{i\lambda x}}{i\lambda}c(\lambda)\,d\lambda.$$

The following theorem is an analogue of the previously mentioned Zygmund's result.

Theorem 15.18 ([57]). *Assume that a function c fulfils $\mathcal{N}_0$. Then the function L is finite almost everywhere, approximately symmetrically continuous everywhere and approximately continuous at each point where L is finite. Moreover, if the trigonometric integral converges at x to s then there exists $L'_{\mathrm{ap.s}}(x)=s$.*

One of the peculiarities arising in the passage from trigonometric series to trigonometric integrals is that to prove the analog of Theorem 15.17 it is not only the Lebesgue theory that is used, but also a Riemann theory for trigonometric integrals with so-called equiconvergence theorems have its share.

The analogue of the Preiss–Thomson Theorem 15.17 for trigonometric integrals sounds as follows:

Theorem 15.19 ([57]). *If a function c is locally Lebesgue integrable and its trigonometric integral converges nearly everywhere to a finite function f, then $f(x)$ and $f(x)e^{-i\mu x}$ are ASH-integrable and for almost all μ, $c(\mu)$ can be recovered as*

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{-i\mu x}\,dx := \lim_{\omega\to\infty}\frac{1}{2\pi\omega}\int_0^{\omega}\left(\int_{-t}^{t} f(x)e^{-i\mu x}\,dx\right)dt,$$

where the integral over x is understood in the ASH sense and the integral over t is understood in the Lebesgue sense.

Corollary 15.20. *If the trigonometric integral of a function c converges nearly everywhere to a finite function f and B is the set of points at which an indefinite ASH-integral of f is finite, then for all $a,b\in B$,*

$$(ASH)\int_a^b f(x)\,dx=(L)\int_{-\infty}^{\infty}\frac{e^{i\lambda b}-e^{i\lambda a}}{i\lambda}c(\lambda)\,d\lambda.$$

15.7 Open problems

1. Is the β-integral of Ridder the *smallest* integral covering the wide Denjoy and approximate Henstock integrals? A related question is, if for every Baire one star approximately continuous function f there is a continuous g such that $f - g$ is *everywhere* approximately differentiable?
2. Provide a 'gauge-free' characterization of indefinite Burkill's *AP*- (or *AH*-) integrals.
3. Prove that Burkill's and approximate Kurzweil–Henstock integrals are equivalent. This boils down to a construction of approximately continuous major and minor functions for any *AH* integrand.
4. Provide a variational characterization to Gordon's $AK_{(N)}$-integral.

Acknowledgement

The first author was partially supported by RFBR-11-01-00321 and by NSh-979.2012.1.

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