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## APPLICATION OF THE WOLF'S ALGORITHM IN CONSTRUCTING EFFECTIVE PORTFOLIOS

**Abstract.** The aim of this paper is to present a method of constructing effective portfolios with application of Wolf's algorithm. The effective portfolios are understood as those which have the lowest risk at give rate of return, and , conversely, which have the highest returns at given risk level. In classic Markowitz model, the rate of return is understood as expected returns which in practice is replaced by a mean return. The variance of the portfolio returns is considered as the risk measure.

Ready-made programs may be used to construct effective portfolios. In practice, using these application causes some problems. In order to calculate the portfolios efficiently, we have written an application in the Delphi programming language using a suitably adapted Wolf's algorithm.

**Key words:** quadratic programming, effective portfolio, risk.

The participants of business processes act in the risk environment. This particularly holds true for the stock exchange investments. In order to reduce risk, the stock exchange investors construct effective investment portfolios; that is they invest monetary resources in stock of many companies quoted at stock exchange thus seeking a portfolio with the lowest risk at given returns or with the highest returns at given risk level. The rate of return is understood here as expected returns, which in practice are replaced by mean returns. In classic Markowitz model, the variance of the portfolio returns is assumed as the risk measure (Markowitz 1952).

General quadratic programming applications (QSB, Operational Research) or a specialized Portfolio program (Elton and Gruber, 1997) may be used for selection of an effective portfolio. Using these application causes some problems. The Portfolio program has limitations in relation to the number of considered companies and observations; moreover it cannot perform the task of returns calculation. Available applications for quadratic or non-linear programming require time-consuming entering of data and

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pre-determination of mean returns and elements of variance-covariance matrix. The problem of constructing effective portfolios has been studied by many scholars (Trzaskalik and Jurek 1995; Kolupa and Plebaniak, 2000) but the absence of an adequate program has made it practically impossible to select effective portfolios from the set of all companies quoted on Warsaw Stock Exchange. In order to calculate the portfolios efficiently (quickly and for any number of companies and observations), we have written an application in the Delphi programming language using a suitably adapted Wolf's algorithm.

The aim of this paper is to present a method of constructing effective portfolios with application of Wolf's algorithm. Classic Markowitz model is based on the following assumptions:

- the portfolio includes  $k$  shares,  $A_1, A_2, \dots, A_k$ ;
- the proportion of shares in the portfolio is varied, which is described by the vector:  $X = (x_1, x_2, \dots, x_k)^T$  with non-negative components ( $x_i \geq 0$ ) normalized to one:

$$\sum_{i=1}^k x_i = 1 \quad (1)$$

in the matrix notation:

$$X^T I_k = 1 \quad (2)$$

where  $I_k = (1, 1, \dots, 1)$  is the  $k$ -dimensional vector with components equal to one;

- each share  $A_j$ ,  $i = 1, 2, \dots, k$  in a specified period of time has  $m$  time units, in which the returns  $z_{it}$  are recorded, where  $t = 1, 2, \dots, m$ .

Mean returns of given shares are calculated on the basis of the following formula:

$$\bar{z}_i = \frac{1}{m} \sum_{t=1}^m z_{it}, \quad i = 1, \dots, k \quad (3)$$

The mean return from the portfolio with  $k$  shares is expressed by the weighted mean:

$$\bar{z}_p = \sum_{i=1}^k x_i \bar{z}_i = X^T \bar{Z} \quad (4)$$

where  $\bar{Z} = (\bar{z}_1, \dots, \bar{z}_k)^T$  is the vector of mean returns.

Thus,  $\bar{z}_p$  depends both on mean returns from given shares and on their proportion in the portfolio. Considering the columns of the  $Z$  matrix with  $z_{it}$  elements, we obtain information on the returns from the shares in question in given moments in time. The proportion of the shares is varied, which is expressed by the vector  $X$ , so it is possible to determine the following scalar products:

$$z_{pt} = X^T Z_t = \sum_{i=1}^k x_i z_{it}, \quad t = 1, \dots, m \quad (5)$$

where  $Z_t = (z_{1t}, \dots, z_{kt})^T$  is the  $t$ -th column of the  $Z$  matrix.

The returns from the portfolio  $z_{pt}$  may be compared to the mean returns  $\bar{z}_p$ . It is assumed that the variance of the returns from the portfolio is the measure of the deviation of  $z_{pt}$  from  $\bar{z}_p$ :

$$s_p^2 = \frac{1}{m-1} \sum_{t=1}^m (z_{pt} - \bar{z}_p)^2 = \frac{1}{m-1} \sum_{t=1}^m X^T (Z_t - \bar{Z}) (Z_t - \bar{Z})^T X = X^T C X \quad (6)$$

where:

$$C = \frac{1}{m-1} \sum_{t=1}^m (Z_t - \bar{Z})(Z_t - \bar{Z})^T \quad (7)$$

is  $(k \times k)$ -dimensional variance-covariance matrix from the sample. The mentioned risk measure  $s_p^2$  for the portfolio may be expressed as follows:

$$s_p^2 = \sum_{i=1}^k x_i^2 c_i^2 + 2 \sum_{1 \leq i < j \leq k} x_i x_j c_{ij} \quad (8)$$

where:

$$c_i^2 = \frac{1}{m-1} \sum_{t=1}^m (z_{it} - \bar{z}_i)^2 \quad (9)$$

is the variance of the returns for the  $i$ -th and  $j$ -th share, whereas:

$$c_{ij} = \frac{1}{m-1} \sum_{t=1}^m (z_{it} - \bar{z}_i)(z_{jt} - \bar{z}_j) \quad (10)$$

is the covariance of the return for the  $i$ -th and  $j$ -th share.

Minimizing the expression:

$$X^T C X \rightarrow \min \quad (11)$$

with the following constraints:

$$Z^T X \geq \gamma \quad (12)$$

$$\sum_{i=1}^k x_i = 1 \quad (13)$$

$$x_i \geq 0, \quad i = 1, \dots, k,$$

where:

$\gamma$  – is the pre-defined return from the whole portfolio, assuming  $\gamma \leq \max \bar{z}_i$ .

In order to use the Wolf's method of quadratic programming (Grabowski 1982), we need to transform the expression (11) and constraints (12), (13) to the following form:

$$\begin{aligned} -PX + X^T C G &\rightarrow \min \\ AX &\leq B \\ X &\geq 0 \end{aligned} \quad (14)$$

where  $P$  is a vector, and  $G$  is a non-negative square matrix.

$X$  is renumbered so that  $x_i$  relating to the company with the highest variance is the last.

$x_k$  – the proportion of shares with highest variance from (13) we receive:

$$x_k = 1 - \sum_{i=1}^{k-1} x_i \quad (15)$$

$x_k$  is substituted to (11) and we receive:

$$-QX + X^T D X \rightarrow \min \quad (16)$$

where:

$D$  –  $[k-1, k-1]$ -dimensional matrix with  $d_{ij}$  elements:

$$d_{ij} = c_{ij} + c_{kk} - c_{kj} - c_{ik} \quad (17)$$

$Q$  –  $(k-1)$ -dimensional vector with the elements:

$$q_i = -2c_{kk} + c_{kj} + c_{ik} \quad (18)$$

Two attributes result from the above-presented formulas:

- elements on the main diagonal of the  $D$  matrix are non-negative.

$$d_{ii} = \sum_{t=1}^m (z_{it} - \bar{z}_i)^2 + \sum_{t=1}^m (z_{jt} - \bar{z}_j)^2 - 2 \sum_{t=1}^m (z_{it} - \bar{z}_i)(z_{jt} - \bar{z}_j) \quad (19)$$

$(a^2 + b^2 - 2ab)$  is non-negative, so elements on the main diagonal the  $D$  matrix are non-negative, too.

- because  $c_{kk}$  is the largest element of the  $C$  matrix, from (18) it is concluded that the elements of the  $D$  vector are negative.

From (12) and (15) we receive:

$$\sum_{i=1}^{k-1} \bar{z}_i x_i + \bar{z}_k \left( 1 - \sum_{i=1}^{k-1} x_i \right) \geq \gamma \quad (20)$$

$$\sum_{i=1}^{k-1} (\bar{z}_i + \bar{z}_k) x_i + \bar{z}_k \geq \gamma \quad (21)$$

$$\sum_{i=1}^{k-1} (\bar{z}_k - \bar{z}_i) x_i \leq \bar{z}_k - \gamma \quad (22)$$

From (14) we receive:

$$1 - \sum_{i=1}^{k-1} x_i \geq 0 \quad (23)$$

$$\sum_{i=1}^{k-1} x_i \leq 1 \quad (24)$$

thus, the problem of selecting an effective portfolio will take the following form:

$$-Qx + x^T D x \rightarrow \min \quad (25)$$

and constraints

$$AX \leq B$$

$$X \geq 0$$

where

$$A = \begin{bmatrix} \bar{z}_1 - \gamma & \bar{z}_2 - \gamma & \cdots & \bar{z}_{k-1} - \gamma \\ 1 & 1 & \cdots & 1 \end{bmatrix} \quad (26)$$

$$B = \begin{bmatrix} \bar{z}_k - \gamma \\ 1 \end{bmatrix} \quad (27)$$

In the Wolf's method [II], introduced are the vector of additional variables  $X^d$  and  $Y^d$  plus the vector  $W$  of the artificial variables with  $w_i$  elements:

$$YX^d + Y^dX = 0$$

$$X^d = B - AX \quad (28)$$

$$Y^d = 2X^TD + YA - Q$$

In the above method, instead of solving the model in the form (25), the sum of artificial variables is minimized with the constraints (30–33):

$$\sum_{i=1}^{k-1} w_i \rightarrow \min \quad (29)$$

$$X^d + AX = B \quad (30)$$

$$2X^TD + YA - Y^d + W = Q \quad (31)$$

$$X \geq 0; X^d \geq 0; Y \geq 0; Y^d \geq 0; W \geq 0 \quad (32)$$

$$x_i y_j^d = 0; x_j^d y_j = 0; i = (1, \dots, k-1); j = (1, 2) \quad (33)$$

According to the Wolf's algorithm, we check the signs of free sets of equations  $X^d + AX = B$  and  $2X^T - D + YA - Y^d + W = Q$ :

- I) if  $b_j \geq 0$ , the  $j$ -th equation is left unchanged, treating  $x_j^d$  as the basis variable,
- II) if  $b_j < 0$ , both sides of the  $j$ -th equation is multiplied by  $-1$  and the artificial variable  $v_j$  (treated as the basis variable) is added to the left side of the equation.
- III) if  $q_i \leq 0$ , both sides of the  $j$ -th equation is multiplied by  $-1$  and the  $y_i^d$  is treated as the basis variable,
- IV) if  $q_i > 0$ , artificial variable  $w_i$  (treated as the basis variable) is added to the left side of the equation.

The starting table will take the following form:

	$x_1$	...	$x_{k-1}$	$y_1^d$	...	$y_{k-1}^d$	$w_1$	...	$w_{k-1}$	$y_1$	$y_2$	$x_1^d$	$x_2^d$	$H_0$	$f.c.$
$x_1^d$	$\bar{z}_1 - \gamma$	...	$\bar{z}_{k-1} - \gamma$	0	...	0	0	...	0	0	0	1	0	$\bar{z}_1 - \gamma$	0
$x_2^d$	1	...	1	0	...	0	0	...	0	0	0	0	1	1	0
$w_1$	$2d_{11}$	...		-1	0	0	1	0	0	$\bar{z}_1 - \gamma$	1	0	0	$p_1$	1
$\vdots$	$\vdots$	$\ddots$	$\vdots$	0	$\ddots$	0	0	$\ddots$	0	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$w_{k-1}$		...	$2d_{k-1, k-1}$	0	0	-1	0	0	1	$\bar{z}_{k-1} - \gamma$	1	0	0	$p_{k-1}$	1

If artificial variable  $v_1$  needs to be added, the starting table will be as follows:

	$x_1$	...	$x_{k-1}$	$y_1^d$	...	$y_{k-1}^d$	$w_1$	...	$w_{k-1}$	$y_1$	$y_2$	$x_1^d$	$x_2^d$	$v_1$	$H_0$	$f.c.$
$x_1^d$	$\bar{z}_1 + \gamma$	...	$\bar{z}_{k-1} + \gamma$	0	...	0	0	...	0	0	0	1	0	1	$\bar{z}_k - \gamma$	0
$x_2^d$	1	...	1	0	...	0	0	...	0	0	0	0	1	0	1	0
$w_1$	$2d_{11}$	...		-1	0	0	1	0	0	$\bar{z}_1 - \gamma$	1	0	0	0	$p_1$	1
$\vdots$	$\vdots$	$\ddots$	$\vdots$	0	$\ddots$	0	0	$\ddots$	0	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$w_{k-1}$		...	$2d$	0	0	-1	0	0	1	$\bar{z}_{k-1} - \gamma$	1	0	0	0	$p_{k-1}$	1

In our case:

- $q_i \leq 0$ , for each  $i = (1, \dots, k-1)$ , so (IV) will not occur,
- $b_2 = 1$ ,
- $b_1$  is positive if the company with the highest mean returns has also the highest return variance. Generally, this assumption is satisfied; it is not, we add the column representing the artificial variable  $v_1$  to the starting table.

In order to construct effective portfolios according to the above-described method, we have written an application in the Delphi language, which has made it possible to prepare a user-friendly interface, easily enter the data and read the results. The program requirements are as follows: a data file (quotations of the companies over a given period of time), specified set of companies (numbers), number of observations, length of the investment period (in days) and pre-defined expected returns. As a result, we receive the names of companies to form our portfolio, the proportion of their shares in the portfolio, variance and other risk measures: variance from the -defined returns, semivariance, and semivariance from pre-defined returns. It is possible to see and save the successive iterations as well as to choose the solution element. The program has an additional feature of constructing an effective portfolio for other risk measures, such as minimizing the semivariance from pre-defined returns.

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## ZASTOSOWANIE ALGORYTMU WOLFA DO WYZNACZANIA PORTFELA EFEKTYWNEGO

Celem artykułu jest przedstawienie metody uzyskiwania portfeli efektywnych przy zastosowaniu algorytmu Wolfa, czyli portfeli, które przy danej stopie zwrotu posiadałyby najniższe ryzyko, zaś dla danego poziomu ryzyka charakteryzowałyby się najwyższą stopą



zwrotu. W klasycznym modelu Markowitza przez stopę zwrotu, rozumie się oczekiwaną stopę zwrotu w praktyce zastępowaną średnią stopą zwrotu, za miarę ryzyka przyjmuje się wariancję stóp zwrotu z portfela.

W celu wyznaczenia portfela efektywnego można posłużyć się gotowymi programami. W praktyce wykorzystanie tych aplikacji stwarza pewne problemy. By sprawnie wyznaczać portfele efektywne napisaliśmy program w Delphi wykorzystujący odpowiednio dostosowany algorytm Wolfa.