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SOME NON-MEASURABLE SETS

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Abstract. This paper contains constructions of some non-measurable sets, based on classical Vitali's and Bernstein's constructions (see for example [6]). This constructions probably belong to mathematical folklore, but as far as we know they are rather hard to be found in literature. It seems that the constructed sets can be used as examples in some interesting situations.

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1. Basic notations and facts

We use standard set theoretic notation. By $\mathbb{Q}$, $\mathbb{R}$ we denote, as usuall, the sets of rationals and reals, respectively. We say that C is a Cantor set if it is homeomorphic with the Cantor cube $\{0,1\}^{\omega}$. Several times we will use the well-known fact that any interval in $\mathbb{R}$ contains a Cantor set of positive Lebesgue measure. By λ we denote Lebesgue measure on $\mathbb{R}$. By λ^* and λ_* we denote outer and inner Lebesgue measures, respectively.

Hashimoto topology is a topology on $\mathbb{R}$ where any open set U is of the form $U = Q \backslash N$, where Q is open in natural topology and N is a nullset.

Let $A \subseteq \mathbb{R}$ be a measurable set. Put

$$\Phi(A) = \left\{ x \in \mathbb{R} : \lim_{h \to \infty} \frac{\lambda(E \cap [x-h, x+h])}{2h} = 1 \right\}.$$

We say that a set $A \subseteq \mathbb{R}$ is open in density topology if $A \subseteq \Phi(A)$.

We say that an operator $\Psi : \mathcal{L} \to P(\mathbb{R})$, where $\mathcal{L}$ is a sigma algebra of all Lebesgue measurable sets, is a density operator if:

i) $\Psi(\emptyset) = \emptyset$, $\Psi(\mathbb{R}) = \mathbb{R}$,

ii) $\forall_{A,B \in \mathcal{L}}\ \Psi(A \cap B) = \Psi(A) \cap \Psi(B)$,

iii) $\forall_{A,B \in \mathcal{L}}\ \lambda(A \triangle B) = 0 \Longrightarrow \Psi(A) = \Psi(B)$,

iv) $\forall_{A \in \mathcal{L}}\ \lambda(A \triangle \Psi(A)) = 0$.

2. Some modification of the Bernstein set

The Bernstein set is non-measurable in every interval. It has also inner measure zero and full outer measure in every subset. It is such called saturated set. We will construct a set that is non-measurable in every interval, but with positive inner measure in every interval. Also its complement will have the same property.

Lemma 1. *There exist Borel pairwise disjoint sets $A, B, C \subseteq \mathbb{R}$ such that:*

$$\left(\forall_{(a,b)\subseteq\mathbb{R}}\right)\left(\lambda\left((a,b)\cap A\right)>0\wedge\lambda\left((a,b)\cap B\right)>0\right)\wedge\lambda\left((a,b)\cap C\right)>0.$$

Proof. Let $\{I_n : n\in\omega\}$ be an enumeration of open intervals with rational endpoints. We will define inductively families $\{C_n : n\in\omega\}$, $\{C'_n : n\in\omega\}$ and $\{C''_n : n\in\omega\}$ such that:

(1) C_n, C'_n and C''_n are pairwise disjoint for any $n\in\omega$.

(2) $(\forall_{n\in\omega})\left(\lambda\Big(I_n\cap\bigcup\limits_{s\leq n}C_s\Big)>0\wedge\lambda\Big(I_n\cap\bigcup\limits_{s\leq n}C'_s\Big)>0\wedge\lambda\Big(I_n\cap\bigcup\limits_{s\leq n}C''_s\Big)>0\right)$,

(3) $\forall_{n\in\omega}\ C_n$, C'_n and C''_n are Borel, nowhere dense sets.

Then $A=\bigcup\limits_{n\in\omega}C_n$, $B=\bigcup\limits_{n\in\omega}C'_n$ and $C=\bigcup\limits_{n\in\omega}C''_n$ will satisfy the thesis. Indeed, let $(a,b)\subseteq\mathbb{R}$. Then there exist $k\in\omega$ such that $I_k\subseteq(a,b)$. We have by (2)

$$\lambda\left((a,b)\cap A\right)\geq\lambda\left(I_k\cap A\right)\geq\lambda\left(I_k\cap\bigcup_{n\leq k}C_n\right)>0.$$

Analogically $\lambda\left((a,b)\cap B\right)>0$ and $\lambda\left((a,b)\cap C\right)>0$.

Let C_0 be a Cantor set such that $C_0\subseteq I_0$ and $\lambda\left(C_0\right)>0$. Since Cantor set is nowhere dense, there exists an interval $(a_0,b_0)\subseteq I_0\backslash C_0$. Let C'_0 be a Cantor set of positive Lebesgue measure such that $C'_0\subseteq(a_0,b_0)$. There exists an interval $(a'_0,b'_0)\subseteq(a_0,b_0)\backslash C'_0$. Let C''_0 be a Cantor set such that $C''_0\subseteq(a'_0,b'_0)$, $\lambda\left(C''_0\right)>0$.

Let $n\in\omega$. Assume that we have defined C_k, C'_k, C''_k for all $k\leq n$. Since $\bigcup_{k\leq n}C_k\cup\bigcup_{k\leq n}C'_k\cup\bigcup_{k\leq n}C''_k$ is a nowhere dense set, as a finite sum of nowhere dense sets, there exists an interval

$$(a_{n+1},b_{n+1})\subseteq I_{n+1}\backslash\left(\bigcup_{k\leq n}C_k\cup\bigcup_{k\leq n}C'_k\cup\bigcup_{k\leq n}C''_k\right).$$

If $\lambda\Big(I_{n+1}\cap\bigcup_{k\leq n}C_k\Big)>0$, then we put $C_{n+1}=\emptyset$. Otherwise, let C_{n+1} be a Cantor set such that $C_{n+1}\subseteq(a_{n+1},b_{n+1})$ and $\lambda\left(C_{n+1}\right)>0$. If $\lambda\Big(I_{n+1}\cap\bigcup_{k\leq n}C'_k\Big)>0$, then we put $C'_{n+1}=\emptyset$. If $\lambda\left(I_{n+1}\cap\bigcup_{k\leq n}C'_k\right)=0$, then there exists an interval

$$(a'_{n+1},b'_{n+1})\subseteq I_{n+1}\backslash\left(\bigcup_{k\leq n+1}C_k\cup\bigcup_{k\leq n}C'_k\cup\bigcup_{k\leq n}C''_k\right).$$

In this case we put C'_{n+1}as a Cantor set such that $C'_{n+1} \subseteq \left(a'_{n+1}, b'_{n+1}\right)$ and $\lambda\left(C'_{n+1}\right) > 0$. If $\lambda\Big(I_{n+1} \cap \bigcup_{k\leq n} C''_k\Big) > 0$, then we put $C''_{n+1} = \emptyset$. If $\lambda\Big(I_{n+1} \cap \bigcup_{k\leq n} C''_k\Big) = 0$, then there exists an interval

$$\left(a''_{n+1}, b''_{n+1}\right) \subseteq I_{n+1} \backslash \left(\bigcup_{k\leq n+1} C_k \cup \bigcup_{k\leq n+1} C'_k \cup \bigcup_{k\leq n} C''_k \right).$$

Then we define C''_{n+1} as a Cantor set such that $C''_{n+1} \subseteq \left(a''_{n+1}, b''_{n+1}\right)$ and $\lambda\left(C''_{n+1}\right) > 0$. Then sets C_0,C_1,. . .,C_{n+1}, C'_0,C'_1,. . .,C'_{n+1} and C''_0, C''_1,. . .,C''_{n+1} obviously satisfy conditions $(1) - (3)$. $\square$

Proposition 1. *There exists a non-measurable set $E \subseteq \mathbb{R}$ such that*

$$\lambda_*\left((a,b) \cap E\right) > 0, \qquad \lambda_*\left((a,b) \cap \mathbb{R}\backslash E\right) > 0$$

and $(a,b) \cap E$ is non-measurable for all $(a,b) \subseteq \mathbb{R}$.

Proof. Let A, B, C be like in lemma 1. Denote $\mathfrak{c} = |\mathbb{R}|$. Let $\mathcal{F} = \{F_\xi : \xi < \mathfrak{c}\}$ be an enumeration of closed subsets of C in subspace topology such that $\lambda_C(F) > 0$ where $\lambda_C(F) = \lambda(F \cap C)$. Let $F \in \mathcal{F}$, then there exists a closed set $F' \subseteq \mathbb{R}$ with $F = F' \cap C$. Since C is a Borel set F is Borel. Moreover, $|F| = \mathfrak{c}$ since F is a Borel set of positive measure. We will define by induction two disjoint sets $\{a_\xi : \xi < \mathfrak{c}\}$ and $\{b_\xi : \xi < \mathfrak{c}\}$ such that

$$F \cap \{a_\xi : \xi < \mathfrak{c}\} \neq \emptyset \neq F \cap \{b_\xi : \xi < \mathfrak{c}\}$$

for all $F \in \mathcal{F}$.

Let a_0, b_0 be two different elements of F_0, and let a_1, b_1 be two different elements of $F_1 \backslash \{a_0, b_0\}$. Assume that a_α, b_α are defined for all $\alpha < \beta < \mathfrak{c}$. Then $\left| F_\beta \backslash \bigcup_{\alpha<\beta} \{a_\alpha, b_\alpha\} \right| = \mathfrak{c}$ because $\left| \bigcup_{\alpha<\beta} \{a_\alpha, b_\alpha\} \right| < \mathfrak{c}$. Hence we can choose two different elements a_β, $b_\beta \in F_\beta \backslash \bigcup_{\alpha<\beta} \{a_\alpha, b_\alpha\}$.

Put $E' = \{a_\xi : \xi < \mathfrak{c}\}$. Let F be a closed set, such that $F \subseteq E' \subseteq C$. Therefore $F = F \cap C$ is closed in C. Then $\lambda_C(F) = 0$. Indeed, if $\lambda_C(F) > 0$, then by the construction there exists $\xi < \mathfrak{c}$ such that $b_\xi \in \mathbb{R}\backslash E' \cap F$. It follows, that $F \cap (\mathbb{R}\backslash E') \neq \emptyset$, so $F \nsubseteq E'$. We have $\lambda_C(F) = \lambda(F \cap C) = \lambda(F) = 0$. Hence

$$\lambda_*\left(E'\right) = \sup\left\{\lambda(F) : F \subset E \text{ and } F \text{ is closed}\right\} = 0.$$

It follows, that $\lambda_*\left(E' \cap (a,b)\right) = 0$ for all $(a,b) \subseteq \mathbb{R}$. We will show that $\lambda^*\left(E' \cap (a,b)\right) > 0$ for all $(a,b) \subseteq \mathbb{R}$. It will follow that E' is non-measurable and that for all $(a,b) \subseteq \mathbb{R}$ set $(a,b) \cap E'$ is non-measurable. Let (a,b) be an open interval and G be an open set such that $E' \cap (a,b) \subseteq G$. Then $G' = G \cap (a,b) \cap C$ is an open set in C. Then $C\backslash G'$ is a closed in C

subset of E'. It follows that $\lambda_C(C\backslash G') = 0$. Indeed, if $\lambda_C(C\backslash G') > 0$ then $\mathbb{R}\backslash G' \cap E \neq \emptyset$. We have

$$\lambda(C) = \lambda_C(C) = \lambda_C\left(C\backslash G'\right) + \lambda_C\left(G'\right) = 0 + \lambda_C\left(G'\right),$$

so $\lambda_C(G') = \lambda(C)$. Then

$$\lambda(G) \geq \lambda(G \cap (a,b) \cap C) \geq \lambda\left(G'\right) = \lambda\left(G' \cap C\right) = \lambda_C\left(G'\right) = \lambda(C).$$

It follows that

$$\lambda^*\left(E'\right) = \inf\left\{\lambda(G) : E' \subseteq G \text{ and } G \text{ is open}\right\} \geq \lambda(C).$$

Therefore E' is non-measurable in every interval.

Put $E = E' \cup A$. Then $E \cap (a,b)$ is non-measurable for any open interval, as a disjoint union of Borel and non-measurable set. Let $(a,b) \subseteq \mathbb{R}$. Then

$$\lambda_*((a,b) \cap E) = \lambda_*\left((a,b) \cap \left(A \cup E'\right)\right) \geq \lambda_*((a,b) \cap A) = \lambda((a,b) \cap A) > 0.$$

It is obvious that $B \subseteq \mathbb{R}\backslash E$. Hence

$$\lambda_*((a,b) \cap (\mathbb{R}\backslash E)) \geq \lambda_*((a,b) \cap B) = \lambda((a,b) \cap B) > 0.$$

□

The sets constructed in lemma 1 and in proposition 1 are examples of sets which have an empty interior in the Hashimoto ([2]) topology and non-empty interior in every interval (a,b) in density topology ([8]). Indeed, let A, B be sets constructed in lemma 1, and let (a,b) be any interval. Then $\lambda((a,b) \cap B) > 0$ and $(a,b) \cap A \cap B = \emptyset$. Therefore A has empty interior in the Hashimoto topology. A has non-empty interior in every interval (a,b) in density topology, because for every interval holds $\lambda((a,b) \cap A) > 0$.

Each of the sets constructed in lemma 1 is also an example proving that

$$\psi(A) = \{x \in \mathbb{R} : \forall_{h>0}\ \lambda((x-h, x+h) \cap A) > 0\}$$

is not a density operator. Indeed, let A, B be sets constructed in lemma 1. Then $\psi(A) = \mathbb{R}$, because for every $x \in \mathbb{R}$ and every $h > 0$ we have $\lambda((x-h, x+h) \cap A) > 0$. But $\lambda(\psi(A) \triangle A) = \lambda(\psi(A) \backslash A) \geq \lambda(B\backslash A) = \lambda(B) > 0$.

3. Some modification of the Vitali set

It is known, that for every set, which has a positive measure or is of second category with a Baire property, we have:

$$0 \in int\,(A - A) \tag{*}$$

This is the Steinhaus or Picard theorem respectively ([7], [5]). Many generalizations of these theorems are known (see f.e. [1], [3], [4]). Often instead of $(*)$ authors write:

$$(**) \qquad int\,(A - A) \neq \emptyset.$$

Let $\langle \mathcal{A}, \mathcal{I} \rangle$ be an algebra $\mathcal{A}$ with an ideal $\mathcal{I} \subset \mathcal{A}$, $(X, +, \tau)$ a topological group. We say that $\langle \mathcal{A}, \mathcal{I}, \tau \rangle$ has the classical Steinhaus property if

$$\forall_{A \in \mathcal{A} \setminus \mathcal{I}} \;\; 0 \in int\,(A - A)\,.$$

We say that $\langle \mathcal{A}, \mathcal{I}, \tau \rangle$ has the week classical Steinhaus property if

$$\forall_{A \in \mathcal{A} \setminus \mathcal{I}} \;\; int\,(A - A) \neq \emptyset.$$

The set constructed below shows that without any additional assumptions, the properties $(*)$ and $(**)$ are not equivalent. Indeed, let A be the set constructed below and $\mathcal{A} = \{\mathbb{R}, \emptyset, A, \mathbb{R} \setminus A\}$, $\mathcal{I} = \{\emptyset\}$, $\tau =$ natural topology. Then $\langle \mathcal{A}, \mathcal{I}, \tau \rangle$ has the weak classical Steinhaus property and $\mathcal{A}$ does not have the classical Steinhaus property.

Theorem 1. *There exists set $A \subseteq \mathbb{R}$ such that $Int\,(A - A) \neq \emptyset$ and $0 \notin Int\,(A - A)$.*

Proof. Let V be a Vitali non-measurable set constructed in $[0, 1]$ such that $0 \in V$. Let $\{a_n\}_{n=1}^{\infty}$ be an enumeration of all rational numbers in $[-1, 1]$. We will construct a sequence $\{V_n\}_{n=0}^{\infty}$ of translated Vitali sets such that $A = \bigcup_{n=0}^{\infty} V_n$ will satisfy the thesis.

Let $V_0 = V$ and

$$V_n = \begin{cases} V + 2n & \text{when } n \text{ is even} \\ V + 2n + a_{\frac{n+1}{2}} & \text{when } n \text{ is odd} \end{cases}$$

for $n \geq 1$.

We will show that $(A - A) \cap (-1, 1) \subseteq (\mathbb{R} \setminus \mathbb{Q}) \cup \{0\}$. It will follow that $0 \notin Int\,(A - A)$. Let $x \in (A - A) \cap (-1, 1)$. Then $x = v_1' - v_2'$ for some $v_1', v_2' \in A$. It is easy to verify that $v_1' \in V_n$, and either $v_2' \in V_{n+1}$, or $v_2' \in V_{n-1}$, for some $n \in \omega$. In any other case, $|v_1' - v_2'| > 1$ which contradicts the choice of x.

Assume $v_1' \in V_n$, $v_2' \in V_{n+1}$ and n is an even number. Let $v_1' = v_1 + 2n$ and $v_2' = v_2 + 2(n + 1) + a_{\frac{n+2}{2}}$ for some $v_1, v_2 \in V$. Then

$$\begin{aligned} v_1' - v_2' = v_1 + 2n - (v_2 + 2n + 2 + a_{\frac{n+2}{2}}) = v_1 - v_2 - 2 - a_{\frac{n+2}{2}} \leq \\ \leq v_1 - v_2 - 2 + 1 = v_1 - v_2 - 1. \end{aligned}$$

Since $x \in (-1, -1)$, so $v_1 - v_2 > 0$. It implies that $v_1 - v_2 \in \mathbb{R} \setminus \mathbb{Q}$. Hence $x \in \mathbb{R} \setminus \mathbb{Q}$. Proofs of remaining cases are similar.

Now we will show that $[2,3] \subseteq A - A$. Let $v' \in V_n$ where n is odd. Then $v' = v + 2n + a_{\frac{n+1}{2}}$ for some $v \in V$. Since $0 \in V$ so $2(n-1) \in V_{n-1}$. Then $v' - 2(n-1) \in A - A$, and

$$v' - 2(n-1) = v + 2n + a_{\frac{n+1}{2}} - 2n + 2 = v + a_{\frac{n+1}{2}} + 2.$$

It follows that $v + a_{\frac{n+1}{2}} + 2 \in A - A$ for all $v \in V$. Hence $V + a_{\frac{n+1}{2}} + 2 \subseteq A - A$. Similarly $V + a_n + 2 \subseteq A - A$ for all $n \in \omega$. It implies that

$$\bigcup_{n=1}^{\infty} (V + a_n + 2) \subseteq A - A$$

and

$$\bigcup_{n=1}^{\infty} (V + a_n + 2) = 2 + \bigcup_{n=1}^{\infty} (V + a_n) = 2 + \bigcup_{q \in \mathbb{Q} \cap [-1,1]} (V + q) \supseteq 2 + [0,1].$$

Therefore $[2,3] \subseteq A - A$, and $(2,3) = Int([2,3]) \subseteq Int(A - A)$. □

Problem 1. *We do not know whether there exists a Borel set with this property.*

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