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Chapter 11
On extensions of quasi-continuous functions

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11.1 Introduction

The Tietze-Urysohn Extension Theorem (see [1, Theorem 2.1.8]) states that every continuous function defined on a closed subspace of a normal topological space has a continuous extension on the whole space. But we may have problems if we want extend functions with other properties. In this chapter we deal with the problem of extension of quasi-continuous functions.

Definition 11.1. A function $f\colon X \to Y$ between topological spaces X and Y is called *quasi-continuous* if for any point $x \in X$, for any neighborhood U of x and for any neighborhood V of $f(x)$ there exists a non-empty open set $U_1 \subseteq U$, such that $f(U_1) \subseteq V$.

In general, this problem may be formulated in such a way.

Problem 11.1. Describe all topological spaces X, Y and Z with $Y \subseteq X$, such that every quasi-continuous function $g\colon Y \to Z$ has a quasi-continuous extension $f\colon X \to Z$.

We also will consider the following specification of the previous problem.

Definition 11.2. Let X and Z be topological spaces, $Y \subseteq Z$, $f\colon X \to Z$, $g\colon Y \to Z$. Define

$$\begin{aligned} C(f) =& \{x \in X\colon f \text{ is continuous at } x\}, \\ \widetilde{C}(g) =& \{x \in \overline{Y}\colon \text{ there exists } z \in Z \text{ such that } g(y) \to z \text{ as } Y \ni y \to x\}, \\ D(f) =& X \setminus C(f) \text{ and } \widetilde{D}(f) = \overline{Y} \setminus \widetilde{C}(f). \end{aligned}$$

Obviously, $\widetilde{C}(g) \cap Y = C(g)$ and $\widetilde{D}(g) \cap Y = D(g)$. So, we have that $C(g) = Y \setminus D(g) = Y \setminus \widetilde{D}(g)$.

Problem 11.2. Describe all topological spaces X, Y and Z with $Y \subseteq X$, such that every quasi-continuous function $g\colon Y \to Z$ has a quasi-continuous extension $f\colon X \to Z$ with $D(f) = \widetilde{D}(g)$.

We know only one article [8] where quasi-continuous extensions was constructed. But there were extensions of bounded continuous functions defined on a subset of the segment [0;1] of the reals.

Theorem 11.1 (Neugebauer, [8, Theorem 4]). *Let $X = [0;1]$ be a segment of the real line, Y be a subspace of X and $g\colon Y \to \mathbb{R}$ be a bounded continuous function. Then there exists an quasi-continuous function $f\colon X \to \mathbb{R}$ such that $f|_Y = g$ and f is continuous at every point of Y.*

The following example shows, that there is an unbounded continuous function without any quasi-continuous extensions.

Remark 11.1. Let $g\colon (0;1] \to \mathbb{R}$ be the function defined by $g(y) = \frac{1}{y}$. Then g has no quasi-continuous extension $f\colon [0;1] \to \mathbb{R}$.

Now we say about the contents of this chapter. In Section 11.2 we obtain the positive answer on Problem 11.2 in the case, where $\overline{Y} = X$ and g is a bounded real-valued quasi-continuous function. The case of compact-valued functions is investigated in Section 11.3. In Section 11.4 we obtain the positive answer on Problem 11.2 in the case, where Y is a closed Baire subspace of a hereditarily normal space X and $Z = \mathbb{R}$.

The rest of this chapter relates with Problem 11.1. So, in Sections 11.5 and 11.6 we discuss a question of the existence of, so called, the universal quasi-continuous extensions. In Sections 11.7 and 11.8 we obtain some auxiliary statements about quasi-clopen sets and quasi-clopen partitions. One of the mains results of this chapter we obtain in Section 11.9. There we give the positive answer on Problem 11.1 in the case, where X is hereditarily normal and Z

is metrizable compact. In the final Section 11.10 we construct two counterexamples and give some open problems on the existence of quasi-continuous extensions.

11.2 Extension of real-valued quasi-continuous functions defined on a dense subspace

In this section we start from some modifications of methods used in the proof of Theorem 11.1 which allows to extent a bounded quasi-continuous function defined on subspace of a hereditarily Baire hereditarily normal space. Firstly, let us introduce some notations and definitions.

Definition 11.3. Let Y be a subspace of a topological space X and let $g\colon Y \to \mathbb{R}$ be a bounded function. *The upper and lower limit functions of* g are said to be functions $g^{\vee}, g^{\wedge}\colon \overline{Y} \to \mathbb{R}$ defined by the formulas

$$g^{\vee}(x) = \limsup_{Y \ni y \to x} g(y) = \inf_{U \in \mathscr{U}_x} \sup_{y \in U \cap Y} g(y) \qquad \text{and}$$

$$g^{\wedge}(x) = \liminf_{Y \ni y \to x} g(y) = \sup_{U \in \mathscr{U}_x} \inf_{y \in U \cap Y} g(y)$$

for any $x \in \overline{Y}$, where $\mathscr{U}_x$ is the system of all neighborhoods of x in X. *The oscillation of* g is said to be a function $\omega_f\colon \overline{Y} \to \mathbb{R}$ defined by

$$\omega_f(x) = \limsup_{\substack{Y \ni y' \to x \\ Y \ni y'' \to x}} |g(y') - g(y'')| = \inf_{U \in \mathscr{U}_x} \sup_{y', y'' \in U \cap Y} |g(y') - g(y'')|.$$

It is easy to see that $\omega_f = g^{\vee} - g^{\wedge}$ and $\widetilde{D}(g) = \{x \in \overline{Y}\colon \omega_g(x) > 0\}$.

Proposition 11.1. *Let Y be a dense subspace of a topological space X and let $g\colon Y \to \mathbb{R}$ be a bounded quasi-continuous function. Then there exists a bounded quasi-continuous function $f\colon X \to \mathbb{R}$ such that $f|_Y = g$ and $D(f) = \widetilde{D}(g)$.*

Proof. Define $f\colon X \to \mathbb{R}$ by the formula

$$f(x) = \begin{cases} g(x), & x \in Y; \\ g^{\vee}(x), & x \in X \setminus Y. \end{cases}$$

Obviously, $f|_Y = g$ and f is bounded. Let us prove that f is quasi-continuous. Fix $x_0 \in X$, $\varepsilon > 0$ and an open neighborhood U_0 of x_0.

Firstly, consider the case, where $x_0 \in Y$. Since g is quasi-continuous at x_0, there exists a non-empty open set $V \subseteq U_0$ such that $|g(x) - g(x_0)| < \varepsilon$ for any $x \in V \cap Y$. But $g(x_0) = f(x_0)$. So, $f(x_0) - \varepsilon < g(x) < f(x_0) + \varepsilon$ for all $x \in V \cap Y$. Since $\overline{Y} = X$, we have that $f(x_0) - \varepsilon < g(x) \le g^{\vee}(x) \le f(x_0) + \varepsilon$ on V. Thus, $|g^{\vee}(x) - f(x_0)| \le \varepsilon$ for $x \in V$ and then f is quasi-continuous at x_0.

Now we consider the case, where $x_0 \in X \setminus Y$. Since $f(x_0) = g^{\vee}(x_0)$, there exists an open neighborhood $U \subseteq U_0$ of x_0 such that $\sup\limits_{y \in U \cap Y} g(y) < f(x_0) + \frac{\varepsilon}{2}$. But

$$\sup_{y \in U \cap Y} g(y) \ge g^{\vee}(x_0) = f(x_0) > f(x_0) - \tfrac{\varepsilon}{2}.$$

So, there is a point $y_0 \in U \cap Y$ with $g(y_0) > f(x_0) - \frac{\varepsilon}{2}$. Then

$$|f(y_0) - f(x_0)| = |g(y_0) - f(x_0)| < \tfrac{\varepsilon}{2}.$$

By the previous case, we have that f is quasi-continuous at y_0. Thus, there is a non-empty open subset V of U such that $|f(x) - f(y_0)| < \frac{\varepsilon}{2}$ on V. Therefore,

$$|f(x) - f(x_0)| \le |f(x) - f(y_0)| + |f(y_0) - f(x_0)| < \varepsilon$$

on V and then f is quasi-continuous at x_0.

Finally, to prove the equality $D(f) = \widetilde{D}(g)$ it is sufficient to show that $\omega_f = \omega_g$. Since $f \le g^{\vee}$ and $g^{\vee}$ is upper semi-continuous, we have that $f^{\vee} \le g^{\vee}$. On the other hand, equalities $f|_Y = g$ and $\overline{Y} = X$ imply that $f^{\vee} \ge g^{\vee}$. So, $f^{\vee} = g^{\vee}$. Analogously, since $f \ge g^{\wedge}$ and $f|_Y = g$, we conclude that $f^{\wedge} = g^{\wedge}$. Therefore, $\omega_f = f^{\vee} - f^{\wedge} = g^{\vee} - g^{\wedge} = \omega_g$. □

11.3 Cluster sets and extension of compact-valued quasi-continuous functions defined on a dense subspace

Now we generalize the result of the previous section to the case of functions ranged in compact spaces.

Definition 11.4. Let X and Z be topological spaces, $x \in X$, $Y \subseteq X$ and $g\colon Y \to \mathbb{R}$. *The cluster set of g at x* is defined by the formula

$$\overline{g}(x) = \bigcap_{U \in \mathscr{U}_x} \overline{g(U \cap Y)},$$

where $\mathscr{U}_x$ is the system of all neighborhoods of x in X. Set

$$D = \{x \in \overline{Y} \colon \overline{g}(x) \ne \emptyset\}.$$

The multifunction $\overline{g}\colon D \multimap Z$ is called *the cluster multifunction of* g. So, the domain $\operatorname{dom}\overline{g}$ of this multifunction $\overline{g}$ is equals to the set D.

It is not hard to show, that the graph $\operatorname{Gr}\overline{g}$ of the cluster multifunction $\overline{g}$ is the closure of the graph $\operatorname{Gr} g$ of g in $X \times Z$. So, $Y \subseteq \operatorname{dom}\overline{g} \subseteq \overline{Y}$ and in the case, where Z is compact we have, that $\operatorname{dom}\overline{g} = \overline{Y}$.

Definition 11.5. A multifunction $F\colon X \multimap Y$ between topological spaces X and Y is called *minimal* if for any $x_0 \in X$ and any open sets U in X and V in Y, such that $x_0 \in U$ and $V \cap F(x_0) \neq \emptyset$, there is a nonempty open set U_1 in U, such that $F(U_1) \subseteq V$.

Proposition 11.2. *Let X be a topological spaces, Z be a regular space, $Y \subseteq X$ and $g\colon Y \to Z$ be a quasi-continuous function. Then the cluster multifunction $\overline{g}$ is minimal.*

Proof. Choose a point $x_0 \in \operatorname{dom}\overline{g}$ and open sets U in X and W in Z, such that $x_0 \in U$ and $W \cap \overline{g}(x_0) \neq \emptyset$. Since $\overline{g(U \cap Y)} \supseteq \overline{g}(x_0)$, we have that $\overline{g(U \cap Y)} \cap W \neq \emptyset$ and then $g(U \cap Y) \cap W \neq \emptyset$. So, there is $y_0 \in U \cap Y$, such that $g(y_0) \in W$. Since Z is regular, there exists an open set W_1 in Z, such that $g(y_0) \in W_1$ and $\overline{W}_1 \subseteq W$. By the quasi-continuity of g, we conclude that there exists an open set $U_1 \subseteq U$, such that $U_1 \cap Y \neq \emptyset$ and $g(U_1 \cap Y) \subseteq W_1$. Fix $x \in U_1$. Then

$$\overline{g}(x) = \bigcap_{U \in \mathscr{U}_x} \overline{g(U \cap Y)} \subseteq \overline{g(U_1 \cap Y)} \subseteq \overline{W}_1 \subseteq W.$$

Therefore, $\overline{g}$ is minimal. □

Proposition 11.3. *Let X and Z be topological spaces, $F\colon X \multimap Z$ be a minimal multifunction and $f\colon X \to Z$ be a function, such that $f(x) \in F(x)$ for any $x \in X$. Then f is quasi-continuous.*

Proof. Fix $x_0 \in X$. Let U and W by open sets, such that $x_0 \in X$ and $f(x_0) \in W$. Since F is minimal and $F(x_0) \cap W \ni f(x_0)$, we have that there exists a nonempty open set $U_1 \in U$ with $F(U_1) \subseteq W$. Therefore, $f(U_1) \subseteq F(U_1) \subseteq W$. □

Theorem 11.2. *Let X be a topological space, Z be a compact, Y be a dense subset of X and $g\colon Y \to Z$ be a quasi-continuous function. Then there exists a quasi-continuous function $f\colon X \to Z$, such that $f|_Y = g$ and $D(f) = \widetilde{D}(g)$.*

Proof. Since Z is a compact and $\overline{Y} = X$, we have that $\operatorname{dom}\overline{g} = X$. Consider a function $f\colon X \to Z$, such that $f(x) \in \overline{g}(x)$ for any $x \in X$ and $f(y) = g(y)$ for any $y \in Y$. By Propositions 11.2 and 11.3 we obtain, that f is a quasi-continuous extension of g.

Let us prove, that $D(f) = \widetilde{D}(g)$. If $x_0 \notin D(f)$, then f is continuous at x_0. So, $\lim\limits_{Y \ni y \to x_0} g(y) = \lim\limits_{Y \ni y \to x_0} f(y) = \lim\limits_{x \to x_0} f(x) = f(x_0)$. Therefore, $x_0 \notin \widetilde{D}(g)$. In the other hand, if $x_0 \notin \widetilde{D}(g)$, then there exists $z_0 = \lim\limits_{Y \ni y \to x_0} g(y) \in Z$. To prove that $x_0 \notin D(f)$ it is sufficient to show that $f(x) \to z_0$ as $x \to x_0$. Fix a neighborhood W of z_0. Then by regularity of the compact Z, there exists an open neighborhood W_0 of z_0 with $\overline{W}_0 \subseteq W$. Since $\lim\limits_{Y \ni y \to x_0} g(y) = z_0 \in W_0$, there is an open neighborhood U_0 of x_0, such that $g(U_0 \cap Y) \subseteq W_0$. So, for any $x \in U_0$ we have that

$$f(x) \in \overline{g}(x) = \bigcap_{U \in \mathscr{U}_x} \overline{g(U \cap Y)} \subseteq \overline{g(U_0 \cap Y)} \subseteq \overline{W}_0 \subseteq W.$$

Therefore, $\lim\limits_{x \to x_0} f(x) = z_0$ and then $x_0 \notin D(f)$. □

Corollary 11.1. *Let X be a topological space, Z be a compact, Y be an open subset of X and $g\colon Y \to Z$ be a quasi-continuous function. Then there exists a quasi-continuous function $f\colon X \to Z$, such that $f|_Y = g$.*

Proof. By Theorem 11.2 there is a quasi-continuous extension $h\colon \overline{Y} \to Z$. Fix an arbitrary point $z_0 \in Z$ and define $f\colon X \to Z$ by the formula

$$f(x) = \begin{cases} h(x) \ , \text{ if } x \in \overline{Y}; \\ \quad z_0 \ , \text{ if } x \in X \setminus \overline{Y}. \end{cases}$$

Obviously, f is an extension of g. Prove that f is quasi-continuous. Fix $x \in X$. In the case, where $x \in X \setminus \overline{Y}$, the quasi-continuity f at x is evident. Consider the case, where $x \in \overline{Y}$. Let U be an open neighborhood of x and W be a neighborhood of $f(x) = h(x)$. The quasi-continuity of h implies that there is a non-empty set V_1 such that V_1 is open in $\overline{Y}$, $V_1 \subseteq U$ and $h(V_1) \subseteq W$. Put $U_1 = V_1 \cap Y$. Since Y is open and dense in $\overline{Y}$, the set U_1 is non-empty and open in X. But $f(U_1) = h(U_1) \subseteq h(V_1) \subseteq W$. So, f is quasi-continuous at x. □

Proposition 11.4. *Let Z be a non-compact topological space. Then there exist a topological space X, a dense subspace Y of X and a quasi-continuous function $g\colon Y \to Z$ which has no quasi-continuous extension $f\colon X \to Z$.*

In the case, where Z is not countably compact we may assume that $X = [0;1]$ and $Y = (0;1]$.

Proof. Firstly, let us prove the second part of this proposition. Since Z is not countably compact, there exists a sequence $(z_n)_{n=1}^{\infty}$ in Z which has no limits point in Z. Denote $X = [0;1]$ and $Y = (0;1]$. Define a quasi-continuous function

$g\colon Y \to Z$ by the formula $g(y) = z_n$ for $y \in \left(\frac{1}{n+1}; \frac{1}{n}\right]$ and $n \in \mathbb{N}$. Let us prove, that g has no quasi-continuous extension.

Assuming the contrary, let $f\colon X \to Z$ be a quasi-continuous extension of g. Let us prove that the point $z_0 = f(0)$ is the limit point of the sequence $(z_n)_{n=1}^{\infty}$. Let W be a neighborhood of z_0 and $n \in \mathbb{N}$. Since f is quasi-continuous at 0 and $U = \left[0; \frac{1}{n}\right)$ is a neighborhood of 0, there exist a non-empty open set $U_1 \subseteq U$, such that $f(U_1) \subseteq W$. Choose $y \in U_1 \cap Y$. Then $f(y) = g(y) = z_m$ for some $m \geq n$. So, $z_m \in W$. And then (z_n) has a limit point in Z, which is impossible.

Now, let us prove the main part of the proposition. Assuming non-compactness of Z, we have that there exist a direct set $(M, \leq)$ and a net $(z_m)_{m \in M}$ without a limit point in Z. Let $X = M \cup \{\infty\}$, where ∞ is some element with $\infty \notin M$. Put $m \leq \infty$ for each $m \in X$. We equip X by the topology generating by the base

$$\big\{\{m\}\colon m \in M\big\} \cup \big\{[m;\infty]\colon m \in M\big\},$$

where $[m;\infty] = \{n \in M\colon m \leq n \leq \infty\}$. Set $Y = M$. Define $g\colon Y \to Z$ by the formula $g(m) = z_m$ for any $m \in M = Y$. Since Y is discrete, g is continuous. Let us prove, that g is needed.

Assuming the contrary, let $f\colon X \to Z$ be a quasi-continuous extension of g. Let us prove that the point $z_0 = f(\infty)$ is the limit point of the net $(z_m)_{m \in M}$. Let W be a neighborhood of z_0 and $m_0 \in M$. Since f is quasi-continuous at ∞ and $U = [m_0;\infty]$ is a neighborhood of ∞, there exist a non-empty open set $U_1 \subseteq U$, such that $f(U_1) \subseteq W$. Choose $m_1 \in U_1 \cap Y$. Then $m_1 \geq m_0$ and $f(m_1) = g(m_1) = z_{m_1}$. So, $z_{m_1} \in W$. And then $(z_m)_{m \in M}$ has a limit point in Z, which is impossible. □

11.4 Extension of real-valued quasi-continuous functions defined on a closed Baire subspace of hereditarily normal space

Proposition 11.5. *Let X be a hereditarily normal topological space, Y be a closed Baire subspace of X and $g\colon Y \to \mathbb{R}$ be a quasi-continuous function. Then there exist a quasi-continuous function $f\colon X \to \mathbb{R}$ such that $f|_Y = g$, $D(f) = D(g)$ and $\sup\limits_{x \in X} |f(x)| = \sup\limits_{y \in Y} |g(y)|$.*

Proof. As it is well known (see for example [4, 7]), since f is quasi-continuous and Y is Baire space, we have that $Y_1 = C(f)$ is dense in Y. Let $g_1 = g|_{Y_1}$. Then g_1 is continuous and it is easy to show that $\sup\limits_{y \in Y_1} |g_1(y)| = \sup\limits_{y \in Y} |g(y)|$. By hereditary normality of X, we have that subspace $X_1 = (X \setminus Y) \cup Y_1$ is normal. But Y_1 is

closed in X_1. So, by the Tietze-Urysohn theorem [1, p. 69] we construct a continuous function $f_1\colon X_1 \to \mathbb{R}$ such that $f_1|_{Y_1} = g_1$ and $\sup\limits_{x\in X_1} |f_1(x)| = \sup\limits_{y\in Y_1} |g_1(y)|$. Define $f\colon X \to \mathbb{R}$ as

$$f(x) = \begin{cases} f_1(x), & x \in X \setminus Y; \\ g(x), & x \in Y. \end{cases}$$

and prove that f is needed. Since $f|_{X_1} = f_1$, we have that $\sup\limits_{x\in X} |f(x)| = \sup\limits_{y\in Y} |g(y)|$.

Fix $x_0 \in X$ and prove that f is quasi-continuous at x_0. If $x_0 \in X \setminus Y$ then this is imply by the continuity of f_1. So, suppose that $x_0 \in Y$. Then $f(x_0) = g(x_0)$. Let $\varepsilon > 0$ and U be an open neighborhood of x. Since g is quasi-continuous, there exists an open set $U_1 \subseteq U$ such that $U \cap Y \neq \emptyset$ and $|g(y) - g(x_0)| < \frac{\varepsilon}{2}$ for any $y \in U_1 \cap Y$. But Y_1 is dense in Y. Thus, there is $x_1 \in U_1 \cap Y_1$. By continuity of f_1 we find an open neighborhood $U_2 \subseteq U_1$ of x_1 such that $|f_1(x) - f_1(x_1)| < \frac{\varepsilon}{2}$ for all $x \in U_2 \cap X_1$. But $f_1(x_1) = g(x_1)$ and $f_1(x) = f(x)$ for all $x \in X \setminus Y$. Therefore, $|f(x) - g(x_1)| < \frac{\varepsilon}{2}$ for each $x \in U_2 \setminus Y$. Then $|f(x) - f(x_0)| = |f(x) - g(x_0)| \leq |f(x) - g(x_1)| + |g(x_1) - g(x_0)| < \frac{\varepsilon}{2}$ for all $x \in U \setminus Y$. But $|f(x) - f(x_0)| = |g(x) - g(x_0)| < \frac{\varepsilon}{2} < \varepsilon$ for every $x \in Y \cap U_2$. So, $|f(x) - f(x_0)| < \varepsilon$ for all $x \in U_2$. Thus, f is quasi-continuous at x_0.

Finally, let us prove that $D(f) = D(g)$. Obviously, $D(g) = Y \setminus C(g) = Y \setminus Y_1$ and $D(g) \subseteq D(f)$. Since f is continuous on $X \setminus Y$, we have that $D(f) \subseteq Y$. For any $y \in Y$ if $y \in Y_1$ then g and f_1 is continuous at y and, so, f is continuous at y. Thus, $D(f) \subseteq Y \setminus Y_1 = D(g)$. Therefore, $D(f) = D(g)$. □

Recall that a topological space X is called *hereditarily Baire* space if each closed subspace Y of X is a Baire space. From Propositions 11.1 and 11.5 we obtain the following result.

Theorem 11.3. *Let X be a hereditarily normal hereditarily Baire topological space, Y be a subspace of X and $g\colon Y \to \mathbb{R}$ be a bounded quasi-continuous function. Then there exist a bounded quasi-continuous function $f\colon X \to \mathbb{R}$ such that $f|_Y = g$, $D(f) = \widetilde{D}(g)$ and $\sup\limits_{x\in X} |f(x)| = \sup\limits_{y\in Y} |g(y)|$.*

11.5 Universal extension of quasi-continuous functions

Definition 11.6. Let X and Z be topological spaces, H be an open subspace of X, $Y = X \setminus H$. A function $h\colon H \to Z$ is called *an universal quasi-continuous extension* if for any function $g\colon Y \to Z$, such that the restriction $g|_{\operatorname{int} Y}$ is quasi-continuous, the function $f = g \cup h\colon X \to Z$, which is defined by the formula

$$f(x) = \begin{cases} g(x)\ , \text{ if } x \in Y; \\ h(x)\ , \text{ if } x \in H, \end{cases} \qquad \text{for any} \quad x \in X,$$

is quasi-continuous.

Firstly, we prove some simple characterization of universal quasi-continuous extensions.

Proposition 11.6. *Let X and Z be topological spaces, H be an open subspace of X, $Y = X \setminus H$ and $h\colon H \to Z$. Then the followings items are equivalent:*

(i) *h is an universal quasi-continuous extension;*
(ii) *h is quasi-continuous and the cluster set $\overline{h}(x) = Z$ for any $x \in \operatorname{fr} H$;*

Proof. $(i) \Rightarrow (ii)$. Prove, that h is quasi-continuous. Let $g_0\colon Y \to Z$ be a constant function. Then (i) implies that $f_0 = g_0 \cup h$ is quasi-continuous. Since H is open, the restriction $h = f_0|_H$ is quasi-continuous too.

Fix $z_0 \in Z$, $x_0 \in \operatorname{fr} H$ and prove that $z_0 \in \overline{h}(x_0)$. Firstly, consider the case, where the only neighborhood of z_0 is Z. Then $z_0 \in \overline{E}$ for any non-empty set $E \subseteq Z$. So, $z_0 \in \overline{h(U \cap H)}$ for any neighborhood U of x_0. Therefore $z_0 \in \overline{h}(x_0)$.

Now assume that there exists a neighborhood W_0 of z_0, such $W_0 \neq Z$. Fix $z_1 \in Z \setminus W_0$. Consider the function $g\colon Y \to Z$ such that $g(y) = z_0$ for all $y \in \operatorname{fr} H = \operatorname{fr} Y = Y \setminus \operatorname{int} Y$ and $g(y) = z_1$ for all $y \in \operatorname{int} Y$. By (i) we conclude that the function $f = g \cup h\colon X \to Z$ is quasi-continuous. Consider open neighborhoods U of x_0 and W of z_0. Since $f(x_0) = z_0$ and f is quasi-continuous, we have that there is a non-empty open set $U_1 \subseteq U$, such that $f(U_1) \subseteq W \cap W_0$. But $f(x) = z_1 \notin W_0$ for any $x \in \operatorname{int} Y$. So, $U_1 \subseteq X \setminus \operatorname{int} Y = \overline{H}$. Therefore, $U_1 \cap H \neq \emptyset$. Choose $x_1 \in U_1 \cap H$. Then $h(x_1) = f(x_1) \in W$. Thus, $W \cap h(U \cap H) \neq \emptyset$. Therefore, $z_0 \in \overline{h(U \cap H)}$ for any $U \in \mathscr{U}_{x_0}$. Then $z_0 \in \overline{h}(x_0)$.

$(ii) \Rightarrow (i)$. Let $g\colon Y \to Z$ be a function, such that the restriction $g|_{\operatorname{int} Y}$ is quasi-continuous, and let $f = g \cup h\colon X \to Z$. We have that the restrictions $f|_{\operatorname{int} Y} = g|_{\operatorname{int} Y}$ and $f|_H = h$ are quasi-continuous. So, it is reminds to show, that f is quasi-continuous at every point of $\operatorname{fr} H$. Fix $x_0 \in \operatorname{fr} H$. Consider open neighborhoods U of x_0 and W of $f(x_0)$. Since $f(x_0) \in Z = \overline{h}(x_0) \subseteq \overline{h(U \cap H)}$, we have that $W \cap h(U \cap H) \neq \emptyset$. Choose $x_1 \in U \cap H$, such that $h(x_1) \in W$. But h is quasi-continuous. So, there exists an non-empty open set $U_1 \subseteq U \cap H$ with $h(U_1) \subseteq W$. Then $f(U_1) = h(U_1) \subseteq W$. Therefore, f is quasi-continuous at x_0. $\square$

The following theorem give some answer to the question on the existing of real-valued universal quasi-continuous extensions.

Theorem 11.4. *Let X be a perfectly normal space and H be an open subspace of X such that $X_0 = \overline{H}$ is locally connected. Then there is a continuous function $h\colon X \setminus Y \to \mathbb{R}$ which is an universal quasi-continuous extension.*

Proof. Set $U = X \setminus Y$, $V = \mathrm{int} Y$, $F = Y \setminus V$. Then $X_0 = U \cup F = X \setminus V$ and F is a nowhere dense subset of X_0. Let $\varphi\colon (0;+\infty) \to \mathbb{R}$ be a function which is defined by the formula $\varphi(t) = \frac{1}{t}\sin\frac{1}{t}$ for any $t > 0$. Obviously, φ is continuous and

$$\varphi\big((0;\delta)\big) = \mathbb{R} \quad \text{for all} \quad \delta > 0. \tag{11.1}$$

By the Vedenissoff theorem [1, p. 45], there exists a continuous function $\psi\colon X \to [0;1]$ such that $\psi^{-1}(0) = F$. Put $h = \varphi \circ \psi$. Obviously, h is continuous.

Prove that h is an universal quasi-continuous extension. Fix $x_0 \in F$ and prove that $\overline{h}(x_0) = \mathbb{R}$. Let U be a neighborhood of x_0. By the local connectedness of X_0, there exists a connected neighborhood U_1 of x_0 in X_0, such that $U_1 \subseteq U$. But F is nowhere dense in X_0. So, there exists a point $x_1 \in U_1 \setminus F$. Then $\psi(x_1) > 0$ and $\psi(x_0) = 0$. Let $\delta = \psi(x_1)$. Since U_1 is connected and ψ is continuous, we have that $\psi(U_1) \supseteq [0;\delta]$. But $\psi(x) = 0$ on F and $U_1 \cap H = U_1 \setminus F$. Therefore, $\psi(U_1 \cap H) \supseteq (0;\delta)$. Then by (11.1) we conclude that $h(U \cap H) \supseteq h(U_1 \cap H) = \varphi\Big(\psi(U_1 \cap H)\Big) \supseteq \varphi\Big((0;\delta)\Big) = \mathbb{R}$. Thus, $\overline{h}(x_0) = \mathbb{R}$. Then Proposition 11.6 implies that h is an universal quasi-continuous extension. □

Proposition 11.5 and Theorem 11.4 imply the following.

Corollary 11.2. *Let X be a perfectly normal space, Y be a subspace of X, such that $X_0 = X \setminus \mathrm{int}\overline{Y}$ is locally connected, and $g\colon Y \to \mathbb{R}$ be a bounded quasi-continuous function. Then there is a quasi-continuous extension $f\colon X \to \mathbb{R}$ of g.*

11.6 Multicellular spaces and existing of universal quasi-continuous extensions

Definition 11.7. For a topological space X the cardinal numbers

$$d(X) = \min\{\mathrm{card} A\colon A \text{ is a dense subset of } X\}$$

and

$$c(X) = \sup\{\mathrm{card}\,\mathscr{U} : \mathscr{U} \text{ is a disjoint system of non-empty open subsets of } X\}$$

are called *the density* and *the cellularity* of X respectively. We call a topological

space X *multicellular* if there exists a disjoint family $\mathscr{U}$ of non-empty open subsets of X with cardinality $\operatorname{card}\mathscr{U} = d(X)$.

Clearly, $c(X) \leq d(X)$. Than $c(X) = d(X)$ for any multicellular space X. But the inverse statement is not true.

Proposition 11.7. *If $\mathfrak{m}$ is weakly inaccessible cardinal (i. e. regular and limit) and $X = \prod_{\mathfrak{n}<\mathfrak{m}} \mathbb{D}(\mathfrak{n})$, where $\mathbb{D}(\mathfrak{n})$ is the discrete space of the cardinality $\mathfrak{n}$, then $d(X) = c(X) = \mathfrak{m}$ but X is not multicellular.*

Proof. I. Juhász [2, 7.6] proves, that $c(X) = \hat{c}(X) = \mathfrak{m}$, that is, for any disjoint open family $\mathscr{U}$ in X we have that $\operatorname{card}\mathscr{U} < \mathfrak{m}$. But from the Hewitt-Marczewski-Pondiczery theorem [1, p. 81] we easily obtain that $d(X) = \mathfrak{m}$. Therefore, X is not multicellular and $c(X) = d(X)$. □

On the other hand, we have the following.

Proposition 11.8. *Let X be a topological space, such that $c(X) = d(X)$ is singular. Then X is multicellular.*

Proof. I. Juhász [2, 4.1] proves, that if $c(X)$ is singular then there exists a disjoint open family $\mathscr{U}$ with $\operatorname{card}\mathscr{U} = c(X)$. So, we have that

$$\operatorname{card}\mathscr{U} = c(X) = d(X).$$

□

Proposition 11.9. *Let X be a Hausdorff separable topological space. Then X is multicellular.*

Proof. Let T be a dense subspace of X, such that $\operatorname{card} T = d(X) \leq \aleph_0$. Firstly, consider the case, where T has an accumulation points. So, there exists a point $a \in X$, such that $a \in \overline{T \setminus \{a\}}$. Let $T_0 = T \setminus \{a\}$. Choose $x_1 \in T_0$. Then we can construct disjoint open neighborhoods U_1 of x_1 and V_1 of a. Since $a \in \overline{T}_0$, there is a point $x_2 \in V_2 \cap T_0$. Thus, there exist disjoint open neighborhoods $U_2 \subseteq V_1$ of x_2 and $V_2 \subseteq V_1$ of a. Then we choose a point $x_3 \in V_2 \cap T_0$. And so on. In this way we can construct a disjoint sequence of non-empty open sets U_n in X.

Now we consider the case, where T has no accumulation points. In this case T is closed and discrete. But $X = \overline{T} = T$. Thus, X is discrete. So, family $\mathscr{U} = \{\{x\} : x \in X\}$ is needed. □

Proposition 11.10. *Let X be a metrizable topological space. Then X is multicellular.*

Proof. As it is well-known the cellularity and the density of a metrizable space are equals [1, p. 255]. Let $d(X) = c(X) = \mathfrak{m}$. By Propositions 11.8 and 11.9 we can assume, that $\mathfrak{m} > \aleph_0$ is regular. Fix some metric ρ generating the topology of X. Applying the Teichmüller-Tukey lemma [1, p. 8], we can choose for any $n \in \mathbb{N}$ a maximal set T_n, such that $\rho(x,y) \geq \frac{1}{n}$ for any distinct points $x, y \in S_n$. So, we have, that $\rho(x,T_n) < \frac{1}{n}$ for any $x \in X$ and $n \in \mathbb{N}$. Then $T = \bigcup\limits_{n=1}^{\infty} T_n$ is dense. Let $\mathfrak{m}_n = \operatorname{card} T_n$. Since $\mathscr{U}_n = \{B(x, \frac{1}{2n}) \colon x \in T_n\}$ is disjoint, we have that $\mathfrak{m}_n = \operatorname{card} \mathscr{U}_n \leq c(X) = \mathfrak{m}$. Thus, $\mathfrak{m} = d(X) \leq \operatorname{card} T = \sup\limits_{n\in\mathbb{N}} \mathfrak{m}_n \leq \mathfrak{m}$. So, $\mathfrak{m} = \sup\limits_{n\in\mathbb{N}} \mathfrak{m}_n$. But $\mathfrak{m}$ is regular. Then cofinality $cf(\mathfrak{m}) = \mathfrak{m} > \aleph_0$. Therefore, there exists n with $\mathfrak{m}_n = \mathfrak{m}$. Then $\mathscr{U}_n$ is a disjoint open family of cardinality $\mathfrak{m}$. Thus, X is multicellular. □

Lemma 11.1. *Let X and Z be topological spaces, H be an open subspace of X, $Y = X \setminus H$, $h \colon H \to Z$ be an universal quasi-continuous extension and $(W_t)_{t\in T}$ be a disjoint family of non-empty open sets W_t in Z. Then there exists a disjoint family $(H_t)_{t\in T}$ of non-empty open sets H_t in H, such that $\operatorname{fr} H \subseteq \overline{H}_t$ for any $t \in T$.*

Proof. Denote $H_t = \operatorname{int} h^{-1}(W_t)$ for $t \in T$. Obviously, the sets H_t are disjoint and open. Fix $t \in T$ and $x \in \operatorname{fr} H$. To prove that $x \in \overline{H}_t$ consider an open neighborhood U of x. By Proposition 11.6 we have that h is quasi-continuous and the cluster set $\overline{h}(x) = Z$. Thus $W_t \cap \overline{h(H \cap U)} \supseteq W_t \cap \overline{h}(x) \neq \emptyset$. Then $W_t \cap h(H \cap U) \neq \emptyset$. Therefore, the quasi-continuity of h implies that there is a non-empty open subset U_1 of $H \cap U$ such that $h(U_1) \subseteq W_t$. So, $U \cap H_t = U \cap \operatorname{int} h^{-1}(W_t) \supseteq U_1 \neq \emptyset$. □

Lemma 11.2. *Let X be a topological spaces, Z be a compact, H be an open subspace of X, $(H_t)_{t\in T}$ be a disjoint family of non-empty open sets H_t in H, such that $\operatorname{fr} H \subseteq \overline{H}_t$ for any $t \in T$ and $\{z_t : t \in T\}$ be a dense subset of Z. Then there exists an universal quasi-continuous extension $h \colon H \to Z$.*

Proof. Let $G = \bigcup\limits_{t\in T} H_t$. Define $g \colon G \to Z$ by the formula $g(x) = z_t$ if $x \in H_t$ for some $t \in T$. Since $\operatorname{fr} H \subseteq \overline{H}_t$ for any $t \in T$, we conclude that $\operatorname{fr} H \subseteq \overline{G}$.

Fix $x \in \operatorname{fr} H$ and prove that $\overline{g}(x) = Z$. Let $z \in Z$, U is a neighborhood of x and W is a neighborhood of z. Since $\overline{\{z_t : t \in T\}} = Z$, we can find $t \in T$ such that $z_t \in W$. But $U \cap H_t \neq \emptyset$. Thus, $g(U \cap H) \supseteq g(U \cap H_t) = \{z_t\}$. Therefore,

$g(U \cap H) \cap W \neq \emptyset$ for any neighborhood W of z. So, $z \in \overline{g(U \cap H)}$ for any neighborhood U of x. Then $z \in \bigcup_{U \in \mathscr{U}_x} \overline{g(U \cap H)} = \overline{g}(x)$ for any $z \in Z$. Therefore, $\overline{g}(x) = Z$.

By Corollary 11.1 we can construct a quasi-continuous extension $h\colon H \to Z$ of the function g. Then for any $x \in \operatorname{fr} H$ we have that $Z \subseteq \overline{g}(x) \subseteq \overline{h}(x)$. So, $\overline{h}(x) = Z$ for such x. Thus, Proposition 11.6 implies that h is an universal quasi-continuous extension. □

From the Lemmas 11.1 and 11.2 we immediately obtain the following.

Theorem 11.5. *Let X be a topological space, Z be a multicellular compact, $\mathfrak{m} = d(Z)$, H be an open subset of X. Then the following conditions are equivalent*

(i) *there exists an universal quasi-continuous extension $h\colon H \to Z$;*
(ii) *there exists a disjoint family $(H_t)_{t \in T}$ of open subsets H_t of H, such that* $\operatorname{card} T = \mathfrak{m}$ *and* $\operatorname{fr} H \subseteq \overline{H}_t$ *for any $t \in T$.*

Using Proposition 11.9 we conclude from previous theorem the following.

Theorem 11.6. *Let X be a topological space, Z be an infinity separable compact, H be an open subset of X. Then the following conditions are equivalent*

(i) *there exists an universal quasi-continuous extension $h\colon H \to Z$;*
(ii) *there exists a disjoint family $(H_n)_{n=1}^{\infty}$ of non-empty open sets H_n in H, such that* $\operatorname{fr} H \subseteq \overline{H}_n$ *for any $n \in \mathbb{N}$.*

Definition 11.8. A topological space X is called *weakly pairwise attainable* if for any open set G and any closed subset F of $\overline{G} \setminus G$ there exist disjoint opens subsets U and V of G such that $F = \overline{U} \setminus G = \overline{V} \setminus G$.

In [6] it was proved the following: if X is a perfectly normal Fréchet-Urysohn space such that for any closed subset F of X there is a strongly σ-discrete set E with $\overline{E} = F$ then X is weakly pairwise attainable. In particular, in [6] (see also [5]) it was proved that every metrizable space is weakly pairwise attainable.

Theorem 11.7. *Let X be a weakly pairwise attainable space, Z be a separable compact, H be an open subset of X. Then there exists an universal quasi-continuous extension $h\colon H \to Z$.*

Proof. Let $F = \operatorname{fr} H$. Since X is weakly pairwise attainable, there are disjoint opens subsets H_1 and G_1 of H such that $F = \overline{H}_1 \setminus H = \overline{G}_1 \setminus H$. In particular,

$F \subseteq \operatorname{fr} G_1$. Then there are disjoint opens subsets H_2 and G_2 of G_1 such that $F = \overline{H}_2 \setminus G_1 = \overline{G}_2 \setminus G_2$. So, we can construct a sequence of nonempty open sets G_n and H_n such that $G_1 \sqcup H_1 \subseteq H$, $G_{n+1} \sqcup H_{n+1} \subseteq G_n$ and $F \subseteq \overline{G}_n \cap \overline{H}_n$ for any $n \in \mathbb{N}$. Therefore, the sets H_n are pairwise disjoint and $F \subseteq \overline{H}_n$ for each n.

By Proposition 11.9 Z is multicellular and $\nu = d(X) \leq \aleph_0$. Put $T = \mathbb{N}$ if $\nu = \aleph_0$ and $T = \{1, 2, \dots \nu\}$ if $\nu < \aleph_0$. Then $(H_n)_{n \in T}$ satisfies the condition (ii) from Theorem 11.5. Thus, there exists an universal quasi-continuous extension $h \colon H \to Z$. □

The Theorems 11.7 and 11.2 immediately imply the following.

Corollary 11.3. *Let X be a weakly pairwise attainable space, Z be a separable compact, $Y \subseteq X$ and $g \colon Y \to Z$ be a quasi-continuous function. Then there exist a quasi-continuous function $f \colon X \to Z$, such that $f|_Y = g$.*

11.7 Quasi-open, quasi-closed and quasi-clopen sets

Definition 11.9. Let X be a topological space and $A \subseteq X$. A set A is called

- *quasi-open* if $A \subseteq \overline{\operatorname{int} A}$;
- *quasi-closed* if $\operatorname{int} \overline{A} \subseteq A$;
- *quasi-clopen* if $\operatorname{int} \overline{A} \subseteq A \subseteq \overline{\operatorname{int} A}$;
- *regularly open* if $A = \operatorname{int} \overline{A}$;
- *regularly closed* if $A = \overline{\operatorname{int} A}$.

It is easy to see that a function $f \colon X \to Y$ between topological spaces X and Y is quasi-continuous if and only if $f^{-1}(V)$ is quasi-open in X for each open set V in Y. In [3] quasi-open sets and quasi-continuous functions are called semi-open and semi-continuous respectively. But in modern articles such functions f is always called quasi-continuous. So, we use term "quasi-open" for such sets. Regularly open (closed) sets are sometimes called open (closed) domain (see [1, p. 20]) or canonically open (closed). Quasi-clopen sets sometimes are called canonical.

Proposition 11.11. *Let X be a topological space and $A \subseteq X$. Then*

(i) *A is quasi-open if and only if $X \setminus A$ is quasi-closed;*
(ii) *A is quasi-closed if and only if $X \setminus A$ is quasi-open;*
(iii) *if A is open then A is quasi-open;*
(iv) *if A is closed then A is quasi-closed;*
(v) *A is quasi-clopen if and only if A is both quasi-open and quasi-closed;*

(vi) *A is regularly open if and only if A is both quasi-clopen and open;*
(vii) *A is regularly closed if and only if A is both quasi-clopen and closed;*
$(viii)$ *A is quasi-clopen if and only if $X \setminus A$ is quasi-clopen;*
(ix) *if A is closed then* $\mathrm{int}A$ *is regularly open;*
(x) *if A is open then $\overline{A}$ is regularly closed.*

Proof. (i) As it is well known, for any $E \subseteq X$ we have that

$$\mathrm{int}(X \setminus E) = X \setminus \overline{E} \qquad \text{and} \qquad \overline{X \setminus E} = X \setminus \mathrm{int}E.$$

By definition of quasi-open sets we have that $A \subseteq \overline{\mathrm{int}A}$. Put $B = X \setminus A$. So,

$$B = X \setminus A \supseteq X \setminus \overline{\mathrm{int}A} = \mathrm{int}(X \setminus \mathrm{int}A) = \mathrm{int}\overline{X \setminus A} = \mathrm{int}\overline{B}.$$

Therefore B is quasi-closed.

(ii) Obviously, $(ii) \Leftrightarrow (i)$.

(iii) Let A be open. Then $A = \mathrm{int}A \subseteq \overline{\mathrm{int}A}$. So, A is quasi-open.

(iv) Let A be closed. Then $A = \overline{A} \supseteq \overline{\mathrm{int}A}$. So, A is quasi-closed.

(v) This item is evident.

(vi) Let A be a regularly open subset of X. Then $A = \mathrm{int}\overline{A}$. So, A is quasi-closed. Openness of the interior implies that A is open. Thus, by (iii), A is quasi-open. Then, by (v), A is quasi-clopen.

Let A be an open subset of X which is quasi-clopen. Then A is quasi-closed. Thus, $\mathrm{int}\overline{A} \subseteq A = \mathrm{int}A \subseteq \mathrm{int}\overline{A}$. Therefore, $\mathrm{int}\overline{A} = A$. So, A is regularly open.

(vii) Let A be a regularly closed subset of X. Then $A = \overline{\mathrm{int}A}$. So, A is quasi-open. Closeness of the closure implies that A is closed. Thus, by (iv), A is quasi-closed. Then, by (v), A is quasi-clopen.

Let A be a closed subset of X which is quasi-clopen. Then A is quasi-open. Thus, $\overline{\mathrm{int}A} \supseteq A = \overline{A} \supseteq \overline{\mathrm{int}A}$. Therefore, $\overline{\mathrm{int}A} = A$. So, A is regularly closed.

$(viii)$ This item follows from (i), (ii) and (v).

(ix) Let $B = \mathrm{int}A$. Then $\mathrm{int}\overline{B} \subseteq \overline{B} \subseteq \overline{A} = A$. So, $\mathrm{int}\overline{B} \subseteq \mathrm{int}A = B$. But $B = \mathrm{int}B \subseteq \mathrm{int}\overline{B}$. Therefore, $B = \mathrm{int}\overline{B}$, that is B is regularly open.

(x) Put $B = \overline{A}$. Then $\overline{\mathrm{int}B} \supseteq \mathrm{int}B \supseteq \mathrm{int}A = A$. So, $\overline{\mathrm{int}B} \supseteq \overline{A} = B$. But $B = \overline{B} \supseteq \mathrm{int}\overline{B}$. Therefore, $B = \overline{\mathrm{int}B}$, that is B is regularly closed. □

Proposition 11.12. *Let X be a topological space and $A \subseteq X$. Then*

(i) *A is quasi-open if and only if there is an open set U and a set $E \subseteq \overline{U} \setminus U$ such that $A = U \cup E$;*
(ii) *A is quasi-closed if and only if there is a closed set F and a set $E \subseteq F \setminus \mathrm{int}F$ such that $A = F \setminus E$;*

(iii) *A is quasi-open if and only if there exists an regularly closed set F and a nowhere dense set $E \subseteq F$ such that $A = F \setminus E$;*
(iv) *A is quasi-closed if and only if there exists an regularly open set U and a nowhere dense set $E \subseteq X \setminus U$ such that $A = U \cup E$;*
(v) *A is quasi-clopen if and only if there is a regularly open set U and a set $E \subseteq \overline{U} \setminus U$ such that $A = U \cup E$;*
(vi) *A is quasi-clopen if and only if there is a regularly closed set F and a set $E \subseteq F \setminus \operatorname{int} F$ such that $A = F \setminus E$.*

Proof. (i) Let A be a quasi-open subset of X. Put $U = \operatorname{int} A$ and $E = A \setminus U$. Then $A = U \cup E$. By the definition of quasi-open sets, we have that $A \subseteq \overline{U}$. So, $E = A \setminus U \subseteq \overline{U} \setminus U$. Therefore, U and E are to be found.

Conversely, let $A = U \cup E$ for some open set U and set $E \subseteq \overline{U} \setminus U$. Since $U = \operatorname{int} U \subseteq \operatorname{int} A$, we conclude that $A = U \cup E \subseteq \overline{U} \subseteq \overline{\operatorname{int} A}$. Thus, A is quasi-open.

(ii) Let A be a quasi-closed subset of X. Put $F = \overline{A}$ and $E = F \setminus A$. Then $A = F \setminus E$. By the definition of quasi-closed sets, we have that $\operatorname{int} F \subseteq A$. So, $E = F \setminus A \subseteq F \setminus \operatorname{int} F$. Therefore, F and E are to be found.

Conversely, let $A = F \setminus E$ for some closed set F and set $E \subseteq F \setminus \operatorname{int} F$. Since $F = \overline{F} \supseteq \overline{A}$, we conclude that $A = F \setminus E \supseteq F \setminus (F \setminus \operatorname{int} F) = \operatorname{int} F \supseteq \operatorname{int} \overline{A}$. Thus, A is quasi-closed.

(iii) Let $U = \operatorname{int} A$. Since A is quasi-open, $A \subseteq \overline{U}$. Proposition 11.11(x) yields that $F = \overline{U}$ is regularly closed. Set $E = F \setminus A$. Then $A = F \setminus E$, $E \subseteq \overline{U} \setminus U = \operatorname{fr} U$ is nowhere dense. So, F and E is to be found.

(iv) Let $F = \overline{A}$. Since A is quasi-closed, $\operatorname{int} F \subseteq A$. Proposition 11.11($ix$) yields that $U = \operatorname{int} F$ is regularly closed. Set $E = A \setminus U$. Then $A = U \cup E$, $E \subseteq F \setminus \operatorname{int} F = \operatorname{fr} F$ is nowhere dense. So, U and E is to be found.

(v) and (vi) Let A be a quasi-clopen subset of X. As in (i) and (ii), put $U = \operatorname{int} A$, $F = \overline{A}$ and $E' = A \setminus U$ and $E'' = F \setminus A$. Then $A = U \cup E' = F \setminus E''$, $E' \subseteq \overline{U} \setminus U$ and $E'' \subseteq F \setminus \operatorname{int} F$. Let us prove that U is regularly open and F is regularly closed. Taking to account that $U \subseteq F$ and by the definition of quasi-clopen subsets $\operatorname{int} F \subseteq A \subseteq \overline{U}$, we obtain that $\operatorname{int} F = \operatorname{int}(\operatorname{int} F) \subseteq \operatorname{int} A = U = \operatorname{int} U \subseteq \operatorname{int} F$ and $\overline{U} \subseteq \overline{F} = F = \overline{A} \subseteq \overline{(\overline{U})} = \overline{U}$. Thus, $\operatorname{int} F = U$ and $\overline{U} = F$. Therefore, $\overline{\operatorname{int} F} = \overline{U} = F$ and $\operatorname{int} \overline{U} = \operatorname{int} F = U$. So, U is regularly open and F is regularly closed.

Let $A = U \cup E$ where U is regularly open subset of X and $E \subseteq \overline{U} \setminus U$. Then by ($i$) we have that A is quasi-open. To prove that A is quasi-closed, observe that $\operatorname{int} \overline{A} = \operatorname{int}(\overline{U} \cup \overline{E}) = \operatorname{int} \overline{U} = U \subseteq A$. Then by Proposition 11.13(v) we have that A is quasi-clopen.

Let $A = F \setminus E$ where F is regularly closed subset of X and $E \subseteq F \setminus \mathrm{int}F$. Then by (ii) we have that A is quasi-closed. Since $F \setminus \mathrm{int}F$ is closed, we have that $\overline{E} \subseteq F \setminus \mathrm{int}F$ and then $\overline{E} \cap \mathrm{int}F = \emptyset$. So, $\mathrm{int}A = \mathrm{int}(F \setminus E) = \mathrm{int}(F \cap (X \setminus E)) = \mathrm{int}F \cap \mathrm{int}(X \setminus E) = \mathrm{int}F \cap (X \setminus \overline{E}) = \mathrm{int}F \setminus \overline{E} = \mathrm{int}F$. Thus, $\overline{\mathrm{int}A} = \overline{\mathrm{int}F} = F \supseteq A$. Therefore, A is quasi-open. Then by Proposition 11.13(v) we have that A is quasi-clopen. □

Proposition 11.13. *Let X be a topological space and $B \subseteq A \subseteq X$. Then*

(i) *if A is quasi-open in X and B is quasi-open in A then B is quasi-open in X;*

(ii) *if A is quasi-closed in X and B is quasi-closed in A then B is quasi-closed in X;*

(iii) *if A is quasi-clopen in X and B is quasi-clopen in A then B is quasi-clopen in X.*

Proof. (i) Put $U = \mathrm{int}A$ and $V = \mathrm{int}_A B$. Then $A \subseteq \overline{U}$ and $B \subseteq \overline{V}$. Since V is open in A and U is dense in A, we have that the set $W = U \cap V$ is dense in V. So, $\overline{W} = \overline{V}$. On the other hand, W is open in U and U is open in X. Therefore, W is open in X. So, $W \subseteq \mathrm{int}B$. Then $B \subseteq \overline{W} \subseteq \overline{\mathrm{int}B}$.

(ii) Put $U = \mathrm{int}\overline{A}$ and $V = \mathrm{int}_A(\overline{B} \cap A)$. Then $U \subseteq A$ and $V \subseteq B$. Since V is open in U, the set $W = U \cap W$ is open in U. But U is open in X. Therefore W is open in X.

Let us prove that $E = A \setminus U$ is nowhere dense in X. Indeed, $E \subseteq X \setminus U$ implies $\mathrm{int}\overline{E} \subseteq \overline{E} \subseteq \overline{X \setminus U} = X \setminus U$. In the other hand, $\mathrm{int}\overline{E} \subseteq \mathrm{int}\overline{A} = U$. Therefore $\mathrm{int}\overline{E} = 0$ and then E is nowhere dense.

Analogously, the set $E_1 = B \setminus V$ is nowhere dense in A. Since U is open, $E_1 \cap U$ is nowhere dense in U. Therefore $E_1 \cap U$ is nowhere dense in X. But $E_1 \setminus U$ is nowhere dense in X as a subset of the nowhere dense set $A \setminus U$. So, E_1 is nowhere dense in X, because $E_1 = (E_1 \setminus U) \cup (E_1 \cap U)$.

Set $G = \mathrm{int}\overline{W}$ and prove that $G = \mathrm{int}\overline{B}$. Clearly, $W \subseteq V \subseteq B$. Then $G = \mathrm{int}\overline{W} \subseteq \mathrm{int}\overline{B}$. To prove the inverse inclusion, we observe that $B \setminus \overline{W}$ is nowhere dense in X, because $B \setminus \overline{W} \subseteq B \setminus W = (B \setminus U) \cup (B \setminus V) \subseteq E \cup E_1$. But $X \setminus \overline{W}$ is open set. So, $\overline{B} \setminus \overline{W} = \overline{B} \cap (X \setminus \overline{W}) \subseteq \overline{B \cap (X \setminus \overline{W})} = \overline{B \setminus \overline{W}}$. Thus, $\overline{B} \setminus \overline{W}$ is nowhere dense in X. So, $\mathrm{int}\overline{B} \subseteq \overline{W}$. Therefore, $\mathrm{int}\overline{B} \subseteq \mathrm{int}\overline{W} = G$.

Thus, we have proved that $\mathrm{int}\overline{B} = G$.

(iii) This item follows from (i), (ii) and Proposition 11.11(v). □

The following proposition implies straitly from the definitions.

Proposition 11.14. *Let X and Y be topological spaces and $f\colon X \to Y$. Then the following statement are equivalent:*

(i) *f is quasi-continuous;*
(ii) *for any open subset G of Y the pre-image $f^{-1}(G)$ is quasi-open;*
(iii) *for any closed subset F of Y the pre-image $f^{-1}(F)$ is quasi-closed.*

11.8 Quasi-clopen partitions and quasi-continuous functions

We need some auxiliary statements.

Lemma 11.3. *Let X be a hereditarily normal space, $Y \subseteq X$ and B be a quasi-clopen subset of Y. Then there exist a quasi-clopen subset A of X with $A \cap Y = B$.*

Proof. Put $V = \operatorname{int}_Y \overline{B}^Y$ and $V' = Y \setminus \overline{B}^Y$. Since B is quasi-clopen in Y, we have that $V \subseteq B \subseteq \overline{V}^Y$. In particular, $\overline{B}^Y = \overline{V}^Y$. Thus, $V = \operatorname{int}_Y \overline{V}^Y$. Obviously, $V \cap \overline{V'}^Y = \overline{V}^Y \cap V' = \emptyset$. So, $V \cap \overline{V'} = \overline{V} \cap V' = \emptyset$, that is V and V' are separated in X. By [1, Theorem 2.1.7.] we have that there exist open subsets U and U' of X, such that $V \subseteq U$, $V' \subseteq U'$ and $U \cap U' = \emptyset$. Then $V \subseteq U \cap Y \subseteq (X \setminus U') \cap Y = Y \setminus (U' \cap Y) \subseteq Y \setminus V' = \overline{B}^Y = \overline{V}^Y$. So, $V \subseteq U \cap Y \subseteq \operatorname{int}_Y \overline{V}^Y = V$. Therefore, $U \cap Y = V$. Put $F = \overline{U}$. By Proposition 11.11(ix), F is a regularly open subset of X. Observe, that $\operatorname{int}_Y \overline{V}^Y = V = U \cap Y \subseteq \operatorname{int} F \cap Y \subseteq F \cap Y \subseteq (X \setminus U') \cap Y \subseteq Y \setminus V' = \overline{V}^Y$. So, $\operatorname{int} F \cap Y = V$ and $F \cap Y = \overline{V}^Y$. Let $E = \overline{V}^Y \setminus B$ and $A = F \setminus E$. Then $E \subseteq \overline{V}^Y \setminus V = (F \cap Y) \setminus (\operatorname{int} F \cap Y) \subseteq F \setminus \operatorname{int} F$. By Proposition 11.13$(vi)$, we have that B is quasi-clopen. Observe, that $A \cap Y = (F \setminus E) \cap Y = (F \cap Y) \setminus E = \overline{V}^Y \setminus (\overline{V}^Y \setminus B) = B$. Therefore, A is to be found. □

Definition 11.10. A system $\mathscr{A}$ of subsets of a topological space X is called *a quasi-clopen partition of* X if $\mathscr{A}$ is a finite system of pairwise disjoint non-empty quasi-clopen subsets of X, such that $\bigcup \mathscr{A} = X$.

Lemma 11.4. *Let X be a hereditarily normal space, $Y \subseteq X$ and $\mathscr{B}$ be a quasi-clopen partition of Y. Then there exist a quasi-clopen partition $\mathscr{A}$ of X such that $\mathscr{B} = \{A \cap Y : A \in \mathscr{A}\}$.*

Proof. Prove this lemma by induction on $\operatorname{card} \mathscr{B}$. If $\operatorname{card} \mathscr{B} = 0$ then $\mathscr{B} = \emptyset$ and $\mathscr{A} = \emptyset$ is needed. Suppose, that for some $n > 0$ we have, that the lemma

holds for any $\mathscr{B}$ with $\operatorname{card}\mathscr{B} = n-1$. Let $\operatorname{card}\mathscr{B} = n$. Since $n > 0$, there exists $B_1 \in \mathscr{B}$. By Lemma 11.3, there exists a quasi-clopen in X set A_1, such that $A_1 \cap Y = B_1$. Put $X' = X \setminus A_1$, $Y' = Y \setminus B_1$ and $\mathscr{B}' = \mathscr{B} \setminus \{B_1\}$. Then $\mathscr{B}'$ is a quasi-clopen partition of a subspace Y' of a hereditarily normal space X' with $\operatorname{card}\mathscr{B}' = n-1$. By the inductive assumption, we have that there exists a quasi-clopen partition $\mathscr{A}'$ of X', such that $\mathscr{B}' = \{A \cap Y' \colon A \in \mathscr{A}'\}$. By Proposition 11.13(*viii*), Y' is quasi-clopen in X. Then Proposition 11.13(*iii*) implies, that for any $A \in \mathscr{A}'$, we have that A is quasi-clopen in X. Therefore, $\mathscr{A} = \mathscr{A}' \cup \{A_1\}$ is needed. □

Lemma 11.5. *Let Y be a topological space, Z be a metric compact, $g \colon Y \to Z$ be a quasi-continuous function and $\varepsilon > 0$. Then there exists a quasi-clopen partition $\mathscr{B}$ of Y such that* $\operatorname{diam} g(B) < \varepsilon$ *for all $B \in \mathscr{B}$.*

Proof. Let $n_\varepsilon(Z)$ be the minimal of numbers $n \in \mathbb{N}$ for which there exists a finite open covering $\{W_1, W_2, \dots, W_n\}$ of Z, such that $\operatorname{diam} W_i < \varepsilon$ for all $i = 1, 2, \dots, n$. Prove our statement by the induction on $n_\varepsilon(Z)$. If $n_\varepsilon(Z) = 1$, then $\operatorname{diam} Z < \varepsilon$. So, $\mathscr{B} = \{Y\} \setminus \{\emptyset\}$ is needed.

Suppose, that for some $n \in \mathbb{N}$ the lemma holds for any Y, Z and g with $n_\varepsilon(Z) < n$. Fix some Y, Z, g with $n_\varepsilon(Z) = n$ and prove the existing of $\mathscr{B}$. Choose a finite open covering $\{W_1, W_2, \dots, W_n\}$ of Z, such that $\operatorname{diam} W_i < \varepsilon$ for all $i = 1, 2, \dots, n$. Set $A_n = g^{-1}(W_n)$, $M_n = g^{-1}(\overline{W}_n)$. By Proposition 11.14, we conclude that A_n is quasi-open and M_n is quasi-closed. Thus, $A_n \subseteq \overline{\operatorname{int} A_n}$ and $\operatorname{int}\overline{M}_n \subseteq M_n$. Let $U_n = \operatorname{int}\overline{A}_n$. Put $B_n = M_n \cap \overline{U}_n$. Then $g(B_n) = g(M_n) \subseteq \overline{W}_n = \overline{W}_n$. So, $\operatorname{diam} g(B_n) \le \operatorname{diam} \overline{W}_n = \operatorname{diam} W_n < \varepsilon$.

Since $A_n \subseteq M_n$, we have that $U_n \subseteq \operatorname{int}\overline{M}_n \subseteq M_n$. Thus, $U_n \subseteq \overline{U}_n \cap M_n = B_n \subseteq \overline{U}_n$. By Proposition 11.11(*ix*), U_n is regularly open. Let $E_n = M_n \setminus U_n$. Then $E_n \subseteq \overline{U}_n \setminus U_n$ and $B_n = U_n \cup E_n$. By Proposition 11.13(*v*), B_n is quasi-clopen.

Put $Y' = Y \setminus B_n$, $Z' = Z \setminus W_n$ and $g' = g|_{Y'}$. Clearly, $n_\varepsilon(Z') \le n-1$. By Proposition 11.11(*viii*), Y' is quasi-clopen. Let us prove, that g' is quasi-continuous. Fix $y_0 \in Y'$. Consider open sets V in Y and W in Z, such that $y_0 \in V$ and $f(y_0) \in W$. Since $y_0 \notin B_n = \overline{U}_n \cap M_n$, we conclude that $y_0 \notin \overline{U}_n$ or $y_0 \notin M_n$.

Firstly, consider the case, where $y_0 \notin \overline{U}_n$. By quasi-continuity of g, we have that there exists nonempty open sets $U \subseteq V \setminus \overline{U}_n$ with $g(U) \subseteq W$. Since $B_n \subseteq \overline{U}_n$, we obtain that $U \subseteq Y \setminus \overline{U}_n \subseteq Y \setminus B_n = Y'$. Then g' is quasi-continuous at y_0.

Now, consider the case, where $y_0 \notin M_n$. Then $g(y_0) \notin \overline{W}_n$. By quasi-continuity of g, we have that there exists a nonempty open set $U \subseteq V$ with $g(U) \subseteq W \setminus \overline{W}_n$. Then $g(U) \subseteq Z \setminus \overline{W}_n$. Thus, $U \subseteq g^{-1}(Z \setminus \overline{W}_n) = Y \setminus g^{-1}(\overline{W}_n) = Y \setminus M_n \subseteq Y \setminus B_n = Y'$. Then g' is quasi-continuous at y_0.

Clearly, Y', Z' and g' satisfy all the required conditions and $n_\varepsilon(Z') < n$. Then, by the inductive assumption, we conclude that there exist a quasi-clopen partition $\mathscr{B}'$ of Y', such that $\operatorname{diam} g'(B) < \varepsilon$ for any $B \in \mathscr{B}'$. By Proposition 11.13(*iii*), since Y' is quasi-clopen in Y, we have that B is quasi-clopen in Y for every $B \in \mathscr{B}'$. Then $\mathscr{B} = \mathscr{B}' \cup (\{B_n\} \setminus \{\emptyset\})$ is to be found. □

The following lemma follows immediately from Lemmas 11.4 and 11.5.

Lemma 11.6. *Let X be a hereditarily normal space, $Y \subseteq X$, Z be a metric compact, $g\colon Y \to Z$ be a quasi-continuous function and $\varepsilon > 0$. Then there exists a quasi-clopen partition $\mathscr{A}$ of X such that $\mathscr{B} = \{A \cap Y : A \in \mathscr{A}\}$ is a quasi-clopen partition of Y and* $\operatorname{diam} g(B) < \varepsilon$ *for all $B \in \mathscr{B}$.*

Lemma 11.7. *Let X be a topological spaces, Y be a T_1-space and $f\colon X \to Y$ be a function with the finite range $f(X)$. Then f is quasi-continuous if and only if $\mathscr{A} = \{f^{-1}(y) : y \in f(X)\}$ is a quasi-clopen partition of X.*

11.9 Extension of quasi-continuous function ranged in a metrizable compact

Theorem 11.8. *Let X be a hereditarily normal space, $Y \subseteq X$, Z be a metrizable compact, $g\colon Y \to Z$ be a quasi-continuous function. Then there exists a quasi-continuous function $f\colon X \to Z$ such that $f|_Y = g$.*

Proof. Consider some metric d on the topological space Z for which $\operatorname{diam} Z < 1$. We will construct a sequence $(\mathscr{A}_n)_{n=1}^{\infty}$ such that for all $n \in \mathbb{N}$ the following conditions are hold:

$$\mathscr{A}_1 = \{X\}; \tag{11.2}$$

$$\text{for all } A \in \mathscr{A}_{n+1} \text{ there exists } B \in \mathscr{A}_n \text{ such that } A \subseteq B; \tag{11.3}$$

$$\mathscr{A}_n \text{ is a quasi-clopen partition of } X; \tag{11.4}$$

$$\mathscr{B}_n = \{A \cap Y : A \in \mathscr{A}_n\} \text{ is a quasi-clopen partition of } Y; \tag{11.5}$$

$$\operatorname{diam} g(B) < \tfrac{1}{n} \text{ for all } B \in \mathscr{B}_n. \tag{11.6}$$

Suppose that for some $n \in \mathbb{N}$ we already construct a family $\mathscr{A}_n$. Let us define a family $\mathscr{A}_{n+1}$. Fix $A_n \in \mathscr{A}_n$. Using Lemma 11.6 for the topological space A_n, its subspace $A_n \cap Y$, the function $g|_{A_n}$ and $\varepsilon = \frac{1}{n+1}$ we obtain a quasi-clopen partition $\mathscr{A}_{n+1}(A_n)$ of A_n such that $\mathscr{B}_{n+1}(A_n) = \{A \cap Y : A \in \mathscr{A}_{n+1}(A_n)\}$ is

a quasi-clopen partition of $Y \cap A_n$ and $\operatorname{diam} g(B) < \frac{1}{n+1}$ for all $B \in \mathscr{B}_{n+1}(A_n)$. Putting $\mathscr{A}_{n+1} = \bigcup\limits_{A_n \in \mathscr{A}_n} \mathscr{A}_{n+1}(A_n)$ we have constructed the required system.

Fix $x \in X$ and $n \in \mathbb{N}$. Choose $A_n(x) \in \mathscr{A}_n$ for which $x \in A_n(x)$. Let $B_n(x) = A_n(x) \cap Y$. Since $B_n(x) \in \mathscr{B}_n$ and $\mathscr{B}_n$ is a quasi-clopen partition of Y, we have that $B_n(x) \neq \emptyset$. Choose an arbitrary point $b_n(x) \in B_n(x)$. Set $f_n(x) = g\big(b_n(x)\big)$. So, we have defined a functions $f_n \colon X \to Z$. Since $\mathscr{A}_n$ is a quasi-clopen partition, we conclude that f_n is quasi-continuous. For any $m < n$ we have that $B_n(x) \subseteq B_m(x)$. So, $b_n(x), b_m(x) \in B_m(x)$. Then $d\big(f_n(x), f_m(x)\big) = d\big(g(b_n(x)), g(b_m(x))\big) < \frac{1}{m}$. Therefore, f_m is uniformly convergent on X. Let $f(x) = \lim\limits_{n\to\infty} f_n(x)$ for all $x \in X$. Then $f \colon X \to Z$ is quasi-continuous. If $x \in Y$ then $y, b_n(y) \in B_n(y)$. So, $d\big(g(y), f_n(y)\big) = d\big(g(y), g(b_n(y))\big) < \frac{1}{n}$. Therefore, $f(y) = g(y)$ on Y. □

11.10 Some counterexamples and open problems

The first counterexample is related with Proposition 11.5 and Theorems 11.4, 11.7 and 11.8.

Theorem 11.9. *Let $\mathbb{L}$ be the Sorgenfrey line and $X = \mathbb{L}^2$. Then X is not a normal space but for any subspace $Y \subseteq X$ and any quasi-continuous function $g \colon Y \to [0;1]$ there exists a quasi-continuous function $f \colon X \to [0;1]$, such that $f|_Y = g$.*

Proof. By Proposition 11.1 wy may assume, that Y is a closed subset of X. Let d be the Euclidean metric on $\mathbb{R}^2 = X$ and $\widetilde{Y}$ be the closure of Y in $\mathbb{R}^2$. Define $f \colon X \to [0;1]$ by the formula

$$f(x) = \begin{cases} g(x)\ , \text{ if } x \in Y; \\ 0\ , \text{ if } x \in \widetilde{Y} \setminus Y; \\ \sin^2 \frac{1}{d(x,Y)}\ , \text{ if } x \in X \setminus \widetilde{Y}. \end{cases}$$

Prove that f is quasi-continuous. Let Y_0 be the interior of Y in X and Y^0 be the interior of Y in $\mathbb{R}^2$. Obviously, f is quasi-continuous at every point of $(Y \setminus \widetilde{Y}) \cup Y_0$. Fix $x_0 = (u_0, v_0) \in Y_1 = \widetilde{Y} \setminus Y_0$ and prove, that f is quasi-continuous at x_0. Consider neighborhoods $W = (f(x_0) - \varepsilon; f(x_0) + \varepsilon)$ of $f(x_0)$ in $\mathbb{R}$ and $U = [u_0; u_0 + \delta) \times [v_0; v_0 + \delta)$ of x_0 in X. Since $x_0 \notin Y_0$, $U \not\subseteq Y$. So, $U \setminus Y$ is a non-empty open set in X. The boundary $\widetilde{Y} \setminus Y^0$ of Y in $\mathbb{R}^2$ is nowhere dense in $\mathbb{R}^2$, and therefore is nowhere dense in X. So, $\widetilde{Y} \setminus Y$ is nowhere dense in X

too. Thus, there exists a non-empty open set V in X, such that $V \subseteq U \setminus Y$ and $V \cap (\widetilde{Y} \setminus Y) = \emptyset$. Therefore, $V \subseteq U \setminus \widetilde{Y}$.

Choose $u_1 < u^*$ and $v_1 < v^*$, such that $\widetilde{V}_1 = [u_1; u^*] \times [v_1; v^*] \subseteq V$. Put $u_t = (1-t)u_0 + tu_1$, $v_t = (1-t)v_0 1 + tv_1$, $V_t = (u_t; u^*) \times (v_t; v^*)$ and $\widetilde{V}_t = [u_t; u^*] \times [v_t; v^*]$ for any $t \in [0;1]$. Obviously, the families $(V_t)_{t\in[0;1]}$ and $(\widetilde{V}_t)_{t\in[0;1]}$ are increasing and $\widetilde{V}_0 \cap \widetilde{Y} \ni x_0$. Let

$$\tau = \sup\{t \in [0;1] \colon \widetilde{V}_t \cap \widetilde{Y} \neq \emptyset\}.$$

Prove that $\widetilde{V}_\tau \cap \widetilde{Y} \neq \emptyset$. Indeed, if we suppose the contrary, then we have that $\tau > 0$ and $\bigcap\limits_{0<t<\tau} \widetilde{V}_t = \widetilde{V}_\tau \subseteq \mathbb{R}^2 \setminus \widetilde{Y}$. But $\widetilde{Y}$ is closed in $\mathbb{R}^2$ and $\widetilde{V}_t$ are compact in $\mathbb{R}^2$. Then there exist $t_0 \in (0;\tau)$, such that $\widetilde{V}_{t_0} \subseteq \mathbb{R}^2 \setminus \widetilde{Y}$. Thus, $\widetilde{V}_{t_0} \cap \widetilde{Y} = \emptyset$. So, $\widetilde{V}_t \cap \widetilde{Y} = \emptyset$ for any $t \in [t_0;1]$. Then $\tau \leq t_0 < \tau$, which is impossible. Therefore, $\widetilde{V}_\tau \cap \widetilde{Y} \neq \emptyset$.

Let us prove, that $\tau < 1$. Note, that $\bigcap\limits_{0<t<1} \widetilde{V}_t = \widetilde{V}_1 \subseteq \mathbb{R}^2 \setminus \widetilde{Y}$. Then there is $t_1 < 1$ with $\widetilde{V}_{t_1 \subseteq \mathbb{R}^2 \setminus \widetilde{Y}}$. So, we have that $\tau \leq t_1 < 1$.

Since $\widetilde{V}_t \subseteq X \setminus \widetilde{Y}$ for any $t \in (\tau;1]$, we have that

$$V_\tau = \bigcup_{t\in(\tau;1]} \widetilde{V}_t \subseteq X \setminus \widetilde{Y}.$$

Therefore, we obtain that $V_\tau \cap \widetilde{Y} = \emptyset$.

Choose $a \in \widetilde{V}_\tau \cap \widetilde{Y}$ and $b \in V_\tau$. Then $d(a,Y) = 0$ and $d(b,Y) = r > 0$. Put $\alpha = \arcsin\sqrt{f(x_0)}$. Choose $n \in \mathbb{N}$ with $\frac{1}{\alpha+2\pi n} < r$. But $d(\cdot,Y)$ is continuous on $\mathbb{R}^2$ and $V_\tau \cup \{a\}$ is connected in $\mathbb{R}^2$. Thus, there is $c \in V_\tau \cup \{a\}$ such that $d(c,Y) = \frac{1}{\alpha+2\pi n}$. Since $d(c,Y) \neq 0$, $c \neq a$. So, $c \in V_\tau$. Observe, that $f(c) = \sin^2(\alpha + 2\pi n) = \sin^2\alpha = f(x_0) \in W$. But f is continuous at c with respect to the Euclidean topology. Then there exists an open in $\mathbb{R}^2$ set $U_1 \subseteq V$, such that $f(U_1) \subseteq W$. Since U_1 is open in X too and $U_1 \subseteq V \subseteq U$, we conclude that f is quasi-continuous at x_0. □

In this regard we have the following problems.

Problem 11.3. Describe all spaces X such that for any subspace Y and any quasi-continuous function $g \colon Y \to [0;1]$ there exists a quasi-continuous extension $f \colon X \to [0;1]$ such that $D(f) = \widetilde{D}(g)$.

Problem 11.4. Describe all spaces X, such that for any subspace Y, any (metrizable) compact Z and any quasi-continuous function $g \colon Y \to Z$, there exists a quasi-continuous extension $f \colon X \to Z$.

Problem 11.5. Describe all spaces X, such that for any closed subspace Y and any (metrizable) compact Z, there exists a universal quasi-continuous extension $h\colon X \setminus Y \to Z$.

Problem 11.6. Let X be a hereditarily normal space, Y be a subspace of X, Z be a compact (non-metrizable) and $g\colon Y \to Z$ be a quasi-continuous function. Is there a quasi-continuous extension $f\colon X \to Z$ of g?

The following counterexample shows, that in Theorem 11.3 we can not replace the hereditarily normality of X by the normality.

Theorem 11.10. *Let $X = [0;1]^{\mathfrak{c}}$ be the Tichonoff's cube and Y be a subspace of X which is homeomorphic to the reminder $\beta\mathbb{N} \setminus \mathbb{N}$ of Čech-Stone compactification of the countable discrete space $\mathbb{N}$. Then there exists a quasi-continuous function $g\colon Y \to [0;1]$ which has no quasi-continuous extension $f\colon X \to [0;1]$ with $D(f) = D(g)$.*

Proof. Suppose, that every quasi-continuous function $g\colon Y \to [0;1]$ has a quasi-continuous extension with $D(f) = D(g)$. Firstly, the Hewitt-Marczewski-Pondiczery theorem [1, p. 81] implies, that X is separable. So, there exist a countable dense subset E of X.

By [1, 3.6.18] choose a disjoint family $(V_t)_{t\in T}$ of non-empty open subset V_t in Y, such that $\operatorname{card} T = \mathfrak{c}$. Fix $A \subseteq T$. Let $V_A = \bigcup\limits_{t\in A} V_t$, $V'_A = X \setminus \overline{V}_A$, $F_A = Y \setminus (V_A \cap V'_A)$. Define $g_A\colon Y \to [0;1]$ by the formulas $g_A(x) = 1$ on $\overline{V}_A$ and $g_A(x) = 0$ on $Y \setminus \overline{V}_A$. Evidently, g_A is quasi-continuous. Since $F_A = \overline{V}_A \cap \overline{V'_A}$, we obtain $D(g_A) = F_A$. Then there exists a quasi-continuous extension $f_A\colon X \to [0;1]$ of g_A, such that $D(f_A) = F_A$. Then f_A is continuous at every point of the sets V_A and V'_A. Therefore, $U_A = \operatorname{int} f_A^{-1}\left((\frac{1}{2};+\infty)\right)$ and $U'_A = \operatorname{int} f_A^{-1}\left((-\infty;\frac{1}{2})\right)$ are disjoint open sets with $V_A \subseteq U_A$ and $V'_A \subseteq U'_A$. Put $E_A = E \cap U_A$.

Let A and B be a distinct subset of T. Prove, that $E_A \neq E_B$. Let $A \not\subseteq B$. Then there exists $t \in A \setminus B$. Fix $y_0 \in V_t$. Then $y_0 \in V_A \cap V'_B \subseteq U_A \cap U'_B$. So, $U_A \cap U'_B \neq \emptyset$. Since $\overline{E} = X$, there exists $x_0 \in E \cap U_A \cap U'_B$. Therefore, $x_0 \in E_A$ and $x_0 \notin E_B$. Thus, $E_A \neq E_B$.

So, we construct a injection $2^T \ni A \mapsto E_A \in 2^E$. Then $2^{\mathfrak{c}} = \operatorname{card} 2^T \leq \operatorname{card} 2^E = 2^{\aleph_0} = \mathfrak{c} < 2^{\mathfrak{c}}$, which is impossible. □

With the previous example is related the following question.

Problem 11.7. Do there exist a normal space X, a closed subspace $Y \subseteq X$ and a quasi-continuous function $g\colon Y \to [0;1]$ without any quasi-continuous extension?

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