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DETECTING SHARP CONTOURS OF IMAGES

Abstract. The theory of wavelets introduced by Daubechies is a developing branch of mathematics with a wide range of potential applications. This paper presents a survey of some methods of detecting sharp cusps of unknown function developed by Wang with examples of their application to detecting sharp contours of images. A simple original algorithm to detect sharp contours of two dimensional images is also proposed and its application is presented. Visual examination allows to state that the results are comparable with the Wang's method.

Key words: sharp contours, function cusps, wavelets theory.

I. INTRODUCTION

We say a function f has an α-cusp at x_0 if there exists a positive constant K such that as h tends to zero from left or right,

$$|f(x_0+h)-f(x_0)| \geqslant K|h|^{\alpha}$$

For the case $\alpha=0$, f has a jump at x_0. We will consider only sharp cusps detection so we restrict ourselves to the case $0<\alpha<1$.

We can observe f from the white noise model

$$Y(dx)=f(x)dx+\tau W(dx), \quad x \in [0, 1] \tag{1}$$

where W is a standard Wiener process, τ is a noise level parameter, and f is an unknown function which may have jumps or cusps. An equivalent way to observe f is the following nonparametric regression model:

$$y_i=f(x_i)+\sigma z_i \quad i=1, ..., n$$

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with $x_i = i/n$, the independent standard normal errors z_i, $\sigma > 0$. The regression process is defined by $\{Y_n(x): x \in [0, 1]\}$, where $x_0 = 0$, $Y_n(0) = 0$ and $y_n(x_i) = y_i + ... + y_i$ for $i = 1, ..., n$ with interpolation by a Wiener process for these x which satisfy $x_i \leqslant x < x_{i+1}$. Then Y_n is a white noise process with $\tau = \sigma/\sqrt{n}$ (Donoho, Johnstone, (1995)).

The problem is to detect sharp cusps and jumps of function f.

II. ONE DIMENSIONAL CASE

Wavelets were introduced by Daubechies (1990). Wavelet in this sense consists of two functions: the scaling functions and the primary wavelet. The scaling function $\varphi(x)$ is solution of the following difference equation

$$\varphi(x) = \sum_{k \in Z} c_k \varphi(2x - k) \tag{2}$$

with the normalizing condition

$$\int_R \varphi(x) dx = 1$$

The primary wavelet $\varphi(x)$ is defined by

$$\varphi(x) = \sum_{k \in Z} (-1)^k c_{k+1} \varphi(2x + k) \tag{3}$$

The coefficients c_k are called the filter coefficients and it is the careful choice of them that ensures the desired properties of wavelets. The condition

$$\sum_k c_k = 2$$

ensures the existence of unique solution to (2) in $L^1(\mathrm{R})$. A wavelet system is the infinite collection of translated and scaled versions of φ and ψ defined by

$$\varphi_{j,k}(x) = 2^{j/2} \varphi(2^j - k) \quad j, \ k \in Z$$

$$\psi_{j,k}(x) = 2^{j/2} \psi(2^j - k) \quad j, \ k \in Z$$

An additional condition on the filter coefficients i.e.

$$\sum_k c_k c_{k+2l} = 2 \quad \text{if} \quad l = 0 \ (0 \text{ if } l \neq 0)$$

with some regularity conditions implies that the family of functions ψ is an orthonormal basis of $L^2(\mathrm{R})$. It seems important to observe that it is possible to construct finite length sequences of filter coefficients that result in compactly supported functions φ and ψ. The simplest example of a wavelet system is the Haar system, given by $c_0 = c_1 = 1$ and all other coefficients equal to zero, but if we want to analyse a very irregular function this system is not sufficient. If ψ is a Daubechies wavelet then we define function

$$\psi_s(x) = (1/\sqrt{n})\psi(x/s)$$

The wavelet transformation of function f is defined as

$$Tf(s,\ x) = \int \psi_s(x-u) f(u) du$$

For wavelets with compact support, the value of $Tf(s,\ x)$ depends upon the value of f in a neighbourhood of x of size proportional to the scale s. At small scales, $Tf(s,\ x)$ provides localised information such as local regularity on $f(x)$. The local regularity is often measured with the help of Lipschitz condition.

A function $f(x)$ is said to be Lipschitz α at x_0 if there exists a positive constant K such that, as h tends to zero,

$$|f(x_0 + h) - f(x_0)| \leqslant K|h|^{\alpha}$$

From the mathematical point of view the global and local Lipschitz regularity can characterised by the asymptotic decay of wavelet transformation at small scales. For example, if f is differentiable at x, $Tf(s,\ x)$ has the order $s^{3/2}$ as s tends to zero, and if f has an α-cusp at x, the maximum of $Tf(s,\ x)$ over a neighbourhood of x of size proportional to s converges to zero at a rate no faster than $s^{\alpha+1/2}$ as s tends to zero. We define the wavelet transformations of the white noise $W(dx)$ and Y in the following way

$$TW(s,\ x) = \int \psi_s(x-u) W(du)$$

$$TY(s,\ x) = \int \psi_s(x-u) Y(du) = Tf(s,\ x) + \tau TW(s,\ x)$$

At a given scale s, $TW(s, x)$is a stationary Gaussian process with zero mean and covariance function

$$E\{TW(s, x)TW(s, x)\} = \int\psi_s(x-u)\psi_s(y-u)du$$

From the above formulas it follows that, at a very fine scale s, $TY(s, x)$ is dominated by $\tau TW(s, x)$, while, at a coarse scale s, $Tf(s, x)$ dominates $TY(s, x)$. The local information on $f(x)$ is provided by $Tf(s, x)$ at fine scales, so the wavelet transformation at finer scales can detect local changes more precisely. The idea is to choose fine scales s_τ such that, at those x where $f(x)$ is differentiable, the orders of $Tf(s_\tau, x)$ and $\tau TW(s_\tau, x)$ are balanced. If function f has sharp cusps, for nearby x, $TW(s_\tau, x)$ will be dominated by $Tf(s_\tau, x)$ and hence significantly larger than the others. Therefore, we can detect sharp cusps by checking the values of $TY(s_\tau, x)$.

Let us consider the following example of applying this approach to testing the null hypotheses H_0: f is differentiable; against H_1: f has sharp cusps. The test statistics is given by the maximum of $|TY(s_\tau, x)|$ over $x \in [0, 1]$. Critical values $C_{\tau,\gamma}$, i.e. values defined by

$$\lim_{\tau \to 0} \Pr\left\{\max_{0 \leqslant x \leqslant 1} |TY(s_\tau, x)| \geqslant C_{\tau,\gamma}\right\} = \gamma$$

where γ is the test's size, are given by the formula

$$C_{\tau,\gamma} = \tau\sqrt{2|\log s_\tau|}\left\{2|\log s_\tau| + \log\left(\left[\int\left\{\psi'(u)\right\}^2 du\right]^{1/2}/(2\pi)\right) - \log\{-\log(1-\gamma)/2\}\right\}$$

If we want to apply this test in practice we have to find the discrete version of wavelet transformations, because we can observe $Y(x)$ only at n discrete values. The discrete version of the continuous wavelet transformation is some orthogonal matrix W (D a u b e c h i e s (1994)). The value $w = n^{1/2} w_{j,k}$ approximates the critical values of the test's statistic for appropriately defined j and k (compare D o n o h o, J o h n s t o n e (1994)). An example of application of this test is presented in Fig. 1 where we try to detect one jump and one sharp cusp of function f.

*$\mathrm{I}(x \leqslant 0.26) - 2|x-0.26|^{3/5}\mathrm{I}(x > 0.26) + \mathrm{I}(x \geqslant 0.78)$, $\varepsilon_i \sim N(0, \sigma^2)$, $\sigma = 0.2$,

$n = 1024$; (a) true curve, (b) critical values w.

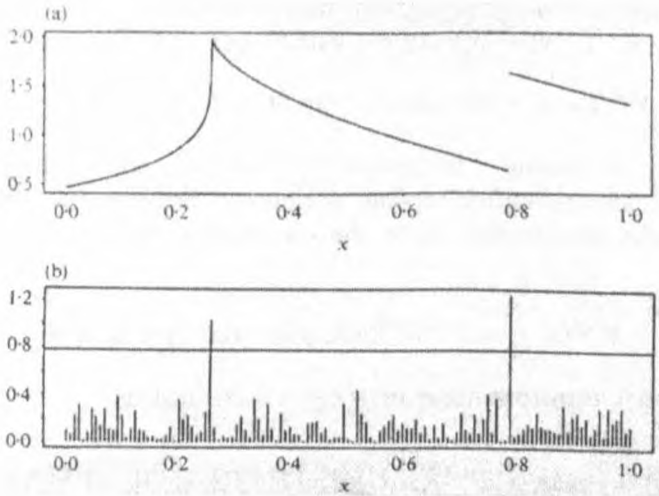

Fig. 1. Data simulated from model $y_i = f(i/n) + \varepsilon_i$, $f(x) = 2 - 2|x - 0.26|^{1/5}$*

III. TWO DIMENSIONAL CASE

In two dimensional case unknown function $f(x, y)$, for $(x, y) \in [0, 1]^2$, is observed from the following model

$$z_{i_1 i_2} = f(x_{i_1}, y_{i_2}) + \varepsilon_{i_1 i_2} \quad i_1, i_2 = 1, \ldots, n = 2^I \tag{4}$$

where $x_{i_1} = i_1/n$, $y_{i_2} = i_2/n$ and $\varepsilon_{i_1 i_2}$ are independent, identically distributed normal errors with mean zero and variance σ^2. The function $f(x, y)$ has sharp cusps along some curves in unit square $(0, 1)^2$ and we want to find them.

We define cusps in two dimensions in the following way: we say that f has α sharp cusps along a continuous curve θ in $(0, 1)^2$, if for each $(x, y) \in \theta$ there exists a $\beta_0 \in [0, 2\pi)$ such that for $\beta \in (\beta_0 - \delta, \beta_0 + \delta)$, as $h \to 0$,

$$|f(x + h\cos\beta, y + h\sin) - f(x, y)| \geqslant K|h|^\alpha$$

where K and δ are positive constants which depend only on θ and f.

If φ and ψ are one dimensional Daubechies wavelets then we define three two dimensional wavelets as follows;

$$\Psi^{h,\beta}(x, y) = \varphi(x \cos\beta - y\sin\beta)\ \psi(x\sin\beta + y\cos\beta)$$

$$\Psi^{v,\beta}(x, y) = \varphi(x \cos\beta - y\sin\beta)\ \varphi(x\sin\beta + y\cos\beta)$$

$$\Psi^{d,\beta}(x, y) = \varphi(x \cos\beta - y\sin\beta)\ \psi(x\sin\beta + y\cos\beta),$$

where h, v, d denote "horizontal", "vertical" and "diagonal" and $\beta \in (0, \pi/4)$ is an orientation-turning parameter. Scale is introduced to the wavelets in the similar way as in the one dimensional case i.e. we define function

$$\Psi_s^{\gamma,\beta}(x, y) = s^{-1}\Psi^{\gamma,\beta}(x/s,\ y/s) \quad \text{for} \quad \gamma = h,\ v,\ d$$

The wavelet transformation of $f(x, y)$ is defined as

$$Tf(s, x, y) = \max_{0 \leqslant \beta \leqslant \pi/4} \left\{|T^{h,\beta}f(s, x, y)|,\ |T^{v,\beta}f(s, x, y)|,\ |T^{d,\beta}f(s, x, y)|\right\},$$

where

$$T^{\gamma,\beta}f(s, x, y) = \int\Psi_s^{\gamma,\beta}(x-u,\ y-u)f(u, v)dudv$$

Similarly to the one dimensional case, at fine scales s, $Tf(s, x, y)$ provides local information on $f(x, y)$, such as Lipschitz regularity. We say that $f(x, y)$ is Lipschitz α at (x_0, y_0) if there exists a positive constant K such that for all β and at $h \to 0$,

$$|f(x + h\cos\beta,\ y + h\sin\beta) - f(x,\ y)| \leqslant K|h|^{\alpha}$$

Similarly as in the one sample case global and local Lipschitz regularities can be characterised by the asymptotic decay of wavelet transformation at small scales. For example, if f is differentiable, then $Tf(s, x, y)$ is of order s^2, if f has α sharp cusps along some curve then the minimum of $Tf(s, x, y)$ along this curve converges to zero at a rate no faster than $s^{\alpha+1}$ as s tends to zero.

If we have data from model (4) we define two processes $Z_n(x, y)$, $W_n(x, y)$ for $(x, y) \in [0, 1]^2$ with the following formulas: $Z_n(0, 0) = W_n(0, 0) = 0$ and

$$Z_n(x, y) = n^{-2} \sum_{i_1 \leqslant nx} \sum_{i_2 \leqslant ny} z_{i_1 i_2}$$

$$W_n(x, y) = n^{-2} \sum_{i_1 \leqslant nx} \sum_{i_2 \leqslant ny} \varepsilon_{i_1 i_2}$$

The wavelet transformations of Z_n and W_n are defined as

$$TW(s, x, y) = \max_{0 \leqslant \beta \leqslant \pi/4} \left\{ |T^{h,\beta} W_n(s, x, y)|, \ |T^{v,\beta} W_n(s, x, y)|, \ |T^{d,\beta} W_n(s, x, y)| \right\},$$

where

$$T^{\gamma,\beta} W_n(s, x, y) = \int \Psi_s^{\gamma,\beta}(x-u, y-u) W_n(du, dv) =$$

$$= n^{-2} \sum_{i_1=1}^{n} \sum_{i_2=1}^{n} \Psi_s^{\gamma,\beta}(x - i_1/n, y - i_2/n) \varepsilon_{i_1 i_2}$$

and

$$TZ_n(s, x, y) = \max_{0 \leqslant \beta \leqslant \pi/4} \left\{ |T^{h,\beta} Z_n(s, x, y)|, \ |T^{v,\beta} Z_n(s, x, y)|, T^{d,\beta} Z_n(s, x, y)| \right\}$$

where

$$T^{\gamma,\beta} Z_n(s, x, y) = \int \Psi_s^{\gamma,\beta}(x-u, y-u) Z_n(du, dv)$$

These transformations are connected with the following formula (Wang, (1998)):

$$T^{\gamma,\beta} Z_n(s, x, y) = T^{\gamma,\beta} f(s, x, y)\{1 + o(1)\} + T^{\gamma,\beta} W_n(s, x, y) \qquad (5)$$

$T^{\gamma,\beta} W_n(s, x, y)$ is a normal random variable with mean 0 and variance of order $1/n$ and $Tf(s, x, y)$ is of order s^2 if f is differentiable and $s^{\alpha+1}$ if f has α sharp cusp at (x, y). Therefore (5) implies that at a very fine scale s, $TZ_n(s, x, y)$ is dominated by $TW_n(s, x, y)$ and at a coarse scale s, $Tf(s, x, y)$ dominates $TZ_n(s, x, y)$. We use the values of $TZ_n(s, x, y)$ which exceed suitably defined threshold values C_n only by sharp cusp curves to detect these curves.

By Θ we denote the class of smooth candidate curves that contains the true sharp cusps curves $\theta_1, \ldots \theta_q$. All curves in Θ along which the minima of $TZ_n(s, x, y)$ exceed threshold C_n we denote by $\{\hat{\theta}\}$, i.e.

$$\{\hat{\theta}\} = \left\{ \vartheta \in \Theta : \min_{(x,y) \in \vartheta} TZ_n(s_n, x, y) \geqslant C_n \right\}$$

and we estimate $\{\theta_i\}_{1 \leqslant i \leqslant q}$ by $\{\hat{\theta}\}$. We measure the distance between two closed subsets $A, B \subset [0, 1]^2$ in the sense of the Hausdorff distance i.e.

$$L(A, B) = \max \left\{ \max_{(x,y) \in A} d(x, y, B), \ \max_{(x,y) \in B} d(x, y, A) \right\}$$

If we apply this distance to measure the distance between the sets $\{\theta_i\}_{1 \leqslant i \leqslant q}$ and $\{\hat{\theta}\}$ then we can find the convergence rate for the distance this distance $L\left(\bigcup\{\hat{\theta}\}, \bigcup_{i=1}^{q}\theta_i\right)$. Figure 2 presents an example of applying this method to detecting sharp contours of two dimensional woman's portrait.

IV. NEW ALGORITHM PROPOSAL

The methods developed by Wang are mathematically elegant but they are relatively complicated and are not satisfactory if we pay attention to the speed of establishing sharp contours of images. In this part we present a simple algorithm which can be used to detect sharp contours of two dimensional images.

Suppose that we have a 800 by 600 pixels image. We have to convert the computer code of this image (it is either a bitmap or vector graphics or other form) to a table each entry of which presents the saturation of one of three basic colours, red, green and blue. Thus we have three, 800 rows by 600 columns tables, each for one colour. Moving along row n we mark the point in which there is a sharp change between this point and the next one in the same row. The same procedure is repeated for the row $n+1$. Then we connect the change point in row n with the change point in row $n+1$ if there is a sharp change in the same column, the previous

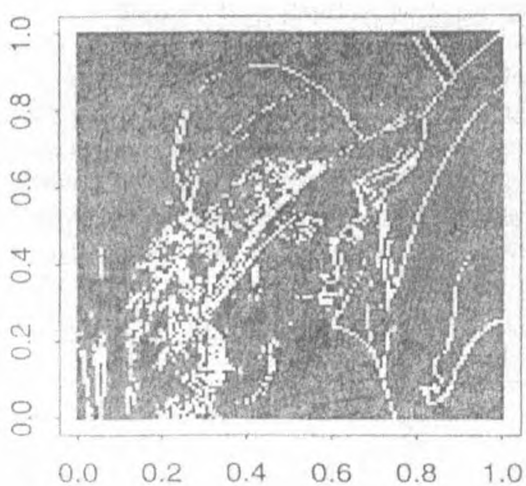

Fig. 2. Sharp contours of a two dimensional image of a woman's photograph

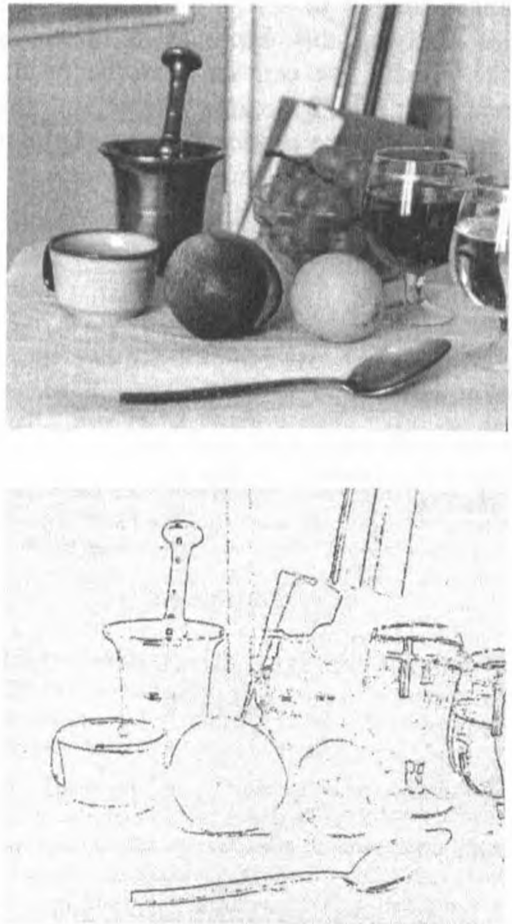

Fig. 3. Two dimensional photograph and its sharp contours established via the new algorithm

one or the next one. The crucial problem is to avoid the theoretical complications connected with detecting sharp changes and to define sharp-change in a simpler way. One of the ways is to say that sharp change occurs when all three absolute values of the differences between the saturations of three basic colours of two compared points exceed certain limit (e.g. 10 for the saturation scale from 0 to 256). Another way, which turned out to be more successful in practice, is to say that sharp change occurs when the smallest of the three differences in saturations exceeds $\alpha - \beta$, the middle of the three differences exceeds α and the highest of the three differences exceeds $\alpha + \beta$, where α is a positive parameter of change (e.g. 10) and β ranges from 0 to α.

The example of applying this approach to detect sharp contours is presented in Fig. 3. Visually we can see that the results are comparable with what is presented in Fig. 2 because sharp contours are correct and are clearly visible and the volume of memory needed for storing this picture is a small fraction of the original volume.

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WYKRYWANIE OSTRYCH KONTURÓW OBRAZÓW

(Streszczenie)

Teoria falek wprowadzonych przez Daubechies jest rozwijającą się częścią matematyki, stosowaną w wielu dziedzinach. W artykule tym przedstawiony jest przegląd metod wykrywania „szpiców" nieznanej funkcji opracowanych przez Wanga. Metody wykrywania szpiców i skoków w przypadku jednowymiarowym są zilustrowane przykładem zastosowania do funkcji rzeczywistej jednej zmiennej, która ma jeden szpic oraz jeden skok (por. rys. 1). W przypadku dwuwymiarowym metody są zilustrowane przykładem zastosowania do wykrywania ostrych konturów obrazu przedstawiającego fotografię kobiety (por. rys. 2). Część 4 artykułu jest wkładem własnym autora. Zaproponowany jest algorytm rozpoznawania ostrych konturów obrazów, którego zaletą jest prostota oraz szybkość działania. Algorytm jest zastosowany do ustalenia ostrych konturów fotografii przedstawiającej martwą naturę. Wzrokowa ocena efektów algorytmu pozwala na stwierdzenie, że wyniki są porównywalne z wynikami uzyskanymi przez Wanga za pomocą metod opartych na skomplikowanym aparacie matematycznym oraz wolno działających.