A C T A   U N I V E R S I T A T I S   L O D Z I E N S I S
FOLIA OECONOMICA 132, 1993

Vyacheslav L. Girko, Alexander Babanin***

NEW LIMIT THEOREMS FOR THE SYSTEM OF LINEAR ALGEBRAIC EQUATIONS WITH RANDOM COEFFICIENTS

Abstract. The existing problems of a linear algebra are discussed. With the help of G - analysis methods new assertions are found for the solutions of the systems of linear algebraic equations (SLAE) with random coefficients. The new class of G_8 - estimates of the solutions of SLAE with random coefficients is developed. Experimental results are provided to compare new G_8 - estimates with traditional ones proposed by A. N. Tikhonov and A. V. Goncharsky.

Key words: limit theorems, random coefficients, regularized pseudosolution, G - conditions.
AMS(MOS) subject classification: 65F05.

1. INTRODUCTION

The solution of system of linear algebraic equations is an important problem that arises in different scientific and engineering applications, such as numerical solution of problems of calculus, differential and integral equations, experimental planning, multivariate statistical analysis etc.

The number of original works in this field is very large. We point out the books [F a d d e y e v, F a d d e y e v a (1963), M o l c h a n o v (1987), F i k h o n o v, A r s e n i n (1986), V o j e v o d i n (1977), V o j e v o d i n, K u z n e t c o v (1984), W i l k i n s o n (1965)] which do not cover the variety

* Professor at the Department of Cybernetics of the University of Kiev.
** Lecturer at the Department of Cybernetics of the University of Kiev.

of different methods of the solution of SLAE and their applications. Notwithstanding a large number of works and new advancements in the solution of SLAE of large dimension nowadays:

1) it is unknown the way of finding of the consistent, the best in some sense estimates of the SLAE solutions if their coefficients are given with certain random errors;

2) the conditions of the existence of the moments of components x_k of the vector x are not found;

3) under general conditions the limit theorems for the distribution functions of values x_k are not proved.

Although $(n \times m)$ matrix A is given exactly under large n and m by virtue of round-off errors of computer calculations we obtain such solutions as if a matrix A is known with some random errors [V o j e v o d i n (1977), p. 286]. We note that under different assumptions a lot of limit theorem for the solutions of SLAE with random coefficients were proved [G i r k o (1970, 1980a, 1980b, 1984, 1986, 1987a, 1987b, 1988a, 1988b), G i r k o, B a b a n i n (1988)]. Among these limit theorems the most important is the so-called "arctangent law" [G i r k o (1980b), p. 333]: let the elements of $(n \times n)$ matrix A and the components of the vector b be independent, their means be 0, variances be 1, their absolute moments of order $4 + \delta$, $\delta > 0$ be bounded. Then

$$\lim_{n\to\infty} P\{x_k < z\} = \pi^{-1} \arctan z + \frac{1}{2},$$

where x_k are the components of the vector x (If $\det A = 0$, then x_k are assumed to be equal to the arbitrary constant). In the case, where variances are bounded, the "arctangent law" was refined [B a b a n i n (1983)]. However with the help of these limit theorems the consistent estimates for the solutions of SLAE of the large order are not obtained.

2. NEW ASSERTIONS FOR THE SYSTEMS OF LINEAR ALGEBRAIC EQUATIONS WITH RANDOM COEFFICIENTS

Definition. The sequence of the estimates $\hat{a}_{mn}$ of some value $\hat{a}_m$ is called G - consistent if

$$\operatorname*{plim}_{m,n\to\infty} [\hat{a}_{mn} - a_m] = 0.$$

2.1. The formulation of the problem

With the help of the independent observations X_i, $i = \overline{1, s}$ under the matrix $A + C_2 \Xi C_1$, $A = (a_{ij})$, $\Xi = (\xi_{ij})$, $i = \overline{1, n}$, $j = \overline{1, m}$ to find the G - consistent estimate of the regularized pseudosolution

$$d'x_\alpha = d'(C_1'C_1\alpha + A'C_2'^{-1}C_2^{-1}A\beta^{-1})^{-1}A'C_2'^{-1}C_2^{-1}b\beta^{-1}$$

of a system of equations $Ax = b$, where C_1, C_2 are nondegenerate $(n \times n)$ and $(m \times m)$ matrices correspondingly, $\alpha \in R^m$, ξ_{ij}, $i = 1, n$, $j = 1, m$ are the independent random elements for every value m and n; if the values σ_n^2, s_n, α_n, β_n depend on n and the G - condition holds:

$$\overline{\lim_{n\to\infty}}\, \sigma_n^2\, n\, \beta_n^{-1} s_n^{-1} = t_1 < \infty,$$

$$\overline{\lim_{n\to\infty}}\, \sigma_n^2\, m_n\, \beta_n^{-1} s_n^{-1} = t_2 < \infty,$$

$$\overline{\lim_{n\to\infty}}\, m_n\, n^{-1} = t_3 < 1, \quad \sigma^2 = \mathrm{Var}\, \xi_{ij}.$$

2.2. Some remarks

Note that t_i can be equal 0. We call the G - consistent estimates as G - estimates. For simplicity index n at values σ_n, m_n, β_n, s_n is omitted.

If $\alpha = 0$, matrix $A'A$ is nondegenerate, then $d'x_0 = d'(A'C_2'^{-1}C_2^{-1}A)^{-1}A'C_2'^{-1}C_2^{-1}b$. If $\alpha = 0$ and A is square, then $d'x_0 = d'A^{-1}b$. Also it is shown that for some G - estimates G_n of values x_α

$$\operatorname*{plim}_{n\to\infty} ||x_\alpha - G_n||^2 = 0.$$

The parameter n is chosen artificially as the parameter of limit transition in all transformations over G - estimates. It was necessary for the proof of the existence of G - estimates of the values $d'x_\alpha$, if the G - conditions holds. As $n\to\infty$ the interesting result is obtained: under certain conditions for each value n it is possible to find the G-estimates of the values $d'x_\alpha$ for the one observation under a matrix $A + \Xi$.

2.3. G_8-estimates class

The G-estimates of the values $d'x_\alpha$ belong to the class of G_8-estimates and denote them as G_8. The following G_8-estimate of G_8-class is found

$$G_8 = \operatorname{Re} d'[C_1C_1(\hat{\theta} + i\varepsilon) + \beta^{-1}(C_2{}^{-1}z_s)'C_2{}^{-1}z_s]^{-1}$$

$$\times (C_2{}^{-1}z_s)'C_2{}^{-1}b\beta^{-1}, \quad z_s = s^{-1}\sum_{k=1}^{s} x_k,$$

$\hat{\theta}$ is any real measurable solution of the equation

$$f_n(\theta) = \alpha, \tag{1}$$

$$f_n(\theta) = \theta \operatorname{Re}[1 + \delta_1\, a(\theta)]^2 - \varepsilon \operatorname{Im}[1 + \delta_1\, a(\theta)]^2$$

$$+ (\delta_1 - \delta_2)[1 + \delta_1 \operatorname{Re} a(\theta)],$$

$$a(\theta) = n^{-1}\operatorname{tr}[I(\theta) + i\varepsilon) + \beta^{-1}(C_2{}^{-1}z_sC_1{}^{-1})'C_2{}^{-1}z_sC_1{}^{-1}]^{-1},$$

$$\delta_1 = \sigma^2 n\beta^{-1}s^{-1},$$

$$\delta_2 = \sigma^2 m\beta^{-1}s^{-1},$$

I is the identity matrix of order m. In general the solution of Eq. (1) is non-unique. It should be chosen random, that means the measurable with respect to the minimal σ-algebra with respect to which random matrices X_i, $i = \overline{1, s}$ are measurable. The $\tilde{G}_8$-estimate is rather distinct of the standard estimates of the form

$$d'x_{stand} = d'(C_1'C_1\alpha + \beta^{-1}(C_2{}^{-1}z_s)'C_2{}^{-1}z_s)^{-1}$$

$$\times (C_2{}^{-1}z_s)'C_2{}^{-1}b\beta^{-1}$$

As $n\to\infty$ the standard estimates have the shifts which do not tend to zero. These shifts can be rather large.

3. NEW LIMITS THEOREMS FOR THE SOLUTIONS OF SLAE WITH RANDOM COEFFICIENTS

Theorem 1. Let for each value $n = 1, 2, \ldots$ the elements ξ_{pl}, $p = \overline{1, n}$, $l = \overline{1, m}$ of the matrix Ξ are independent, $E\,\xi_{pl} = 0$, $\operatorname{Var} \xi_{pl} = \sigma^2$ the G-condition holds,

$$\lambda_m + \alpha \geq h,$$

where $h \geq 0$ is some number, $\lambda_1 \geq \ldots \geq \lambda_m$ are the eigenvalues of the matrix $\tilde{A}'\tilde{A}\beta^{-1}$, $\tilde{A} = C_2^{-1}AC_1^{-1}$,

$$\overline{\lim_{n\to\infty}} [(\tilde{b}'\tilde{b} + \tilde{d}'\tilde{d})\beta^{-1/2} + \sup_{k=\overline{1,n}} a'_k a_k (b'b\, d'd)^{1/2}\beta^{-1} < \infty,$$

where a_k are the vector-rows of the matrix $\tilde{A}$, $\tilde{b} = C_2^{-1}b$, $\tilde{d} = C_1^{-1}d$

$$\sup_n \lambda_1 < \infty,$$

for some $\delta > 0$

$$\sup_n \sup_{p=1,n,l=1,m} E|\xi_{pl}|^{4+\delta} < \infty.$$

Then if $\varepsilon \neq 0$

$$\operatorname{plim}_{n\to\infty} [\tilde{G}_8 - \operatorname{Re} d'x_{\alpha+i\gamma(\varepsilon)}] = 0,$$

where

$$\gamma(\varepsilon) = \varepsilon \operatorname{Re}[1 + \delta_1 a(\hat{\theta})]^2 + \tilde{\theta} \operatorname{Im}[1 + \delta_2 a(\tilde{\theta})]^2 + (\delta_1 - \delta_2)\delta_1 \operatorname{Im} a(\hat{\theta}).$$

Lemma 1. If the conditions of Theorem 1 hold

$$\psi(v,y) = \operatorname{Re}[a(v,y) + \nu(v,y)],$$

where the function $a(v,y) = E\operatorname{tr}[I(y + i\varepsilon) + Q'Q]$ is satisfying the equation

$$a(v,y) = E\sum_{k=1}^{m} [(y + i\varepsilon)(1 + \delta_1 n^{-1}a(v,y)) + \delta_1 - \delta_2 + \lambda_k(v)(1 + \delta_n n^{-1}a(v,y))^{-1} + \varepsilon_{kn}(v,y)]^{-1},$$

$\lambda_k(v)$ are the eigenvalues of the matrix $K'(v)K(v)$,
$K(v) = (\tilde{A} + v\tilde{b}\tilde{d}')\beta^{-1/2}$, $Q = \Lambda(v) + n^{-1/2}\delta_1^{1/2}\Xi$,

$\Lambda(v) = (\delta_{ij}\lambda_i^{1/2}(v))_{i,j=1}^{m}$, $\tilde{\Xi} = (\tilde{\xi}_{ij})$, $i = \overline{1,n}$, $j = \overline{1,m}$

is the random matrix whose elements $\tilde{\xi}_{ij}$ are independent among themselves, do not depend on matrices X_i and distributed normally $N(0, 1)$, δ_{ij} is a Kronecker symbol,

$$\varepsilon_{kn}(v,y) = q_k q'_k - \delta_1 - \lambda_k(v) - q_k Q'_k L_k Q_k q'_k + \lambda_k(v)\mathrm{tr}\, L_k T_k^{k}$$
$$+ n^{-1}\delta_1 \mathrm{tr}\, L_k Q'_k Q_k + (y + i\varepsilon)\delta_1 n^{-1}\mathrm{tr}(L - EL)$$
$$+ \lambda_k(v)\{[1 + q_k^{(k)} L_k^{(k)} q_k^{(k)}]^{-1} - [1 + n^{-1}\delta_1 E\,\mathrm{tr}\, L]^{-1}\},$$

$$L = (l_{ij}) = [I(y + i\varepsilon) + Q'Q]^{-1}, \quad L_k = [I(y + i\varepsilon) + Q'_k Q_k]^{-1},$$

the matrix Q_k is obtained from the matrix Q by deleting the K-th row q_k,

$$L_k^{(k)} = [I(y + i\varepsilon) + \sum_{p\neq k} T_p^{(k)}]^{-1},$$

$$T_p^{(k)} = (q_{pi}\, q_{pj})_{i,j\neq k}^{m},$$

$$q_k^{(k)} = (q_{k,1}, \ldots, q_{k,k-1}, q_{k,k+1}, \ldots q_{k,m}),$$

$$\nu(v,y) = \sum_{k=1}^{n} \{(E_{k-1} - E_k)\frac{\partial}{\partial y}\ln[1 + p'_k T_k(v,y)p_k]$$
$$+ E\frac{\partial}{\partial y}\ln[1 + p'_k R_k(v,y)p_k] - E\frac{\partial}{\partial y}\ln[1 + g'_k R_k(u,v)g_k]\},$$

$$R_k(v,y) = [I(y + i\varepsilon) + \sum_{s=1}^{k-1} g_s g'_s + \sum_{s=k+1}^{n} p_s p'_s]^{-1},$$

$$T_k(v,y) = [I(y + i\varepsilon) + \sum_{s=1, s\neq k}^{n} p_s p'_s]^{-1},$$

g_s is s th vector-column of the matrix

$[(\tilde{A} + v\tilde{b}\tilde{d}')\beta^{-1/2} + n^{-1/2}\delta_1^{1/2}\tilde{\Xi}]'$, p_s is s-th

vector-column of the matrix $B'(v)$, E_k is the conditional expectation under fixed minimal σ-algebra with respect to which the random vectors p_s, $s = \overline{k+1, n}$ are measurable.

Lemma 2. The partial derivative of the 1st order with respect to the variable V of the function a(v, y) as V = 0 exists. It is possible to express this derivative as values

$$\frac{\partial \lambda_k(v)}{\partial v}\Big|_{v=0}, \quad \frac{\partial \lambda_k^{1/2}(v)}{\partial v}\Big|_{v=0}$$

which are equal to $\varphi_k B \varphi_k$, $\tilde{b}' H \varphi_k d_k' \varphi_k \beta^{-1}$ correspondingly, where $B = [(\tilde{b}\tilde{d}')'\tilde{A} + \tilde{A}'\tilde{b}\tilde{d}']\beta^{-1}$ are the orthonormalized eigenvectors of matrix $\tilde{A}'\tilde{A}$, corresponding to the eigenvalues $\lambda_k(0)$, $H = \tilde{A}(\tilde{A}'\tilde{A})^{-1/2}$, if $\det \tilde{A}'\tilde{A} \neq 0$ and H is some orthogonal matrix, if $\det \tilde{A}'\tilde{A} = 0$.

Lemma 3. If the conditions of Theorem 1 hold then for every y

$$\lim_{n\to\infty} \sup_{k=\overline{1,m}} E|\varepsilon_{kn}(0, y)|^2 = 0,$$

$$\lim_{n\to\infty} \sum_{k=1}^{m} |E \frac{\partial}{\partial v} \varepsilon_{kn}(v, y)|_{v=0}| = 0,$$

$$\lim_{n\to\infty} \sum_{k=1}^{m} |E(\varepsilon_{kn}(0, y) - E\,\varepsilon_{kn}(0, y)) \frac{\partial \varepsilon_{kn}(v,y)}{\partial v}|_{v=0}| = 0,$$

for some $\delta_1 > 0$

$$\lim_{n\to\infty} \sum_{k=1}^{m} |\varepsilon_{kn}(0, y)|^{1+\delta_1} \frac{\partial \varepsilon_{kn}(v,y)}{\partial v}|_{v=0}| = 0.$$

Lemma 4. If the conditions of Theorem 1 hold

$$\operatorname{plim}_{n\to\infty} \frac{\partial \nu_n(v,y)}{\partial v}\Big|_{v=0} = 0.$$

Lemma 5. There are measurable real solutions $\tilde{\theta}$ of the Eq. (1). If in this equation the function a(y) is changed by Ea(y), then such an equation has real solutions $\tilde{\theta}$ also and they are distinct beginning from some $n \geqslant n_o$. If the solutions of Eq. (1) and the solutions $\tilde{\theta}$ are arranged by increasing $\hat{\theta}_1 \leqslant \hat{\theta}_2 \leqslant \ldots$, $\tilde{\theta}_1 \leqslant \tilde{\theta}_2 \leqslant \ldots$ then for every $\varepsilon \neq 0$, $k = 1, 2, \ldots$

$$\operatorname{plim}_{n\to\infty} |\hat{\theta}_k - \tilde{\theta}_k| = 0.$$

Lemma 6. If the conditions of Theorem 1 hold for every $k = 1, 2, \ldots$

$$\operatorname{plim} \varkappa_n = 0,$$

where

$$\varkappa_n = \operatorname{Re} d'\{[I(\hat{\theta}_k + i\varepsilon) + Z_s' Z_s \beta^{-1}]^{-1}$$

$$- [I(\tilde{\theta}_k + i\varepsilon) + Z_s' Z_s \beta^{-1}]\} Z_s' b \beta^{-1}.$$

Lemma 7. If the conditions of Theorem 1 hold

$$\lim_{n\to\infty} [\sum_{k=1}^{m} b_k(y)\, a_k^{-2}(y) - \sum_{k=1}^{m} [b_k(y) + \partial\varepsilon_k(v,y)/\partial v|_{v=0}]$$

$$\times [a_k(y) + \varepsilon_k(0, y)]^{-2} = 0.$$

Theorem 2. If in addition to the conditions of Theorem 1

$$\alpha + \lambda_m \geqslant 2\delta_2 + c,$$

$$2\delta_2(1 + \delta_2\tau)^2[\alpha + |\delta_1 - \delta_2|(1 + \delta_2\tau)]$$

$$\times (\alpha + \lambda_m - 2\delta_2)^2 + |\delta_1 - \delta_2|\delta_2\tau^2 \leqslant h < 1,$$

where $\tau = (\alpha + \lambda_m - \delta_2)^{-1}$, $c > 0$,

then $\lim_{\varepsilon\downarrow 0} \operatorname{plim}_{n\to\infty} |\partial(\varepsilon)| = 0.$

Corollary 1. If the conditions of Theorem 2 hold

$$\lim_{\varepsilon\downarrow 0} \operatorname{plim}_{n\to\infty} [G_8 - d'x_\alpha] = 0.$$

Corollary 2. If the conditions of Theorem 1 hold, $m = n$ and $\lambda_m \geqslant 2\delta_2 + h$, $h > 0$, $\alpha = 0$, then

$$\lim_{\varepsilon\downarrow 0} \operatorname{plim}_{n\to\infty} [G_8 - d'A^{-1}b] = 0.$$

4. NUMERICAL EXPERIMENTS

The computer program for comparison of the standard estimate and G_8-estimate was written in FORTRAN. The program was run at BESM-6 computer which is known of its good accuracy of calculations. Some experiments were described in [G i r k o, B a b a n i n (1989)].

The following example is not an analog of from [T i k h o n o v, G o n c h a r s k y, S t e p a n o v, J a g o l a (1983)], where $A = (a_{ij})$, $i = \overline{1, n}$, $j = \overline{1, m}$, $m = n = 41$,

$$a_{ij} = [1 + 100(x_i - s_j)^2]^{-1},$$

$$x_i = (i - 1)/(n - 1); \quad s_j = (j - 1)/(m - 1).$$

The exact model solution was given by the equality [T i k h o n o v, G o n c h a r s k y, S t e p a n o v, J a g o l a (1983), p. 109]. The right hand vector was calculated by direct multiplication of a matrix and the exact solution. The observation matrix X was modelled in such a way $X = A + \Xi$, where $\Xi = (n^{-1/2}\xi_{ij})$, $i = \overline{1, n}$, $j = \overline{1, m}$ is a matrix of a pseudorandom numbers. They are generated with the help of a standard subroutine NORMCO and normally N(0, 3.E(-7)) distributed. The value of errors in a matrix is greater than in [T i k h o n o v, G o n c h a r s k y, S t e p a n o v, J a g o l a (1983)] on 2 orders and equals 2.24. E(-5); the value of errors in the right hand is 1.E(-8). The deviation was calculated by the formula $\mathrm{dev}(x) = \ A x - b||^2_R \, m$. First we obtained the standard estimate and the regularization parameter α. After that we carried out the double regularization: the initial meaning of $\theta = \theta_o = \alpha$. Then using the method of successful approximations by the formula

$$\theta_k = \frac{\theta_{k-1}}{(1 + a(\theta_{k-1}))^2},$$

where $\theta(\alpha)$ is a solution of equation

$$\theta(\alpha)[1 + \sigma^2 a(\theta(\alpha))]^2 + \sigma^2(1 - mn^{-1})(1 + \sigma^2 a(A(\alpha))) = \alpha,$$

$$\alpha > 0, \quad a(\theta(\alpha)) = n^{-1}\,\mathrm{tr}[I\theta(\alpha) + X'X)^{-1}$$

in correspondance with the condition of the iteration stop

$$|\theta_k - \theta_{k-1}| \leqslant 0.0001$$

we find the desired meaning of θ. Further this number was used in the formula for the G_8-solution

$$G_8 = [I\theta^*(\alpha) + X'X]^{-1}X'b.$$

The G_8-deviation = 9.168E(-5) consists of 93.35% of the standard deviation 9.822E(-5).

REFERENCES

Babanin A. S. (1983): *A generalization of arctangent law*, Theory of Probability and Math. Statistics, 28, p. 3-5.

Cato T. (1972): *Perturbation Theory of Linear Operators*, Mir, Moscow.

Faddeyev D. K., Faddeyeva V. N. (1963): *Computational Methods of Linear Algebra*, Fizmatgiz, Moscow.

Gantmacher F. R. (1987): *Theory of Matrices*, Nauka, Moscow.

Girko V. L. (1970): *On the distribution of solutions of systems of linear equations with random coefficients*, Theory of Probability and Math. Statistics, 2, p. 41-44.

- (1980a): *An arctangent law*, Reports of Academy of Sciences of Ukrainian SSR, A, 4, p. 7-9.

- (1980b): *Theory of Random Determinants*, Vyshcha Shkola, Kiev.

- (1984): *A circle law*, "Theory of Probability and their Applications", XXIX, 4, p. 669-679.

- (1986): *G-analysis of observations of large dimension*, "Calculative and Applied Mathematics", 60, p. 11-121.

- (1987a): *An introduction to the general statistical analysis*, "Theory of Probability and their Applications", XXXII, 2, p. 250-265.

- (1987b): *G_2-estimate of inverse covariance matrix*, "Modelling and Optimization of Complex Systems", 6, p. 40-41.

- (1988a): *Multivariate Statistical Analysis*, Vyshcha Shkola, Kiev.

- (1988b): *Spectral Theory of Random Matrices*, Nauka, Moscow.

- (1990): *Consistent G-estimates of solutions of systems of linear algebraic equations*, "Theory of Probability and Math. Statistics", p. 42-43.

Girko V. L., Babanin A. S. (1988): *On the G_8-estimate of the solutions of empirical systems of linear algebraic equations*, "Theory of Probability and Math. Statistics", 39, p. 29-33.

-, -: *Systems of linear algebraic equations with random coefficients when the regularization matrices are used*, Proceedings of DIANA-III, Prague (to be appear).

-, - (1989): *On the estimate of solutions of empirical systems of linear algebraic equations*, "Modelling and Optimization of Complex Systems", 8, p. 13-16.

Molchanov I. N. (1987): *Computer Methods of Solution of Applied Tasks*. Algebra, the Approximation of Functions, Naukova Dumka, Kiev.

Tikhonov A. N., Arsenin V. Ya. (1986): *Methods of Solution of Ill-Posed Problems*, Nauka, Moscow.

Tikhonov A. N., Goncharsky A. V., Stepanov V. V., Jagola A. G. (1983): *Regularization algorithms and apriori information*, Nauka, Moscow.

Vojevodin V. V. (1977): *Computational Bases of Linear Algebra*, Nauka Moscow.

Vojevodin V. V., Kuznetcov Yu. A. (1984): *Matrices and Computations*, Nauka, Moscow.

Wilkinson J. H. (1965): *The Algebraic Eigenvalue Problem*, Clarendon Press, Oxford.

Vyacheslav L. Girko, Alexander Babanin

NOWE TWIERDZENIA OGRANICZONE DLA SYSTEMÓW LINIOWYCH RÓWNAŃ ALGEBRAICZNYCH Z WSPÓŁCZYNNIKAMI LOSOWYMI

Omówione zostają istniejące problemy algebry liniowej. Metody analizy G pozwalają odkryć nowe twierdzenia dla rozwiązań systemów liniowych równań algebraicznych (SLAE) z współczynnikami losowymi. Rozwinięto nową klasę estymatorów G_8 rozwiązań SLAE z współczynnikami losowymi. Podane są eksperymentalne wyniki w celu porównania nowych estymacji G_8 z tradycyjnymi, zaproponowanymi przez A. N. Tikhonova i A. V. Goncharskyego.