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STOCHASTIC ORDERS IN DISCRETE DYNAMIC PROGRAMMING

Abstract

This paper deals with a problem of dynamic optimization with values of criteria function in the set of the random variables. Precisely, there is a dynamic model with finite number of stages, states and decision variables described. Such a dynamic process is evaluated regarding values of the random variables. The random variables have to fulfil some conditions, if they are to be applied to dynamic optimization. These conditions are described in presented paper. Moreover, there is given a review of stochastic orders, which can be used in the model.

Key words: dynamic programming, partially ordered set, stochastic orders.

I. INTRODUCTION

The theory of dynamic programming was introduced by R. Bellman (1957). Next Brown and Strauch (1965) generalized Bellman's principle to a class of multi-criteria dynamic programming with a lattical order. Then, the use of optimality principle was the interest of Mitten (1974) who considered preferences relation, and Henig (1985) – infinite dynamic process with values of criteria function in a partially ordered set. Others who took interest in the use of multicriterial methods in dynamic programming have been: Trzaskalik (1998), Sobel (1975), Li and Haimes (1989). The continuous dynamic decision model with the methodology of multi criteria-decision making is analyzed in Glaser (2002). In the meantime theory of comparing random variables have developed as well, Rolski (1976), Shaked and Shanthikumar (1993). It enabled us to use such structures in our dynamic model.

Our work considers the dynamic discrete decision making model with returns in partially ordered set. Bellman's principle of optimality for such

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a problem will be shown. Moreover the methods of narrowing the set of the solutions will be described. These methods are analogical to multi-criteria programming methods, but here, additionally, the dynamic aspect of the model must be taken to consideration. Such a general approach, i.e. structure of partially ordered set, let us to apply a lot mathematical structures to describe practical problems. As an interesting sample of this, we will show combination of dynamic programming and random variables with stochastic dominance.

II. MODEL

We consider a discrete dynamic process, which consist of T periods. Let for $t = 1, ..., T$

- Y_t is the set of all feasible state variables at the beginning of period t,
- Y_{T+1} is the set of all states at the end of the process,
- $X_t(y_t)$ is the set of all feasible decision variables for period t and state $y_t \in Y_t$.

We assume that all above sets are finite.

- $D_t = \{d_t = (y_t, x_t)\colon y_t \in Y_t,\ x_t \in X_t(y_t)\}$ is the set of all period realizations in period t,
- $\Omega_t\colon D_t \to Y_{t+1}$ are given transformations,
- $D = \{d \doteq (d_1, ..., d_T)\colon \forall_{t\in\{1,...,T\}}\ y_{t+1} = \Omega_t(y_t, x_t)$ and $x_T \in X_T(y_T)\}$ is the set of all process realizations d,
- $D_t(y_t) = \{(y_t, x_t)\colon\ x_t \in X_t(y_t)\}$ is the set of all realizations in period t which begin at y_t,
- $d(y_t) = (y_t, x_t, ..., y_T, x_T)$ is the partial realization for given realization d, which begin at y_t,
- $D(y_t) = \{d(y_t)\colon\ d \in D\}$ is the set of all partial realizations, which begin at y_t,
- $D(Y_t) = \{D(y_t)\colon y_t \in Y_t\}$ is the set of all partial realizations, which begin at period t,
- $P = \{(Y_t, Y_{T+1}1),\ X_t(y_t),\ \Omega_t\colon\ t = 1, ..., T\}$ denotes discrete dynamic process, where sets $Y_1, ..., Y_{T+1}, X_1(y_1), ..., X_T(y_T)$, functions $\Omega_1, ..., \Omega_T$ are identified.

We consider the following structure, functions and operators to describe multi-period criteria function of process realization.

- $(W, \leqslant, \circ)$ – structure, where $(W, \leqslant)$ is partially ordered set (poset), and operator $\circ\colon W \times W \to W$ satisfies conditions

$$1)\ a \circ (b \circ c) = (a \circ b) \circ c;\ a,b,c \in W.$$

$$2)\ a \leqslant b \Rightarrow a \circ c \leqslant b \circ c \quad \text{and} \quad c \circ a \leqslant c \circ b;\ a,b,c \in W.$$

We denote relation as follows

$$a < b \Leftrightarrow a \leqslant b \quad \text{and} \quad a \neq b.$$

Remark 1. It is enough to assume transitive condition from the relation $\leqslant$ in the set W. In this case, through proper operations, we obtain a poset; Birkhoff (1973).

For each finite subset $A \subset W$ we define

$$\max(A) = \{a^* \in A: \ \sim \exists_{a \in A} \ a^* < a\}.$$

– $f_t : D_t \rightarrow W$, for $t = 1, ..., T$, are period criteria functions with returns in partially ordered set W.

We assume that for each period exist following operators:

– $\circ_t : W \times W \rightarrow W$ are monotone operators ($t = 1, ..., T-1$).

$$\forall_{a,b,c \in W} \ a \leqslant b \Rightarrow c \circ_t a \leqslant c \circ_t b.$$

– $F_t : D(Y_t) \rightarrow W$ are the functions defined in the following way

$$F_t = f_t \circ_t (f_{t-1} \circ_{t-1} (...(f_{T-1} \circ_{T-1} f_T))), \ t = \mathrm{T}, \ ..., \ 1.$$

– $F = F_1$ – is called multi-period criteria function.

In further consideration we postulate to maximize function F (in the sense of relation $\leqslant$)

– (P, F) denotes discrete dynamic decision process. It is given, if there are defined discrete dynamic process P and multi-period criteria function F.

– Realization $d^* \in D$ is said to be efficient, if

$$F(d^*) \in \max \ F(D).$$

III. PRINCIPLE OF OPTIMALITY

In the discrete dynamic decision process (P, F) there holds following theorem:

Theorem 1. Let (P, F) be decision dynamic process,

For all $t = T-1, ..., 1$ and all $y_t \in Y_t$ holds

a) $\max \ \{F_t(D(y_t)\} = \max \{f_t(d_t) \circ_t \max (F_{t+1}(d(\Omega_t(d_t)))): \ d_t \in D_t(y_t)\}$.

b) $\max \{F(D)\} = \max \{\max F_1(d(y_1)) : y_1 \in Y_1\}$.

Proof. Trzaskalik and Sitarz (2000).

IV. NARROWING THE SET OF SOLUTIONS

Solution of such a dynamic problem, in the sense of maximal values, may consist of very numerous set. To narrow the set of maximal values, we may use method presented below. Sources of the method are taken from the analogical methods of multicriterial programming, however, here, the procedure will proceed by stages. Moreover the methods are much more general considering the fact that set of values is much broader than vector of real numbers.

We will narrow the set of maximal values with the help of a new relation $\leqslant'$ fulfilling the conditions:

1. $(W, \leqslant', \circ)$ is ordered structure.
2. $(a < b \Rightarrow a <' b)$.

Having in mind the fact, that the relation $\leqslant'$ defines other maximal elements, let's mark it max'. We will mark the relation $\leqslant'$ fulfilling above conditions 1 and 2 (in reference to the relation $\leqslant$) as follows $\leqslant \subset \leqslant'$. The corollary of the condition 2 is that for each $A \subset W$

$$\max' A \subset \max A.$$

Remark 2. Let's notice that if certain relation $\leqslant'$ fulfils weaker condition then condition 2, namely

3. $a \leqslant b \Rightarrow a \leqslant'' b$,

then introducing the relation $\leqslant'$ of the form

$$a \leqslant' b \Leftrightarrow (a \leqslant'' b) \wedge \sim (b < a)$$

we will get $\leqslant \subset \leqslant'$.

Theorem 2. If the relation $\leqslant'$ fulfills the conditions 1 and 2, then applying the procedure describes in the point 3, we'll get a subset of maximal values of the process i.e.

$$\max' F(R) \subset \max F(R).$$

Proof. Correctness of the procedure guarantees the condition 1, whereas the inclusion of maximal elements gives us the condition 2.

Remark 3. We use procedure of calculating a narrowed set of maximal values used to earlier obtained sets of maximal values in such a way that we solely consider maximal values obtained with the help of the procedure (we ignore other values of stages criterion function).

V. STOCHASTIC ORDERS

Now we present the notation of stochastic dominance and establish basic properities, which are usefull in the dynamic programming (Ogryczak, 1997; Rolski, 1976; Baccelli, 1991).

In the stochastic dominance approach, random variables are compared by the pointwise comparision of their distribuation functions. For real random variables ξ function $F_\xi^{(1)}$ is the right-continous cumulative distribution function

$$F_\xi^{(1)}(x) = \int_{-\infty}^{x} f_\xi(t)dt = P(\xi \leqslant x), \;\; x \in R.$$

Moreover, the k-th function $F_\xi^{(k)}$ is defined as follows:

$$F_\xi^{(k)}(x) = \int_{-\infty}^{x} F_\xi^{(k-1)}(t)dt, \;\; x \in R.$$

The relation of k-th degree stochastic dominance is understood as follows:

$$\xi \leqslant_{(n)} \eta \Leftrightarrow \forall_{x \in R} F_\xi^{(k)}(x) \leqslant F_\eta^{(k)}(x).$$

The folowing properities of relation $\leqslant_{(k)}$ are usefull in dynamic programming. Let ξ, η, γ are random variables for which exist functions $F_\xi^{(k)}$, $F_\eta^{(k)}$, $F_\gamma^{(k)}$, then

$$\xi \leqslant_{(k)} \eta \wedge \eta \leqslant_{(k)} \gamma \Rightarrow \xi \leqslant_{(k)} \gamma$$

$$\xi \leqslant_{(k)} \eta \Rightarrow (\xi + \gamma) \leqslant_{(k)} (\eta + \gamma)$$

For more details on stochastic dominance, properties and proofs see Rolski (1976).

As we can see above, the properties of stochasic dominance let us use such a structure in dynamic programming. We can adopt a structure $(W, \leqslant, \circ)$ preseneted in section II, which is the following triple:

- W is the set of random varaibles ξ for which exist $F_\xi^{(k)}$,
- relation $\leqslant$ of this structure is k-th degree stochastic dominance,
- operator $\circ$ of this structure is adding random variables (or convolution of distrubuation function).

VI. CONCLUSIONS

A multi-stage process with finite number of periods, states and decision variables was considered. Dynamic discrete decision making model with returns in partially ordered set and Bellman's principle for such a problem were shown. The model can be applied for many multi-stage, multi-criteria decision making problems. In the numerical example, given in the paper, we considered orderd structure, in which values of criteria function were given by random variable. The order in the structure was defined by the stochastic dominance.

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PORZĄDKI STOCHASTYCZNE W DYSKRETNYM PROGRAMOWANIU DYNAMICZNYM

Streszczenie

W pracy rozważane jest zadanie optymalizacji dynamicznej z wartościami funkcji kryterium będącymi zmiennymi losowymi. Ściślej opisany jest model dynamiczny ze skończoną liczbą etapów, stanów oraz decyzji. Proces taki oceniany jest ze względu na osiągane wartości zmiennych losowych. Aby można było zastosować zmienne losowe w optymalizacji dynamicznej, muszą one spełniać odpowiednie warunki, co opisane jest w pracy. Podany jest przykład możliwych do wykorzystania porządków stochastycznych, tzw. dominacji stochastycznych.