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MULTIVALUED MARKOV PROCESSES

Abstract. Multivalued random variables and stochastic processes can be use in integral geometry, mathematical economics or stochastic optimization. Using the methods of selection operators we can give the selection characterization of identically distributed multivalued random variables. In this paper the regular selections and Markov selections for multivalued stochastic processes will be studied.

Key words: mutivalued random variable, mutivalued stochastic processes.

1. INTRODUCTION

We present a concept of selection operators for multivalued random variables. For multivalued stochastic processes the some clue problem is the question of existing the vector-valued selection processes. In this paper we continue our work on properties of multivalued random variables and stochastic process (Trzpiot 1999, 2000, 2004). First sections contain basic definition, next we remain characterizations of identically distributed multivalued random variables and the selection problem of multivalued random variables converging in distribution. Finally, we study the selection problem for multivalued Markov processes.

2. MULTIVALUED RANDOM VARIABLE

Given a probability measure space (Ω, A, μ) random variable in classical definition is a mapping from Ω to **R**. Multivalued random variable is a mapping from Ω to all closed subset of X.

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We have a real Banach space X with metric d. For any nonempty and closed sets $A, B \subset X$ we define the Hausdorff distance $h(A, B)$ of A and B.

Definition 1. The excess for two nonempty and closed sets be defined by

$$e(A, B) = \sup_{x \in A} d(x, B), \quad \text{where} \quad d(x, B) = \inf_{y \in B} \|x - y\|,$$

the Hausdorff distance of A and B is given by

$$h(A, B) = \max\{e(A, B), e(B, A)\},$$

the norm $\| A \|$ of set A we get as

$$\| A \| = h(A, \{0\}) = \sup_{x \in A} \| x \|.$$

The set of all nonempty and closed subsets of X is a metric space with the Hausdorff distance. The set of all nonempty and compact subsets of X is a complete, separable metric space with the metric h.

Definition 2. A **multivalued function** $\varphi : \Omega \to 2^X$ with nonempty and closed values, is said to be (weakly) **measurable**, if φ satisfies the following equivalent conditions:

a) $\varphi^{-1}(C) = \{\omega \in \Omega : \varphi(\omega) \cap G \neq \emptyset\} \in A$ for every G open subset of X,

b) $d(x, \varphi(\omega))$ is measurable in ω for every $x \in X$,

c) there exists a sequence $\{f_n\}$ of measurable functions $f_n : \Omega \to X$ such that

$$\varphi(\omega) = \mathrm{cl}\{f_n(\omega)\} \text{ for all } \omega \in \Omega.$$

Definition 3. A measurable multivalued function $\varphi : \Omega \to 2^X$ with nonempty and closed values is called a **multivalued random variable**.

A multivalued function φ is called strongly measurable, if there exist a sequence $\{\varphi_n\}$ of simple functions (measurable functions having a finite number of values in 2^X), such that $h(\varphi_n(\omega), \varphi(\omega)) \to 0$ a.e.

Since set of all nonempty and compact (or convex and compact) subsets of X is a complete separable metric space with the metric h, so multifunction $\varphi : \Omega \to 2^X$ is measurable, if and only if is strongly measurable. This is equivalent to the Borel measurability of φ.

Let $K(X)$[1] denote all nonempty and closed subsets of X. As the σ-field on $K(X)$, we get the σ-field generated by $\varphi^{-1}(G) = \{\omega \in \Omega : \varphi(\omega) \cap G \neq \emptyset\}$,

[1] $KC(X)$ – denote all nonempty, closed and convex subsets of X.

for every open subset G of X. The smallest σ-algebra containing these $\varphi^{-1}(G)$ we de noted by $A\varphi$

1. Two multifunctions φ and ψ are independent, if $A\varphi$ and $A\psi$ are independent.

2. Two multifunctions φ and ψ are identically distributed, if $\mu(\varphi^{-1}(C)) = \mu(\psi^{-1}(C))$ for all closed $C \subset X$.

Definition 4. We say that a sequence of multivalued random variables $\varphi_n : \Omega \to 2^{K(X)}$ is independent, if so is $\{\varphi_n\}$ considered as measurable functions from (Ω, A, μ) to $(K(X), G)$.

Definition 5. Two multivalued random variables $\varphi, \psi : \Omega \to 2^{K(X)}$ are identically distributed, if $\varphi(\omega) = \psi(\Omega)$ a.e.

Particularly for φ_n with compact values independence (identical distributedness) of $\{\varphi_n\}$ coincides with that considered as Borel measurable functions to all nonempty, compact subsets of X.

Definition 6. A selection of the measurable multifunction $\varphi : \Omega \to 2^X$ is a measurable function $f : \Omega \to X$, such that $f(\omega) \in \varphi(\omega)$ for all $\omega \in \Omega$.

Let $L^p(\Omega, A)$, for $1 \leqslant p \leqslant \infty$, denote the X - valued, L^p - space. We introduce the multivalued L^p space.

Definition 7. The multivalued space $L^p[\Omega, K(X)]$, for $1 \leqslant p \leqslant \infty$ denote the space of all measurable multivalued functions $\varphi : \Omega \to 2^{K(X)}$, such that $\| \varphi \| = \| \varphi(\cdot) \|$ is in L^p.

Then $L^p[\Omega, K(X)]$ becomes a complete metric space with the metric H_p given by

$$H_p(\varphi, \psi) = \{\int_\Omega h(\varphi(\omega), \psi(\omega))^p \mathrm{d}\mu\}^{1/p} \quad \text{for} \quad 1 \leqslant p \leqslant \infty,$$

$$H_\infty(\varphi, \psi) = \operatorname*{esssup}_{\omega \in \Omega} h(\varphi(\omega), \psi(\omega)),$$

where φ and ψ are considered to be identical, if $\varphi(\omega) = \psi(\omega)$ a.e.

We can define similarly other L^p space for set of different subsets of X (convex and closed, weakly compact or compact). We denote by $L^p[\Omega\, K(X)]$ the space of all strongly measurable functions in $L^p[\Omega, K(X)]$. Then all this space is complete metric space with the metric H_p.

Definition 8. The mean $\mathrm{E}(\varphi)$, for a multivalued random variables $\varphi : \Omega \to 2^{K(X)}$ is given as the integral $\int_\Omega \varphi \mathrm{d}\mu$ of φ defined by

$$\mathrm{E}(\varphi) = \int_{\Omega} \varphi d\mu = \{\int_{\Omega} f \mathrm{d}\mu : f \in S(\varphi)\},$$

where

$$S(\varphi) = \{f \in L^1[\Omega, X] : f(\omega) \in \varphi(\omega) \text{ a.e.}\}.$$

The mean $\mathrm{E}(\varphi)$ exist, if $S(\varphi)$ is nonempty. Multifunction φ is an integrable, if $\|\varphi(\omega)\|$ is an integrable. If φ have an integral, then $\mathrm{E}(\varphi)$ is compact. If μ is atomless, then $\mathrm{E}(\varphi)$ is convex. If φ have an integral[2] and $\mathrm{E}(\varphi)$ is nonempty, then $\mathrm{coE}(\varphi) = \mathrm{E}(\mathrm{co}\varphi)$ (co – denote convex hull of the set). Now we present some properties of mean of multivalued random variables.

Let $\varphi, \psi : \Omega \to 2^{K(X)}$ be two multivalued random variables with nonempty $S(\varphi)$ and $S(\psi)$, then:

1) $\mathrm{cl}\ \mathrm{E}(\varphi \cup \psi) = \mathrm{cl}(\mathrm{E}(\varphi) + \mathrm{E}(\psi))$, where $(\varphi \cup \psi)(\omega) = \mathrm{cl}(\varphi(\omega) + \psi(\omega))$,
2) $\mathrm{cl}\ \mathrm{E}(\overline{\mathrm{co}}\, \varphi) = \overline{\mathrm{co}}\, \mathrm{E}(\varphi)$, where $(\overline{\mathrm{co}}\, \varphi)(\omega) = \overline{\mathrm{co}}\, \varphi(\omega)$, the closed convex hull,
3) $h(cl\,\mathrm{E}(\varphi), \mathrm{cl}\,\mathrm{E}(\psi)) = H_1(\varphi, \psi)$.

Lemat 1 (Auman 1965) Let $\varphi : \Omega \to 2^{K(X)}$ and $1 \leqslant p \leqslant \infty$. If

$$S^p(\varphi) = \{f \in L^p[\Omega, X] : f(\omega) \in \varphi(\omega) \text{ a.e.}\}.$$

then exists a sequence $\{f_n\}$ contained in $S^p(\varphi)$ such that $\varphi(\omega) = \mathrm{cl}\{f_n(\omega)\}$ for all $\omega \in \Omega$.

Lemat 2 (Auman 1965). Let φ, $\psi : \Omega \to 2^{K(X)}$ and $1 \leqslant p \leqslant \infty$. If $S^p(\psi) = S^p(\psi) \neq 0$ then $\varphi(\psi) = \psi(\omega)$ a.e.

3. MULTIVALUED STOCHASTIC PROCESS

Let T denote the set of positive integers or nonnegative real numbers.

Definition 9. Multivalued stochastic process is a family of multivalued random variables indexed by $T\{\varphi_t, t \in T\}$.

Supposing that **P** are the certain properties of stochastic processes.

Definition 10. A vector valued stochastic process $\{f_t, t \in T\}$ will be called a **P** selection of $\{\varphi_n, n \geqslant 1\}$, if $\{f_t, t \in T\}$ has the properties **P** and $f_t \in \varphi_t$, a.e. for each $t \in T$.

Let $\{A_t, t \in T\}$ be an increasing family of sub-σ-algebras of A.

A multivalued stochastic process $\{\varphi_t, t \in T\}$ is said to be integrable, if for each $t \in T$ is integrable bounded (respectively, A_t measurable).

[2] The multivalued integral was introduced by R. J. Auman (1965). For detailed arguments concerning the measurability and integration of multifuction we refer to (Castaing, Valadier 1977; Debreu 1967; Rockefellar 1976).

Definition 11. Let X be a separable Banach space. The map $\Gamma: K(X) \to X$ is called a **selection operator**, if $\Gamma(A) \in A$, for all $A \in K(X)$.

1) Γ is called a continuous selection operator (or measurable operator), if Γ is continuous with respect to topology on $K(X)$ generated by the subbase $\{A \in K(X),\ a < d(x, A) < b\}$ $(a, b \in \mathbf{R}, x \in X\}$. Denote Borel σ-algebra of this topology by $\mathscr{B}$. This is separable and completely mertizable topology space $(K(X), W)$.

2) Γ is called a linear selection operator, if for any A, $B \in K(X)$

$$\Gamma(\alpha_1 A + \alpha_2 B) = \alpha_1 \Gamma(A) + \alpha_2 \Gamma(B),$$

3) Γ is called a Lipschitz selection operator, if there exists a constant $k > 0$ such that for any A, $B \in K(X)$

$$\|\Gamma(A) - \Gamma(B)\| \leqslant kd(A, B).$$

Theorem 1 (Trzpiot 2004). Let X be a separable Banach space. Then there exists a sequence of measurable selection operators $\{\Gamma_n,\ n \geqslant 1\}$ such that for each $A \in K(X)$

$$A = \mathrm{cl}\{\Gamma_n(A),\ n \geqslant 1\}.$$

G. Salinetti and R. Wets (1979) studied the distribution theory of multivalued random variables in finite-dimensional Banach spaces and they proved that:

- Multivalued random variables φ_1 and φ_2 are identically distributed, if and only if the real-valued stochastic process $\{d(x, \varphi_1),\ x \in X\}$ and $\{d(x, \varphi_2),\ x \in X\}$ have the same finite dimensional distribution.
- If a sequence of multivalued random variables $\{\varphi_n,\ n \geqslant 1\}$ converges in distribution to φ, then there exist selections $\{f_n,\ n \geqslant 1\}$ of $\{\varphi_n,\ n \geqslant 1\}$ such that $\{\|f_n\| n \geqslant 1\}$ converges in distribution to $\|f\|$, where f is a vector valued random variables with $f \in \varphi$ a.e.

Theorem 2 (Trzpiot 2004). Let X be a finite-dimensional Banach space, and let φ_1 and φ_2 be two multivalued random variables. Then the following are equivalent:

1) φ_1 and φ_2 are identically distributed,

2) there exist selection sequences $\{f_n^1,\ n \geqslant 1\}$ and $\{f_n^2,\ n \geqslant 1\}$ of φ_1 and φ_2, such that $\varphi_i(\tilde{\omega}) = \mathrm{cl}\{f_n^i(\tilde{\omega}),\ n \geqslant 1\}$, $i = 1, 2$,

3) the real-valued stochastic process $\{d(x, \varphi_1),\ x \in X\}$, and $\{d(x, \varphi_2),\ x \in X\}$ have the same finite dimensional distribution.

Given a complete probability space (Ω, A, μ) and increasing family of sub-σ-algebra of A: $\{A_t,\ t \in \mathbf{R}^+\}$. A multivalued stochastic process $\{\varphi_t,\ t \in T\}$ is said to be regular and right-continuous with respect to topology space $(K(X), W)$, if it is adapted and for each $\omega \in \Omega$, has a left-hand limit and is right-continuous with respect to topology space $(K(X), W)$ for every $t \in \mathbf{R}^+$.

Theorem 3 (Trzpiot 2004). Let X be a separable Banach space and let $\{\varphi_t,\ t \in \mathbf{R}^+\} \subset L^p[\Omega, K_c(X)]$ be a regular and right-continuous with respect to topology space $(K(X), W)$. Then $\{\varphi_t,\ t \in \mathbf{R}^+\}$ has a regular and right--continuous selection.

Based on Theorem 3 (Trzpiot 2004) we can write the following theorem, which is a multivalued version of Theorem 2 (Trzpiot 2004).

4. MULTIVALUED MARKOV PROCESSES

Now we will study the selection problem for Markov process. As in Definition 11, the topology on $K(X)$ is generated by the subbase $\{A \in K(X),\ a < d(x, A) < b\}$ $(a, b \in \mathbf{R}, x \in X)\}$. Denote Borel σ-algebra of this topology by B. This is separable and completely mertizable topology space $(K(X), W)$. So we start from Definition 12.

Definition 12. Multivalued stochastic process is called a Markov process, if it is a Markov process with the measurable space $(K(X), W)$ being its state-space.

Denote by D an index set and by $\sigma\{\varphi_d,\ d \in D\}$ the σ-algebra generated by the family of multivalued random variables. Denote by (Z^N, Y^N) the countable product space of measurable space (Z, Y).

Theorem 4. Let X be a separable Banach space and let $\{\varphi_t,\ t \in \mathbf{R}^+\}$ be a multivalued Markov process and $A_t = \sigma\{\varphi_s,\ s \leqslant t\}$. Then exist the family of vector valued stochastic process $\{f_t^n,\ t \in \mathbf{R}^+,\ n \geqslant 1\}$ such that

$$\varphi_t(\omega) = \mathrm{cl}\{f_t^n(\omega),\ t \in \mathbf{R}^+,\ n \geqslant 1\}.$$

Additionally if we write $x_t(\omega) = (f_t^1, f_t^2, \ldots, f_t^n, \ldots, \ldots)$, then $\{x_t,\ t \in \mathbf{R}^+\}$ is a Markov process with the state space $(X^N, \mathscr{B}^N)$.

Proof. Let $\{\Gamma_n,\ n \geqslant 1\}$ be a sequence of measurable selection operators on $K(X)$ (Theorem 1) and define for each $t \in \mathbf{R}^+$, $n \geqslant 1$ a sequence

$f_t^n(\omega) = \Gamma_n(\varphi_t(\omega))$. So we have a family of adapted process which satisfying the first thesis.

To proof that $\{x_t,\ t \in \mathbf{R}^+,\ n \geqslant 1\}$ is a Markov process it is sufficient to show that for each bounded measurable function $h: (X^N, \mathscr{B}^N) \to \mathbf{R}$, one has

$$E[h(x_t)|A_s] = E[h(x_t)|\sigma(x_s)],\ t > s,\ t,\ s \in \mathbf{R}^+,$$

where $E[h\,|\,A]$ denotes the conditional expectation with respect to A.

Define $H: (K(X), W) \to \mathbf{R}$ as follows

$$H(A) = h(\Gamma_1(A), \Gamma_2(A), \ldots, \Gamma_n(c), \ldots),\ A \in K(X).$$

Then $H(\cdot)$ is a bounded measurable function on $K(X)$.
From the Markov property of $\{\varphi_t,\ t \in \mathbf{R}^+\}$ follows that

$$E[H(\varphi_t)\,|\,A_s] = E[H(\varphi_t)\,|\,\sigma(\varphi_s)],\ \text{a.e. } t > s,\ t,\ s \in \mathbf{R}^+,$$

that means

$$E[h(x_t)\,|\,A_s] = E[h(x_t)\,|\,\sigma(\varphi_s)],\ \text{a.e. } t > s,\ t,\ s \in \mathbf{R}^+,$$

and because $\varphi_t(\omega) = \mathrm{cl}\{f_t^n(\omega),\ t \in \mathbf{R}^+,\ n \geqslant 1\}$ we have

$$\sigma(\varphi_s) = \sigma(f_s^n,\ n \geqslant 1) = \sigma(x_s\ \mathrm{s}).$$

According Theorem 2 the statistical law of φ_t is completely determinated by that of $\{f_t^n,\ t \in \mathbf{R}^+,\ n \geqslant 1\}$, so we can call the Theorem 4 the discretization theorem for multivalued Markov process.

If the multivalued Markov process takes the values in a finite subset Z of positive integer, then the process $\{x_t,\ t \in \mathbf{R}^+,\ n \geqslant 1\}$ presented on Theorem 4 may be regarded as a model of interacting particle systems.

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WIELOWARTOŚCIOWE PROCESY MARKOWA

Wielowartościowe zmienne losowe i wielowartościowe procesy stochastyczne znajdują zastosowanie w geometrii różniczkowej, w matematycznej ekonomii oraz w zadaniach stochastycznej optymalizacji. Wykorzystując operatory selekcyjne możliwa jest charakterystyka ciągu wielowartościowych zmiennych losowych o takim samym rozkładzie. Przedmiotem badań zaprezentowanych w artykule są selektory wielowartościowych procesów Markowa.