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Chapter 12
Weak convergence with respect to category

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12.1 Introduction

We consider a new kind of convergence of a sequence of real functions having the Baire property called in this paper "weak convergence in category". The main result of the paper says that the convergence in category is equivalent to weak convergence in category.

12.2

Throughout the paper $\mathscr{B}$ will stand for a σ-algebra of subsets of a real line $\mathbb{R}$ having the Baire property (that is, of the form $G \triangle I$, where G is open and I is of the first category, both with respect to the natural topology), $\mathscr{I}$ – a σ-ideal of sets of the first category. We shall say that some property holds $\mathscr{I}$ – almost everywhere (in abbr. $\mathscr{I}$-a.e.) if it holds at each point except a set of the first category and that a function $f\colon [0,1] \to \mathbb{R}$ has the Baire property if it is measurable with respect to $\mathscr{B}$.

A classical Riesz theorem says that a sequence $\{f_n\}_{n\in\mathbb{N}}$ of Lebesgue measurable real functions defined on $[0,1]$ converges in measure to a measurable function f if and only if for each increasing sequence $\{n_m\}_{m\in\mathbb{N}}$ of positive integers there exists a subsequence $\{n_{m_p}\}_{p\in\mathbb{N}}$ such that $\{f_{n_{m_p}}\}_{p\in\mathbb{N}}$ converges a.e. to f. Using this theorem E. Wagner in [4] has introduced the notion of the convergence in category in the following form: we say that a sequence $\{f_n\}_{n\in\mathbb{N}}$ of real functions defined on $[0,1]$ and having the Baire property converges in category to a function f having the Baire property if and only if for each increasing sequence $\{n_m\}_{m\in\mathbb{N}}$ of positive integers there exists a subsequence $\{n_{m_p}\}_{p\in\mathbb{N}}$ such that $\{f_{n_{m_p}}\}_{p\in\mathbb{N}}$ converges $\mathscr{I}$-a.e. to f.

In this paper we shall introduce and study the notion of the weak convergence in category. It is worth mentioning that the weak convergence in measure may be useful in defining the density of a set at a point (compare [5], Prop. 3 and [3]). The definition below is constructed in the spirit of [1], p. 149, 9.13 or [2], Cor. V 3.14. Namely, if the σ-ideal of null sets is used in the definition below in the place of the σ-ideal of sets of the first category, we obtain the definition of the weak convergence in measure.

All real functions under consideration are assumed to be defined on $[0,1]$ and to have the Baire property.

Definition 12.1. We say that a sequence $\{f_n\}$ of functions converges weakly with respect to $\mathscr{I}$ to a function f if and only if for each increasing sequence $\{n_m\}_{m\in\mathbb{N}}$ of natural numbers there exists a subsequence $\{n_{m_p}\}_{p\in\mathbb{N}}$ and there exists a sequence $\{\phi_p\}_{p\in\mathbb{N}}$ of convex combinations of the following form:

$$\phi_p = \sum_{j=0}^{k_p} a_{p,j}\cdot f_{n_{m_{p+j}}},$$

where k_p is some natural number, $a_{p,j}\geq 0$ for $p\in\mathbb{N}$ and $j=0,1,\ldots,k_p$ and $\sum_{j=0}^{k_p} a_{p,j} = 1$ such that $\phi_p \to f$ $\mathscr{I}$-almost everywhere. (Remark: ϕ_p starts at $f_{n_{m_p}}$ for $p\in\mathbb{N}$).

Theorem 12.1. *If $\{f_n\}_{n\in\mathbb{N}}$ converges to f with respect to $\mathscr{I}$, then $\{f_n\}_{n\in\mathbb{N}}$ converges weakly to f with respect to $\mathscr{I}$.*

Proof. Obvious. □

Theorem 12.2. *If $\{f_n\}_{n\in\mathbb{N}}$ converges weakly to f with respect to $\mathscr{I}$, then $\{f_n\}_{n\in\mathbb{N}}$ converges to f with respect to $\mathscr{I}$.*

Proof. Suppose that $\{f_n\}_{n\in\mathbb{N}}$ does not converge to f with respect to $\mathscr{I}$. We shall show that there exist a subsequence $\{n_m\}_{m\in\mathbb{N}}$, a number $\varepsilon_0>0$ and an interval $(a,b) \subset [0,1]$ such that for each $x \in (a,b)$ we have

$$\lim_{m\to\infty}[\operatorname{dist}(x, Op\{t\colon f_{n_m}(t) - f(t) > \varepsilon_0\})] = 0 \tag{12.1}$$

or for each $x \in (a,b)$,

$$\lim_{m\to\infty}[\operatorname{dist}(x, Op\{t\colon f_{n_m}(t) - f(t) < -\varepsilon_0\})] = 0 \tag{12.2}$$

where $\operatorname{dist}(x,A)$ is, as usual, the distance between x and A and $Op(A)$ is an arbitrary open set which differs from A by a set of first category (A has the Baire property, in particular $Op(A)$ can be a regular open part of A).

Suppose that it is not the case. So we have

$$\mathop{\forall}_{\{n_m\}} \mathop{\forall}_{\varepsilon>0} \mathop{\forall}_{(a,b)\subset[0,1]} \mathop{\exists}_{x\in(a,b)} \overline{\lim_{m\to\infty}}[\operatorname{dist}(x, Op\{t\colon f_{n_m}(t) - f(t) > \varepsilon\})] > 0 \tag{12.3}$$

and

$$\mathop{\forall}_{\{n_m\}} \mathop{\forall}_{\varepsilon>0} \mathop{\forall}_{(a,b)\subset[0,1]} \mathop{\exists}_{x\in(a,b)} \overline{\lim_{m\to\infty}}[\operatorname{dist}(x, Op\{t\colon f_{n_m}(t) - f(t) < -\varepsilon\})] > 0. \tag{12.4}$$

From the supposition that $\{f_n\}_{n\in\mathbb{N}}$ does not converge to f we conclude that there exists a subsequence $\{f_{n_m}\}_{m\in\mathbb{N}}$ without a subsequence $\{f_{n_{m_p}}\}_{p\in\mathbb{N}}$ convergent $\mathscr{I}$-a.e. to f. Take this subsequence $\{f_{n_m}\}_{m\in\mathbb{N}}$ and fix $\varepsilon > 0$.

Let $\{(a_i,b_i)\}_{i\in\mathbb{N}}$ be a basis of $(0,1)$ in the natural topology. From (12.3) there exists a point $x_1' \in (a_1,b_1)$ such that

$$\overline{\lim_{m\to\infty}}[\operatorname{dist}(x_1', Op\{t\colon f_{n_m}(t) - f(t) > \varepsilon\})] > 0,$$

so there exist a subsequence $\{n_m^{(1)'}\}$ of $\{n_m\}$ and an interval

$$(x_1' - \delta_1', x_1' + \delta_1') \subset (a_1,b_1)$$

such that for each $m \in \mathbb{N}$,

$$(x_1' - \delta_1', x_1' + \delta_1') \cap Op\{t\colon f_{n_m^{(1)'}}(t) - f(t) > \varepsilon\} = \emptyset.$$

From (12.4) for $\{n_m^{(1)'}\}$, the same ε and for the interval $(x_1' - \delta_1', x_1' + \delta_1')$ similarly we find a point $x_1 \in (x_1' - \delta_1', x_1' + \delta_1')$, a subsequence $\{n_m^{(1)}\}$ of $\{n_m^{(1)'}\}$ and an interval $(x_1 - \delta_1, x_1 + \delta_1) \subset (x_1' - \delta_1', x_1' + \delta_1') \subset (a_1,b_1)$ such that for each $m \in \mathbb{N}$,

$$(x_1 - \delta_1, x_1 + \delta_1) \cap Op\{t\colon f_{n_m^{(1)}}(t) - f(t) < -\varepsilon\} = \emptyset.$$

So finally we have

$$(x_1-\delta_1,x_1+\delta_1)\cap Op\{t\colon |f_{n_m^{(1)}}(t)-f(t)|>\varepsilon\}=\emptyset$$

for each $m\in\mathbb{N}$ (because $Op(A\cap B)$ differs from $Op(A)\cap Op(B)$ by a first category set for A,B having the Baire property).

Suppose that for some $i\in\mathbb{N}$ we have chosen a subsequence $\{n_m^{(i)}\}$ of $\{n_m\}$ and the interval $(x_i-\delta_i,x_i+\delta_1)\subset(a_i,b_i)$ such that

$$(x_i-\delta_i,x_i+\delta_i)\cap Op\{t\colon |f_{n_m^{(i)}}(t)-f(t)|>\varepsilon\}=\emptyset$$

for each $m\in\mathbb{N}$.

From (12.3) for $\{n_m^{(i)}\}$, the same ε and for the interval (a_{i+1},b_{i+1}) there exist a subsequence $\{n_m^{(i+1)'}\}$ and an interval $(x'_{i+1}-\delta'_{i+1},x'_{i+1}+\delta'_{i+1})\subset(a_{i+1},b_{i+1})$ such that

$$(x'_{i+1}-\delta'_{i+1},x'_{i+1}+\delta'_{i+1})\cap Op\{t\colon f_{n_m^{(i+1)'}}(t)-f(t)>\varepsilon\}=\emptyset$$

for each $m\in\mathbb{N}$. From (12.4) for $\{n_m^{(i+1)'}\}$, the same ε and for the interval $(x'_{i+1}-\delta'_{i+1},x'_{i+1}+\delta'_{i+1})$ there exist a subsequence $\{n_m^{(i+1)}\}$ and an interval $(x_{i+1}-\delta_{i+1},x_{i+1}+\delta_{i+1})\subset(x'_{i+1}-\delta'_{i+1},x'_{i+1}+\delta'_{i+1})$ such that

$$(x_{i+1}-\delta_{i+1},x_{i+1}+\delta_{i+1})\cap Op\{t\colon f_{n_m^{(i+1)}}(t)-f(t)<-\varepsilon\}=\emptyset$$

for each $m\in\mathbb{N}$.

So finally we have

$$(x_{i+1}-\delta_{i+1},x_{i+1}+\delta_{i+1})\cap Op\{t\colon |f_{n_m^{(i+1)}}(t)-f(t)|>\varepsilon\}=\emptyset$$

for each $m\in\mathbb{N}$ (and also $(x_{i+1}-\delta_{i+1},x_{i+1}+\delta_{i+1})\subset(a_{i+1},b_{i+1})$).

Now let us consider a decreasing sequence $\{\{f_{n_m^{(i)}}\}_{m\in\mathbb{N}}\}_{i\in\mathbb{N}}$ of subsequences of $\{f_{n_m}\}_{m\in\mathbb{N}}$ and let $\{g_m^\varepsilon\}_{m\in\mathbb{N}}$ be a diagonal sequence (i.e. $g_m^\varepsilon=f_{n_m^{(m)}}$). We shall show that $\limsup_m\{t\colon |g_m^\varepsilon(t)-f(t)|>\varepsilon\}=E_\varepsilon$ is of the first category. Suppose that this is not the case. Then there exists an interval $(c,d)\subset[0,1]$ such that E_ε is residual on this interval (E_ε has the Baire property). Let i_0 be a natural number for which $(a_{i_0},b_{i_0})\subset(c,d)$. Then we have

$$(x_{i_0}-\delta_{i_0},x_{i_0}+\delta_{i_0})\cap Op\{t\colon |g_m^\varepsilon(t)-f(t)|>\varepsilon\}=\emptyset$$

for almost all $m\in\mathbb{N}$ ($m\geq i_0$). Hence $(x_{i_0}-\delta_{i_0},x_{i_0}+\delta_{i_0})\cap E_\varepsilon$ is a set of the first category – a contradiction, because $(x_{i_0}-\delta_{i_0}+\delta_{i+0})\subset(a_{i_0},b_{i_0})\subset(c,d)$.

Now suppose that $\varepsilon = 1$. By virtue of the above reasoning we obtain a subsequence $\{g_m^1\}_{m\in\mathbb{N}}$ of $\{f_{n_m}\}$. Repeat the argument for $\varepsilon = \frac{1}{2}$ and $\{g_m^1\}$ to obtain a subsequence $\{g_m^{\frac{1}{2}}\}_{m\in\mathbb{N}}$ of $\{g_m^1\}$ and proceed further by induction. From the decreasing sequence $\{\{g_m^{\frac{1}{k}}\}_{m\in\mathbb{N}}\}_{k\in\mathbb{N}}$ of subsequence $\{f_{n_m}\}$ we choose a diagonal sequence $\{g_m\}_{m\in\mathbb{M}}$ (i.e. $g_m = g_m^{\frac{1}{m}}$). We shall show that $\{g_m\}_{m\in\mathbb{N}}$ converges $\mathscr{I}$-a.e. to f. Indeed, let $E = \bigcup_{k\in\mathbb{N}} E_{\frac{1}{k}}$, ($E_{\frac{1}{k}}$ is E_ε for $\varepsilon = \frac{1}{k}$). The set E is of the first category. Observe that $F_k = \limsup_m \{t\colon |g_m(t) - f(t)| > \frac{1}{k}\} \subset E_{\frac{1}{k}}$ because $\{g_m\}_{m\in\mathbb{N}}$ is almost a subsequence of $\{g_m^{\frac{1}{k}}\}$. Hence $E_0 = \bigcup_{k\in\mathbb{N}} F_k \subset E$ is also of the first category. Observe that

$$\begin{aligned}
[0,1]\setminus E_0 &= \bigcap_{k\in\mathbb{N}} ([0,1]\setminus F_k) = \\
&= \bigcap_{k\in\mathbb{N}} \liminf_m \big([0,1]\setminus \{t\colon |g_m(t) - g(t)| > \tfrac{1}{k}\}\big) = \\
&= \bigcap_{k\in\mathbb{N}} \Big(\liminf_m \{t\colon |g_m(t) - g(t)| \le \tfrac{1}{k}\}\Big)
\end{aligned}$$

is a residual set and for $x \in [0,1]\setminus E_0$ we have $g_m(x) \underset{m\to\infty}{\longrightarrow} f(x)$. It is a contradiction because $\{g_m\}_{m\in\mathbb{N}}$ is a subsequence of $\{f_{n_m}\}$ and $\{g_m\}$ should not converge $\mathscr{I}$-a.e. to f.

So we proved the existence of $\{n_m\}$, $\varepsilon_0 > 0$ and $(a,b) \subset [0,1]$ such that (12.1) or (12.2) holds. We shall consider the first case. In the second the argument is similar.

Observe first that if for $\{n_m\}$, (a,b) and $\varepsilon_0 > 0$ we have the following property:

$$\text{for each } x \in (a,b),\ \lim_{m\to\infty} [\operatorname{dist}(x, Op\{t\colon f_{n_m}(t) - f(t) > \varepsilon_0\})] = 0,$$

then also for each $x \in (a,b)$,

$$\lim_{m\to\infty} [\operatorname{dist}(x, (a,b) \cap Op\{t\colon f_{n_m}(t) - f(t) > \varepsilon_0\})] = 0. \tag{12.5}$$

Observe also that if $G \subset (a,b)$ is an open set, $G = \bigcup_{j=1}^\infty (a_j, b_j)$ and for some $\delta > 0$ we have $\operatorname{dist}(x, G) < \delta$ for each $x \in (a,b)$, then there exists $j_0 \in \mathbb{N}$ such that $\operatorname{dist}(x, (a_{j_0}, b_{j_0})) < \delta$ for each $x \in (a,b)$. Subtracting, if necessary, a finite set, we obtain an open set $G_0 \subset G$ with the following properties:

a) G_0 has a finite number of components,
b) the length of each component of G_0 is less than δ,
c) $\operatorname{dist}(x, G_0) < \delta$ for each $x \in (a,b)$.

Now we shall choose from $\{f_{n_m}\}$ a subsequence $\{f_{n_{m_p}}\}_{p\in\mathbb{N}}$ which does not include a subsequence with convex combinations of the required form convergent $\mathscr{I}$-a.e. to f (which will end the whole proof).

Put $\delta_1 = \frac{1}{3}$. From (12.5) we conclude that there exists $n_{m_1} \in \mathbb{N}$ such that for each $x \in (a,b)$,

$$\operatorname{dist}(x, (a,b) \cap Op\{t\colon f_{n_{m_1}}(t) - f(t) > \varepsilon_0\}) < \delta_1$$

(this is for all x simultaneously, which easily follows from (12.5) by using finite set $\{x_1, \ldots, x_k\} \subset (a,b)$ forming a $\frac{\delta_1}{2}$ net).

Let $G_1 \subset (a,b) \cap Op\{t\colon f_{n_{m_1}}(t) - f(t) > \varepsilon_0\}$ be an open set with properties a), b), c) for δ_1, i.e. G_1 has a finite number of components, each of length less than δ_1 and such that $\operatorname{dist}(x, G_1) < \delta_1$ for each $x \in (a,b)$.

Let Δ_1 be the length of the shortest component of G_1. Obviously $\Delta_1 < \delta_1$. Put $\delta_2 = \frac{\Delta_1}{4}$.

Let $n_{m_2} > n_{m_1}$ be such natural number that

$$\operatorname{dist}(x, (a,b) \cap Op\{t\colon f_{n_{m_2}}(t) - f(t) > \varepsilon_0\}) < \delta_2$$

for each $x \in (a,b)$ (the existence follows from (12.5) again). Choose an open set $G_2 \subset (a,b) \cap Op\{t\colon f_{n_{m_2}}(t) - f(t) > \varepsilon_0\}$ having a finite number of components, each of length less than δ_2 and such that $\operatorname{dist}(x, G_2) < \delta_2$ for each $x \in (a,b)$.

Observe that from the fact that $\delta_2 = \frac{\Delta_1}{4}$ if follows that each component of G_1 includes some component of G_2.

Suppose that we have defined an increasing finite subsequence $n_{m_1}, \ldots, n_{m_p}$ of natural numbers and finite sequence of open sets $G_1, \ldots, G_p$ such that each has only finite number of components, $G_i \subset (a,b) \cap Op\{t\colon f_{n_{m_i}}(t) - f(t) > \varepsilon_0\}$, $\operatorname{dist}(x, G_i) < \delta_i$ for each $x \in (a,b)$ and the length of each component of G_i is less than δ_i for $i = 1, \ldots, p$, moreover $\delta_{i+1} < \frac{\delta_i}{4}$ for $i = 1, \ldots, p-1$.

Let Δ_p be the length of the shortest component of G_p. Obviously $\Delta_p < \delta_p$. Put $\delta_{p+1} = \frac{\Delta_p}{4}$.

Let $n_{m_{p+1}} > n_{m_p}$ be such natural number that

$$\operatorname{dist}(x, (a,b) \cap Op\{t\colon f_{n_{m_{p+1}}}(t) - f(t) > \varepsilon_0\}) < \delta_{p+1}$$

for each $x \in (a,b)$.

Choose an open set $G_{p+1} \subset (a,b) \cap Op\{t\colon f_{n_{m_{p+1}}}(t) - f(t) > \varepsilon_0\}$ having a finite number of components, each of length less than δ_{p+1} and such that $\operatorname{dist}(x, G_{p+1}) < \delta_{p+1}$ for each $x \in (a,b)$. Observe that from the fact that $\delta_{p+1} = \frac{\Delta_p}{4}$ it follows that each component of G_p includes some component of G_{p+1}.

Thus by induction we have defined an increasing sequence $\{n_{m_p}\}_{p\in\mathbb{N}}$ of natural numbers and the sequence $\{G_p\}_{p\in\mathbb{N}}$ of open sets fulfilling the following conditions:

(i) for each $p \in \mathbb{N}$ $\quad G_p \subset (a,b) \cap Op\{t\colon f_{n_{m_p}}(t) - f(t) > \varepsilon_0\}$;
(ii) $\operatorname{dist}(x, G_p) < \frac{1}{4^p}$ for each $p \in \mathbb{N}$ and for each $x \in (a,b)$;
(iii) for each $p \in \mathbb{N}$ each component of G_p has length less than $\frac{1}{4^p}$;
(iv) for each $p \in \mathbb{N}$ each component of G_p includes some component of G_{p+1}.

Let $\phi = \sum_{i=p}^{p+k} a_i f_{n_{m_i}}$ be a convex combination. We have

$$Op\{t\colon \phi(t) - f(t) > \varepsilon_0\} \supset \bigcap_{i=p}^{p+k} (G_i \setminus P_i) = H,$$

where P_i are sets of the first category.

From (iv) it follows that each component of G_p includes some component of $Op(H)$. From (ii) and (iii) it follows that $\operatorname{dist}(x, Op(H)) < \frac{2}{4^p}$ for each $x \in (a,b)$.

If we take an arbitrary subsequence $\{f_{n_{m_{p_r}}}\}$ of $\{f_{n_{m_p}}\}$ and an arbitrary sequence $\{\phi_r\}$ of convex combinations of the required form, i.e. $\phi_r = \sum_{i=0}^{k_r} a_{r,i} \cdot f_{n_{m_{p_{r+1}}}}$ and if H_r is a set described above attached to ϕ_r, then we have

$$\limsup_r \phi_r(t) - f(t) > \varepsilon_0$$

$\mathscr{I}$-a.e. on the set $A = \bigcap_{r=1}^{\infty} \bigcup_{i=r}^{\infty} H_i$. But $\bigcup_{i=r}^{\infty} Op(H_i)$ is an open set dense on (a,b), so A is residual on (a,b). This means that $\{f_n\}$ is not weakly convergent to f with respect to $\mathscr{I}$. □

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