

INTEGRABLE FUNCTIONS VERSUS A GENERALIZATION OF LEBESGUE POINTS IN LOCALLY COMPACT GROUPS

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Abstract. Here in this paper we intend to deal with two questions: How large is a “Lebesgue Class” in the topology of Lebesgue integrable functions, and also what can be said regarding the topological size of a “Lebesgue set” in \mathbb{R} ?, where by a Lebesgue class (corresponding to some $x \in \mathbb{R}$) is meant the collection of all Lebesgue integrable functions for each of which the point x acts as a common Lebesgue point, and, by a Lebesgue set (corresponding to some Lebesgue integrable function f) we mean the collection of all Lebesgue points of f .

However, we answer these two questions in a more general setting where in place of Lebesgue integration we use abstract integration in locally compact Hausdorff topological groups.

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1. INTRODUCTION

We begin by introducing the following classical notions and a historically important result.

Definition 1. A point x is called “a Lebesgue point” of a Lebesgue integrable function f if

$$\limsup_{I \rightarrow x} \frac{\int_I |f(t) - f(x)| d\mu}{\mu(I)} = 0,$$

where the left hand side expresses the quantity

$$\sup_{\{I_k\}} \left\{ \limsup_{k \rightarrow \infty} \frac{\int_{I_k} |f(t) - f(x)| d\mu}{\mu(I_k)} \right\},$$

the supremum being taken over all sequences $\{I_k\}$ of all non-degenerate intervals in \mathbb{R} such that $x \in I_k$ for all k and $\mu(I_k) \rightarrow 0$. For any Lebesgue integrable function f we write $L(f) = \{x \in \mathbb{R} : x \text{ is a Lebesgue point of } f\}$. As mentioned in the abstract, the above set $L(f)$ may be termed as the “Lebesgue set” corresponding to f .

Now instead of the function, if we keep the point as fixed, we obtain in turn the following subclass of Lebesgue integrable functions.

Definition 2. $L(x) = \{f : f \text{ has a Lebesgue point at } x\}$ which as mentioned in the abstract may be termed as the “Lebesgue class” corresponding to x . This class is a subclass of the class of all Lebesgue integrable functions which forms a topological space, the topology being induced by the usual metric on the class of Lebesgue integrable functions defined by $\rho(f, g) = \int |f - g| d\mu$. The class $L(x)$ is actually the dual of $L(f)$, where the roles of ‘point’ and ‘function’ are interchanged.

The following classical theorem is known as the Lebesgue density Theorem: The set of Lebesgue points of any given Lebesgue integrable function is of full (Lebesgue) measure in \mathbb{R} .

Thus in connection with Lebesgue’s work, $\mu(\mathbb{R} \setminus L(f)) = 0$ which means that the complement of $L(f)$ in \mathbb{R} is a set of Lebesgue measure zero (such sets are often called “sets of Lebesgue full-measure in \mathbb{R} ”). It can also be shown that “given any Lebesgue integrable function, each of its Lebesgue point is also its point of approximate continuity”. The converse holds provided f is bounded and measurable.

We already know by virtue of Lebesgue’s theorem that for any Lebesgue integrable function f , $L(f)$ is a set of full-measure in \mathbb{R} and hence is measure-theoretically very large. Can there be a Lebesgue integrable function f for which the set $L(f)$ is also topologically large in \mathbb{R} ? So if such functions exist, then it is worth investigating the topological size of that subclass in the topology of integrable functions. Besides this, for any $x \in \mathbb{R}$ we may also enquire regarding the size of the class $L(x)$; whether $L(x)$ is topologically large in the same topology; or stands in opposition to the measure-theoretic largeness of $L(f)$.

In this paper, we propose to deal with these two questions. But instead of Lebesgue integrable functions defined on the real line, we prefer treating the entire thing in a more general setting which refers to abstract integration in locally compact groups.

We therefore start by supposing that G is a locally compact, Hausdorff topological group, with e as the identity element. Let S_1 denote the σ -ring generated by compact sets [6] and S denote the σ -ring generated by S_1 and subsets of sets in S_1 of μ -measure zero, where μ is a non-zero, σ -finite, diffused (this property of Haar measure is equivalent to the non-discreteness of the group) and complete, regular left Haar measure on S . The diffusedness of the measure μ may also be stated in other words as follows: for each $\epsilon > 0$, there exists an open set V containing e such that $\mu(V) < \epsilon$. For any $E(\subseteq G)$ let the outer measure induced by μ be given by

$\mu^*(E) = \inf\{\mu(F) : E \subseteq F \in \mathcal{S}\}$. Moreover, $L^1(G)$ denotes the class of all real valued μ -integrable functions on G . It is the class of all μ -measurable real valued functions f on G for which $\int_G f d\mu$ is finite. On $L^1(G)$ the topology induced by the standard norm is considered and we express by writing $C(L^1(G))$ the class of all real valued continuous functions on the space $L^1(G)$. Although S_1 has been defined in [6] as the class of Borel sets, we will not adopt this terminology. Instead, by Borel sets we mean the members of the σ -algebra generated by open subsets of G .

Apart from these, we will also be using in the sequel notations such as

- (i) \mathbb{N} for the set of all positive integers and χ_A for the characteristic function of a set A .
- (ii) $E_{(x)}$ and $E^{(y)}$ for the x -section ($x \in X$) and y -section ($y \in Y$) of any set $E \subseteq X \times Y$.
- (iii) $f(x, \cdot)$ and $f(\cdot, y)$ for the x -section ($x \in X$) and the y -section ($y \in Y$) of any function $f : X \times Y \rightarrow Z$.

A definition of “density of a set at a point” with respect to Haar measure in topological groups was introduced by Lahiri [8]. It is based on the notion of demi-spheres, the credit for formulation of which goes to Comfort and Gordon [4]. This concept was used by the present author [3] in extending some results of Steinhaus. However, we do not use here the same notion of demi-spheres in extending the classical definition of “Lebesgue point” from the real line to this general setting.

Definition 3. A family \mathcal{C} of compact subsets of G is called *admissible* (or *nice*) if the following conditions are fulfilled.

- (i) $e \in S$ and $\mu(S) > 0$ for every $S \in \mathcal{C}$;
- (ii) for every open neighbourhood V of the identity element e there is g with $e \in gS \subseteq V$;
- (iii) for every sequence $\{g_n S_n\}_{n=1}^{\infty}$ satisfying $e \in g_n S_n$ and $\lim_{n \rightarrow \infty} \mu(S_n) = 0$ and every open neighbourhood V of the identity element e , we have $g_n S_n \subseteq V$ for sufficiently large n .

It may be noted that conditions (ii) and (iii) are equivalent to the following ones

- (ii)' if $x \in V$ (open), there exist $g \in G$ and $S \in \mathcal{C}$ such that $x \in gS \subseteq V$;
- (iii)' for every sequence $\{g_n S_n\}_{n=1}^{\infty}$ satisfying $x \in g_n S_n$ for every n and $\lim_{n \rightarrow \infty} \mu(S_n) = 0$, if $x \in V$ (open) then $g_n S_n \subseteq V$ for all n sufficiently large.

However, in the Euclidean n -space, with respect to the normal n -dimensional metric and the Lebesgue measure, we may choose the family of closed balls

centered at the origin as an admissible family of sets. But with respect to another metric, the corresponding family of closed balls can form an admissible family only when an appropriate measure is chosen.

We now introduce the notion of ‘‘Generalized Lebesgue point with respect to \mathcal{C} ’’ or \mathcal{C} -point.

Definition 4. For each $f \in L^1(G)$, let us write

$$\limsup_{gS \rightarrow x} \frac{\int_{gS} |f(y) - f(x)| d\mu}{\mu(S)}$$

to express the quantity

$$\sup \left\{ \limsup_{n \rightarrow \infty} \frac{\int_{g_n S_n} |f(y) - f(x)| d\mu}{\mu(S_n)} \right\},$$

where the supremum is taken over all sequences $\{g_n S_n\}_{n=1}^{\infty}$, ($g_n \in G$, $S_n \in \mathcal{C}$) such that $x \in g_n S_n$ for all n and $\lim_{n \rightarrow \infty} \mu(S_n) = 0$.

Likewise, we write

$$\liminf_{gS \rightarrow x} \frac{\int_{gS} |f(y) - f(x)| d\mu}{\mu(S)}$$

to express the quantity

$$\inf \left\{ \liminf_{n \rightarrow \infty} \frac{\int_{g_n S_n} |f(y) - f(x)| d\mu}{\mu(S_n)} \right\},$$

where the infimum is taken over all sequences $\{g_n S_n\}_{n=1}^{\infty}$, ($g_n \in G$, $S_n \in \mathcal{C}$) such that $x \in g_n S_n$ for all n and $\lim_{n \rightarrow \infty} \mu(S_n) = 0$.

Now as an extension of the notion of ‘‘Lebesgue point’’ from \mathbb{R} to the present setting, we define a point $x \in G$ as a \mathcal{C} -point of some function $f \in L^1(G)$ provided

$$\limsup_{gS \rightarrow x} \frac{\int_{gS} |f(y) - f(x)| d\mu}{\mu(S)} = 0.$$

We write

$$\mathcal{C}(f) = \{x \in G : x \text{ is a } \mathcal{C}\text{-point of } f\}$$

(which is a natural generalization of the Lebesgue set $L(f)$ in the present context), and also after interchanging the role of f and x , obtain the class $\mathcal{C}(x) = \{f \in L^1(G) : f \text{ has an } \mathcal{C}\text{-point at } x\}$ which is a natural generalization of the Lebesgue class $L(x)$.

As may be noted from the preceding paragraph, the notion of \mathcal{C} -point which is a generalization of Lebesgue point in \mathbb{R} or \mathbb{R}^n depends on the

choice of the admissible family \mathcal{C} . Thus a given point may be a \mathcal{C} -point of a given function with respect to one admissible family but not with respect to another. For example in the usual metric of \mathbb{R}^2 and the two dimensional Lebesgue measure, the family \mathcal{C}_1 of all closed balls centered at the origin is an admissible family, and, likewise in the pseudometric d^* defined by $d^*((x_1, y_1), (x_2, y_2)) = |x_1 - x_2|$ and the measure γ defined on the σ -algebra $M_\gamma = \{A : \text{the intersection of } A \text{ with the } x - \text{axis is lebesgue measurable}\}$ by the formula $\gamma(A) = \text{the lebesgue measure of the common part of } A \text{ and the } x\text{-axis}$, the corresponding family \mathcal{C}_2 of all closed balls centered at the origin is also an admissible family (according to Definition 3). But the origin $(0, 0)$ of \mathbb{R}^2 although being not a lebesgue point of the function $f = \chi_{E \setminus (0,0)}$ (where E is the set constructed in Theorem 14.4, pg. 54, [9]) with respect to \mathcal{C}_1 is a Lebesgue point of f with respect to \mathcal{C}_2 .

We have already shown in [2] that if apart from its defining condition given by Definition 3, the class \mathcal{C} also satisfies the following condition

(*) For every set $A \subseteq G$ of σ -finite measure, there exists a sequence $\{g_n S_n\}_{n=1}^\infty$ ($g_n \in G, S_n \in \mathcal{C}$) such that the sets $g_n S_n$ are mutually disjoint and $\mu\left(A \setminus \bigcup_{n=1}^\infty g_n S_n\right) = 0$, then for any $f \in H$, μ -almost every point of G is a \mathcal{C} -point of f , in symbols $\mu(G \setminus \mathcal{C}(f)) = 0$.

It will follow from Theorem 1 that in opposition to the measure-theoretic largeness of $\mathcal{C}(f)$, the class $\mathcal{C}(x)$ is topologically small in the topology of $L^1(G)$.

This answers the second question.

2. THEOREMS ETC.

The class \mathcal{C} once chosen is kept fixed throughout. As we do not want to complicate our symbols unnecessarily, so in what follows no reference to the symbol \mathcal{C} is given in introducing any of our notations except in places where we have written $\mathcal{C}(f)$ and $\mathcal{C}(x)$. We also assume in both Theorem 1 and 2 that our topological group G is having \mathcal{C} as a class of compact sets satisfying conditions given by Definition 3.

Theorem 1. *For each $x \in G$, the class $\mathcal{C}(x)$ is meager in the topology of $L^1(G)$.*

Before we start proving Theorem 1 (which is one of the two main results of this paper), we introduce the following set of functions and state and prove a set of propositions.

$$\bar{\mathbb{H}}(\cdot, x) : L^1(G) \rightarrow (-\infty, \infty] \text{ given by } \bar{\mathbb{H}}(f, x) = \limsup_{gS \rightarrow x} \frac{\int_{gS} f d\mu}{\mu(S)}$$

and

$$\mathbb{H}(\cdot, x) : L^1(G) \rightarrow [-\infty, \infty) \text{ given by } \mathbb{H}(f, x) = \liminf_{gS \rightarrow x} \frac{\int_{gS} f d\mu}{\mu(S)}$$

where $\limsup_{gS \rightarrow x} \frac{\int_{gS} f d\mu}{\mu(S)}$ (resp. $\liminf_{gS \rightarrow x} \frac{\int_{gS} f d\mu}{\mu(S)}$) expresses the quantity $\sup \left\{ \limsup_{n \rightarrow \infty} \frac{\int_{g_n S_n} f d\mu}{\mu(S_n)} \right\}$ (resp. $\inf \left\{ \liminf_{n \rightarrow \infty} \frac{\int_{g_n S_n} f d\mu}{\mu(S_n)} \right\}$), the supremum (resp. infimum) being taken over all sequences $\{g_n S_n\}_{n=1}^{\infty}$ ($g_n \in G$, $S_n \in \mathcal{C}$) such that $x \in g_n S_n$ and $\lim_{n \rightarrow \infty} \mu(S_n) = 0$.

We also denote their common value (in case when it exists) by $\mathbb{H}(f, x)$ which is obviously finite.

Moreover, for those $n \in \mathbb{N}$, for which there exists $S \in \mathcal{C}$ such that

$\frac{1}{n+1} < \mu(S) \leq \frac{1}{n}$, let $\overline{\mathbb{H}}^{(n)}(\cdot, x) : L^1(G) \rightarrow \mathbb{R}$ be given by

$$\overline{\mathbb{H}}^{(n)}(f, x) = \sup \left\{ \frac{\int_{gS} f d\mu}{\mu(S)} : x \in gS \text{ where } g \in G, S \in \mathcal{C} \text{ and } \frac{1}{n+1} < \mu(S) \leq \frac{1}{n} \right\}.$$

(Note that for any $f \in L^1(G)$, $|\overline{\mathbb{H}}^{(n)}(f, x)| \leq (n+1) \int |f| d\mu$)

Now fix a sequence $\{S_n^{(0)}\}_{n=1}^{\infty} \subseteq \mathcal{C}$ such that $\lim_{n \rightarrow \infty} S_n^{(0)} = 0$, and define a function $\mathbb{H}^{(n)}(\cdot, x) : L^1(G) \rightarrow \mathbb{R}$ by the formula $\mathbb{H}^{(n)}(f, x) = \frac{\int_{x S_n^{(0)}} f d\mu}{\mu(S_n^{(0)})}$.

Along with this, let us also define the following classes

$$\mathbb{G}(x) = \{f \in L^1(G) : \overline{\mathbb{H}}(f, x) + \overline{\mathbb{H}}(1-f, x) = 1\}$$

$$\mathbb{A}(x) = \{f \in L^1(G) : \mathbb{H}(f, x) < \infty\}$$

$$\mathbb{A}_\gamma(x) = \{f \in \mathbb{A}(x) : \mathbb{H}(f, x) = \gamma\}.$$

It may be observed that $\mathbb{A}_\gamma(x) \subseteq \mathbb{A}(x) \subseteq \mathbb{G}(x)$. Likewise, upon interchanging the roles of f and x , let

$$\mathbb{G}(f) = \{x \in G : \overline{\mathbb{H}}(f, x) + \overline{\mathbb{H}}(1-f, x) = 1\},$$

$$\mathbb{A}(f) = \{x \in G : \mathbb{H}(f, x) \text{ exists}\}$$

$$\mathbb{A}_\gamma(f) = \{x \in \mathbb{A}(f) : \mathbb{H}(f, x) = \gamma\}.$$

Before we proceed to prove the following propositions, we may note that

$\overline{\mathbb{H}}(f, x) = \limsup_{n \rightarrow \infty} \overline{\mathbb{H}}^{(n)}(f, x)$ the proof of which is only a routine exercise.

Proposition 1. $\mathbb{A}_\gamma(x)$ is a G_δ subset of $\mathbb{A}(x)$.

Proof. We first claim that for any $n \in \mathbb{N}$ and $x \in G$, $\mathbb{H}^{(n)}(\cdot, x) \in C(L^1(G))$. This follows since for any $f, g \in L^1(G)$,

$$\left| \mathbb{H}^{(n)}(f, x) - \mathbb{H}^{(n)}(g, x) \right| = \left| \frac{\int_{xS_n^{(0)}} f d\mu}{\mu(S_n^{(0)})} - \frac{\int_{xS_n^{(0)}} g d\mu}{\mu(S_n^{(0)})} \right| \leq \frac{1}{\mu(S_n^{(0)})} \int_G |f - g| d\mu.$$

Also note that for $f \in \mathbb{A}(x)$, $\mathbb{H}(f, x) = \lim_{n \rightarrow \infty} \mathbb{H}^{(n)}(f, x)$. So the assertion follows from well-known facts on limits of sequences of continuous functions. \square

Proposition 2. For each $n \in \mathbb{N}$, $\overline{\mathbb{H}}^{(n)}(\cdot, x) \in C(L^1(G))$ and $\mathbb{A}(x) = \mathbb{G}(x)$.

Proof. We first claim that for each $n \in \mathbb{N}$, $\overline{\mathbb{H}}^{(n)}(\cdot, x) \in C(L^1(G))$. This follows since for any $f, g \in L^1(G)$

$$\begin{aligned} \left| \overline{\mathbb{H}}^{(n)}(f, x) - \overline{\mathbb{H}}^{(n)}(g, x) \right| &= \left| \sup \left\{ \frac{\int_{gS} f d\mu}{\mu(S)} : x \in gS \text{ and } \frac{1}{n+1} < \mu(S) \leq \frac{1}{n} \right\} \right. \\ &\quad \left. - \sup \left\{ \frac{\int_{gS} g d\mu}{\mu(S)} : x \in gS \text{ and } \frac{1}{n+1} < \mu(S) \leq \frac{1}{n} \right\} \right| \\ &\leq \sup \left\{ \frac{\int_{gS} |f-g| d\mu}{\mu(S)} : x \in gS \text{ and } \frac{1}{n+1} < \mu(S) \leq \frac{1}{n} \right\} \leq (n+1) \int_G |f - g| d\mu. \end{aligned}$$

Next note that the inclusion $\mathbb{A}(x) \subseteq \mathbb{G}(x)$ is obvious. Conversely, as for any $f \in L^1(G)$, the identity $\frac{\int_{gS} f d\mu}{\mu(S)} + \frac{\int_{gS} (1-f) d\mu}{\mu(S)} = 1$ implies the inequality $\overline{\mathbb{H}}(f, x) \geq 1 - \overline{\mathbb{H}}(1-f, x)$, so for those $f \in L^1(G)$ for which $\overline{\mathbb{H}}(f, x) + \overline{\mathbb{H}}(1-f, x) = 1$ is satisfied, $\overline{\mathbb{H}}(f, x) \geq \overline{\mathbb{H}}(f, x) \geq 1 - \overline{\mathbb{H}}(1-f, x) = \overline{\mathbb{H}}(f, x) + \overline{\mathbb{H}}(1-f, x) - \overline{\mathbb{H}}(1-f, x) = \overline{\mathbb{H}}(f, x)$. Therefore $f \in \mathbb{A}(x)$. Hence $\mathbb{G}(x) \subseteq \mathbb{A}(x)$. \square

Proposition 3. Both $\mathbb{A}_0(x)$ and $\mathbb{A}_1(x)$ are dense in $L^1(G)$.

Proof. Let $f \in L^1(G)$ and $\varepsilon > 0$ be given. Since the class

$$\left\{ \sum_{i=1}^n \lambda_i \chi_{A_i} : \lambda_i \in \mathbb{R}, A_i \text{ are } \mu\text{-measurable of finite } \mu \text{ measure, } n \in \mathbb{N} \right\}$$

of simple functions is dense in $L^1(G)$, there exist $m \in \mathbb{N}$, $\lambda'_i \in \mathbb{R}$ and μ -measurable sets A'_i for $i = 1, 2, \dots, m$ such that the simple function $\sum_{i=1}^m \lambda'_i \chi_{A'_i}$

belongs to the above class and $\int \left| f - \sum_{i=1}^m \lambda'_i \chi_{A'_i} \right| d\mu < \varepsilon/2$. We now choose

an open set V_1 such that $x \in V_1$ and

$$\mu(V_1) < \frac{\varepsilon/2 - \int \left| f - \sum_{i=1}^m \lambda'_i \chi_{A'_i} \right| d\mu}{\left(1 + \sum_{i=1}^m |\lambda'_i| \right)}.$$

The choice of the set V_1 is justified by virtue of the fact that μ being diffused, we can choose an open set containing e of sufficiently small measure.

Set $C'_i = A'_i \setminus V_1$, and define $g = \sum_{i=1}^m \lambda'_i \chi_{C'_i}$ and $h = \sum_{i=1}^m \lambda'_i \chi_{C'_i} + \chi_{V_1}$.

Clearly $\tau(f, g) < \varepsilon$, $\tau(f, h) < \varepsilon$, where τ denotes the usual metric induced by the standard norm on $L^1(G)$. It is also easy to note that $g \in \mathbb{A}_0(x)$, $h \in \mathbb{A}_1(x)$ which follows from Definition 3. \square

Proof of Theorem 1. We know that $L^1(G)$ is a topologically complete metrizable space and therefore is of second category by Baire's theorem. Now if possible, let $\mathbb{G}(x)$ be a set of second category in $L^1(G)$. Then $\mathbb{G}(x)$ is also a second category subspace of itself (by Th 1, pg 83, [7]). Since by Propositions 1 and 2 both $\mathbb{A}_0(x)$ and $\mathbb{A}_1(x)$ are G_δ subsets of $\mathbb{A}(x)$ and $\mathbb{A}(x) = \mathbb{G}(x)$, so both are G_δ subsets of $\mathbb{G}(x)$. Moreover, both these sets are dense in $L^1(G)$, so they are also dense in $\mathbb{G}(x)$. Consequently, both $\mathbb{G}(x) \setminus \mathbb{A}_0(x)$, $\mathbb{G}(x) \setminus \mathbb{A}_1(x)$ and so also their union are meager in $\mathbb{G}(x)$. Therefore some f exists such that $f \in \mathbb{A}_0(x) \cap \mathbb{A}_1(x)$ which is absurd.

Now as $\mathcal{C}(x) \subseteq \mathbb{G}(x)$, the class $\mathcal{C}(x)$ is meager in the topology of $L^1(G)$.

This proves Theorem 1. \square

Remark 1. Here we may note that not only $\mathcal{C}(x)$ is meager but it is also contained in an $F_{\sigma\delta}$ meager set. Indeed,

$$\begin{aligned} L^1(G) \setminus \mathbb{G}(x) &= \{f \in L^1(G) : \overline{\mathbb{H}}(f, x) > 1 - \overline{\mathbb{H}}(1 - f, x)\} = \\ &= \bigcup_{k=1}^{\infty} (\{f \in L^1(G) : \overline{\mathbb{H}}(f, x) > r_k\} \cap \{f \in L^1(G) : \overline{\mathbb{H}}(1 - f, x) > 1 - r'_k\}). \end{aligned}$$

Therefore

$$\begin{aligned} L^1(G) \setminus \mathbb{G}(x) &= \bigcup_{k=1}^{\infty} \left(\left[\bigcap_{n=1}^{\infty} \bigcup_{p=n}^{\infty} \{f \in L^1(G) : \overline{\mathbb{H}}^{(p)}(f, x) > r_k\} \right] \cap \right. \\ &\quad \left. \cap \left[\bigcap_{n=1}^{\infty} \bigcup_{p=n}^{\infty} \{f \in L^1(G) : \overline{\mathbb{H}}^{(p)}(1 - f, x) > 1 - r'_k\} \right] \right) \end{aligned}$$

where $\{(r_k, r'_k) : k \geq 1\}$ is the set of all pairs (p, q) of rationals such that $p > q$, which is $G_{\delta\sigma}$ (in $L^1(G)$) by virtue of Proposition 2, and consequently

$\mathbb{G}(x)$ is $F_{\sigma\delta}$ (in $L^1(G)$). But it is also meager as shown in the proof of the above theorem.

Our next Theorem 2, answers our first question.

Theorem 2. *If G is second countable, then the class of all functions (in $L^1(G)$) for which $\mathcal{C}(f)$ is meager in the topology (of G) is co-meager in the topology of $L^1(G)$.*

By (th. 3.3.1, [5]) locally compact spaces are completely regular (or Tychonoff or $3\frac{1}{2}$) and by (th. 4.2.9, [5]) second countable regular spaces are metrizable. Hence if a topological group is a second countable locally compact space then it is completely metrizable. But as assumed in the beginning, since the topological group we are considering in this article is equipped with a class \mathcal{C} of compact sets satisfying conditions given by Definition 3, it follows (by virtue of the deduction laid down in remark 3) that our topological group G is always metrizable, irrespective of whether it is second countable or not.

As in Theorem 1, here also, we state and prove a set of propositions and then finally give a proof of the Theorem.

The following result is the dual of the second part of Proposition 2, a proof of which may be given on the same lines as before.

Proposition 4. *For any $f \in L^1(G)$,*

$$\mathbb{A}(f) = \{x \in G : \overline{\mathbb{H}}(f, x) + \overline{\mathbb{H}}(1 - f, x) = 1\}.$$

Let $\pi = \{(x, f) \in G \times L^1(G) : \overline{\mathbb{H}}(f, x) + \overline{\mathbb{H}}(1 - f, x) = 1\}$. Then $\pi_{(x)} = \mathbb{G}(x)$ and $\pi^{(f)} = \mathbb{G}(f)$, are the two sections of π in $G \times L^1(G)$. We now show that π is a set with the Baire property by showing that it is Borel.

For any $n \in \mathbb{N}$, let us define two functions $\mathbb{F}^{(n)} : G \times L^1(G) \rightarrow \mathbb{R}$ and $\mathbb{J}^{(n)} : G \times L^1(G) \rightarrow \mathbb{R}$ by setting

$$\mathbb{F}^{(n)}(x, f) = \overline{\mathbb{H}}^{(n)}(f, x) \text{ and } \mathbb{J}^{(n)}(x, f) = \overline{\mathbb{H}}^{(n)}(1 - f, x).$$

It follows by our first claim (Proposition 2) that for each $x \in G$ both $\mathbb{F}^{(n)}(x, \cdot)$, $\mathbb{J}^{(n)}(x, \cdot) \in C(L^1(G))$.

Proposition 5. *For each $f \in L^1(G)$, both $\mathbb{F}^{(n)}(\cdot, f)$ and $\mathbb{J}^{(n)}(\cdot, f)$ are lower semi-continuous.*

Proof. We prove this fact for $\mathbb{F}^{(n)}(\cdot, f)$ only.

The case for $\mathbb{J}^{(n)}(\cdot, f) = \mathbb{F}^{(n)}(\cdot, 1 - f)$ will follow similarly.

Let $\epsilon > 0$ be given. Then by the absolute continuity of μ -integral there exists $\delta > 0$ such that $\int_E |f| d\mu < \epsilon/4(n + 1)$ whenever E is μ -measurable with $\mu(E) < \delta$.

Now let $x \in G$ be chosen arbitrarily. Then from the definition of $\mathbb{F}^{(n)}(x, f)$ it follows that there exists gS with $\frac{1}{n+1} < \mu(S) \leq \frac{1}{n}$ such that $x \in gS$ and

$$\mathbb{F}^{(n)}(x, f) - \frac{\epsilon}{2} < \frac{\int_{gS} f(z) d\mu(z)}{\mu(S)}.$$

Since gS is compact, by the regularity of μ it follows that an open set W exists such that $gS \subseteq W$ and $\mu(W \setminus gS) < \delta$. Let V be any symmetric open neighbourhood (i.e., $V = V^{-1}$) of the identity element e such that $VgS \subseteq W$. If $yx^{-1} \in V$, then $\mu(yx^{-1}gS \setminus gS) \leq \mu(VgS \setminus gS) \leq \mu(W \setminus gS) < \delta$. Also $\mu(gS \setminus yx^{-1}gS) \leq \mu(W \setminus yx^{-1}gS) = \mu(W) - \mu(gS) = \mu(W \setminus gS) < \delta$. Hence

$$\begin{aligned} & \left| \frac{\int_{yx^{-1}gS} f(z) d\mu}{\mu(S)} - \frac{\int_{gS} f(z) d\mu}{\mu(S)} \right| < \\ & < (n+1) \left\{ \int_{yx^{-1}gS \setminus gS} |f| d\mu + \int_{gS \setminus yx^{-1}gS} |f| d\mu \right\} < \epsilon/2 \end{aligned}$$

whenever $yx^{-1} \in V$ and therefore $\mathbb{F}^{(n)}(x, f) - \epsilon < \mathbb{F}^{(n)}(y, f)$ whenever $yx^{-1} \in V$. \square

The argument behind the following proposition runs similar to the argument for a Carathéodory function of two variables (see [1], pg 156, Th 20.15 and [7], pg 378, Th 2).

Proposition 6. *Both $\mathbb{F}^{(n)}$ and $\mathbb{J}^{(n)}$ are Borel measurable.*

Proof. As in Proposition 5, we prove this fact for $\mathbb{F}^{(n)}$ only. Since G is second countable, the space of all continuous real valued functions with compact support is separable. Again as this space is also dense in $L^1(G)$, it follows that $L^1(G)$ is also separable. Let $\{f_1, f_2, f_3, \dots, f_m, \dots\}$ be a countable dense subset of $L^1(G)$. Now for any $a \in \mathbb{R}$, $(x, f) \in (\mathbb{F}^{(n)})^{-1}[a, \infty)$ if and only if $\mathbb{F}^{(n)}(x, f) \in (a - \frac{1}{k}, \infty)$ for each $k \in \mathbb{N}$. But this is again equivalent to the assertion that for each $k \in \mathbb{N}$, there exists f_m with $f \in B_\tau(f_m, \frac{1}{m})$ such that $\mathbb{F}^{(n)}(x, f_m) \in (a - \frac{1}{k}, \infty)$ (since $\mathbb{F}^{(n)}(x, \cdot) \in C(L^1(G))$) by the first claim of Proposition 2) where $B_\tau(f_m, \frac{1}{m})$ is the open ball in $L^1(G)$ with center at f_m and radius $\frac{1}{m}$. Thus we may write

$$(\mathbb{F}^{(n)})^{-1}[a, \infty) = \bigcap_{k=1}^{\infty} \bigcup_{m=1}^{\infty} \left[\left\{ x \in G : \mathbb{F}^{(n)}(x, f_m) \in \left(a - \frac{1}{k}, \infty \right) \right\} \times B_\tau \left(f_m, \frac{1}{m} \right) \right]$$

Now $\mathbb{F}^{(n)}(\cdot, f_m)$ being lower semi-continuous and therefore Borel measurable, the Proposition follows. \square

Proof of Theorem 2. From Proposition 6, it follows that the functions $\mathbb{F} : G \times L^1(G) \rightarrow \mathbb{R}$ and $\mathbb{J} : G \times L^1(G) \rightarrow \mathbb{R}$ defined by $\mathbb{F}(x, f) =$

$\limsup_{n \rightarrow \infty} \mathbb{F}^{(n)}(x, f)$ and $\mathbb{J}(x, f) = \limsup_{n \rightarrow \infty} \mathbb{J}^{(n)}(x, f)$ are both Borel measurable, and so π is a Borel measurable subset of $G \times L^1(G)$ because from the identities $\mathbb{F}^{(n)}(x, f) = \overline{\mathbb{H}}^{(n)}(f, x)$, $\mathbb{J}^{(n)}(x, f) = \overline{\mathbb{H}}^{(n)}(1 - f, x)$ it follows that π can be expressed as $\pi = \{(x, f) : \mathbb{F}(x, f) + \mathbb{J}(x, f) = 1\}$. Hence it has the property of Baire.

We have already established (in course of proving Theorem 1) that for each $x \in G$ the set $\mathbb{G}(x)$ (or, equivalently, the section $\pi(x)$) is meager in the topology of $L^1(G)$. Hence (by Theorem 15.4, pg. 57, [9] which is a converse of the famous Kuratowski-Ulam theorem) it follows that π is meager.

But then (by the Kuratowski-Ulam theorem, pg. 56, [9]), the sections $\pi^{(f)}$ can be proved as meager (in G) for all $f \in L^1(G)$ except those which constitute a meager subset of $L^1(G)$. That we may apply the Kuratowski-Ulam theorem in the present situation is justified since $L^1(G)$ is separable.

This proves Theorem 2. \square

Remark 2. *The defining condition (Definition 3) for the class \mathcal{C} makes the collection $\{gS : g \in G, S \in \mathcal{C}\}$ an “indefinitely fine system” of sets in any locally compact Hausdorff topological group. Further, under the effect of condition (*), this indefinitely fine system also turns out to be a “Vitali system” of sets.*

Remark 3. *We now prove that under the condition given by Definition 3, our topological group G becomes first countable and hence metrizable.*

Assume that a locally compact group G admits a collection \mathcal{C} given by Definition 3.

Let $\{S_n\}_{n=1}^{\infty}$ be any sequence of members of \mathcal{C} such that $\lim_{n \rightarrow \infty} \mu(S_n) = 0$. For each n , let U_n be the interior of $S_n S_n^{-1}$.

Claim 1. *U_n is a neighbourhood of the natural element e of G .*

Proof. For simplicity, put $S = S_n$. By Theorem XII. 61A of [6], the function $g : G \rightarrow \mathbb{R}$ given by $g(x) = \mu(xS \setminus S) + \mu(S \setminus xS)$ is continuous. So, the set $V = \{x \in G : g(x) < \mu(S)\}$ is open in G . Notice that $e \in V$. Moreover, if $y \in V$, then $\mu(S \setminus xS) < \mu(S)$ which implies that $S \cap xS \neq \emptyset$. Consequently, $x \in SS^{-1}$. This shows that $V \subset SS^{-1}$ and we are done. \square

Claim 2. *There exists a sequence $\{g_n\}_{n=1}^{\infty}$ ($g_n \in G$) such that for every open neighbourhood W of e there is k such that $W_k \subseteq W$, where $W_k = g_k U_k g_k^{-1}$.*

Proof. Let V be an open neighbourhood of e for which $VV^{-1} \subseteq W$. We infer from Definition 3 and the non-discreteness of G that there exists a sequence $\{g_n S_n\}_{n=1}^{\infty}$ ($g_n \in G, S_n \in \mathcal{C}$) such that $e \in g_n S_n$ for each n and

$\lim_{n \rightarrow \infty} \mu(S_n) = 0$ (see the part within bracket at the end of Definition 3). Moreover, from the same Definition it follows that there exists k such that $g_k S_k \subset V$. But then $g_k U_k g_k^{-1} \subseteq (g_k S_k)(g_k S_k)^{-1} \subset VV^{-1} \subseteq W$. Now setting $W_k = g_k U_k g_k^{-1}$ finishes the proof. \square

Theorem 1 and 2, which are the two main results of the paper are related to the notion of \mathcal{C} -points. To define \mathcal{C} -points and also to prove Theorem 1 (upon which the proof of Theorem 2 also depends heavily), we find from above that our topological group G should necessarily be first countable and therefore metrizable. In these circumstances, the condition of second countability (used in Theorem 2) becomes equivalent to the σ -compactness of G .

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