ACTA UNIVERSITATIS LODZIENSIS
FOLIA MATHEMATICA 4, 1991

Jacek Hejduk

SOME REMARKS ON EGOROFF'S THEOREM

We define uniform convergence with respect to a small system $\{\mathcal{N}_n\}_{n\in N}$. Some connections with uniform convergence with respect to the σ-ideal $\bigcap_{n=1}^{\infty} \mathcal{N}_n$ are examined. There are also proved the equivalent conditions to uniform convergence with respect to the small system $\{\mathcal{N}_n\}_{n\in N}$ by assumption of upper semicontinuity of the small system $\{\mathcal{N}_n\}_{n\in N}$.

Let X denote a nonempty abstract set and $\mathcal{S}$ a σ-field of subsets of X.

DEFINITION 1. We shall say that a sequence $\{\mathcal{N}_n\}_{n\in N}$ of subfamilies of $\mathcal{S}$ is a small system on $\mathcal{S}$ if

(i) $\emptyset \in \mathcal{N}_n$ for each $n \in N$;

(ii) for any $n \in N$, there exists a sequence $\{k_i\}_{i\in N}$ of positive integers such that if $A_i \in \mathcal{N}_{k_i}$ for $i \in N$, then $\bigcup_{i=1}^{\infty} A_i \in \mathcal{N}_n$.

(iii) for any $n \in N$, $A \in \mathcal{N}_n$ and $B \in \mathcal{S}$ such that $B \subset A$, we have that $B \in \mathcal{N}_n$;

(iv) for any $n \in N$, $A \in \mathcal{N}_n$ and $B \in \bigcap_{m=1}^{\infty} \mathcal{N}_m$, we have $A \cup B \in \mathcal{N}_n$;

(v) $\mathcal{N}_{n+1} \subset \mathcal{N}_n$ for each $n \in N$.

It is not difficult to check (cf. [2]) that the family $\mathcal{N} = \bigcap_{n=1}^{\infty} \mathcal{N}_n$ forms a σ-ideal of $\mathcal{S}$-measurable sets, i.e. $\mathcal{N}$ is closed under countable unions and any $\mathcal{S}$-measurable subset of a set from $\mathcal{N}$ is a member of $\mathcal{N}$. Further, $\mathcal{N}$ will also denote the σ-ideal $\bigcap_{n=1}^{\infty} \mathcal{N}_n$.

DEFINITION 2. A small system $\{\mathcal{N}_n\}_{n\in N}$ is called upper semicontinuous if, for an arbitrary nonincreasing sequence $\{A_n\}_{n\in N}$ of $\mathcal{S}$-measurable sets such that there exists $m \in N$ such that $A_n \notin \mathcal{N}_m$ for any $n \in N$, we have $\bigcap_{n=1}^{\infty} A_n \notin N$.

In the sequel, we shall assume that $\{\mathcal{N}_n\}_{n\in N}$ fulfils (i) - (v). If it proves necessary, we shall additionally require that $\{\mathcal{N}_n\}_{n\in N}$ be upper semicontinuous. For a quite arbitrary σ-ideal $\mathcal{J}$ of subsets of $\mathcal{S}$-measurable set, we shall say that a property holds $\mathcal{J}$-almost everywhere (abbreviation $\mathcal{J}$-a.e.) if the set of points not having this property belongs to $\mathcal{J}$.

Let $M[\mathcal{S}, \mathcal{N}]$ denote the family of $\mathcal{N}$-a.e. finite $\mathcal{S}$-measurable real functions.

DEFINITION 3. We shall say that a sequence $\{f_n\} \subset M[\mathcal{S}, \mathcal{N}]$ converges $\mathcal{N}$-a.e. to a function $f \in M[\mathcal{S}, \mathcal{N}]$ if

$$\{x: \lim_{n\to\infty} f_n(x) \nrightarrow f(x)\} \in N.$$

It is obvious to consider this kind of convergence with respect to an arbitrary σ-ideal $\mathcal{J}$ of subsets of X.

For an arbitrary operator of convergence of a sequence of real functions defined on X, we can look for a possibly "large" set on which this sequence is uniformly convergent. Egoroff's theorem concerns this problem in the space (X, S, κ) with a finite measure κ over X and the convergence κ-a.e. There are some examples proving that it is sometimes impossible to find such a "large" set for the uniform convergence (cf. [4], [8]).

DEFINITION 4. We shall say that a sequence $\{f_k\}_{k\in N} \subset M[\mathcal{S}, \mathcal{N}]$ converges uniformly in the sense of Egoroff with respect of the small system $\{\mathcal{N}_n\}_{n\in N}$ to a function $f \in M[\mathcal{S}, \mathcal{N}]$ (abbreviation $(\mathcal{N}_n)$-uniformly convergent) if, for an arbitraty $n \in \mathcal{N}$ there exists a set $A_n \in \mathcal{N}_n$ such that the sequence $\{f_{k|X-A_n}\}_{k\in N}$ converges uniformly to the function $f_{|X-A_n}$.

THEOREM 1. Convergence $\mathcal{N}$-a.e. is equivalent to $(\mathcal{N}_n)$-uniform convergence if and only if the small system $\{\mathcal{N}_n\}_{n\in N}$ is upper semicontinuous.

P r o o f. Sufficiency. It is proved in [2] that any sequence of $\mathcal{S}$-measurable and finite functions convergent everywhere is $(\mathcal{N}_n)$-uniformly convergent if the small system $\{\mathcal{N}_n\}_{n\in N}$ is upper semicontinuous. This proof can easily be repeated in our case. Moreover, $(\mathcal{N}_n)$-uniform convergence implies $\mathcal{N}$-a.e. convergence. This ends the proof of sufficiency.

Necessity. Let us suppose that the small system $\{\mathcal{N}_n\}_{n\in N}$ is not upper semicontinuous. There exist $i_o \in N$ and a nonincreasing sequence $\{E_i\}_{i\in N}$ of $\mathcal{S}$-measurable sets, such that $E_i \notin \mathcal{N}_{i_o}$ for each $i \in N$, and $\bigcap_{i=1}^{\infty} E_i \in \mathcal{N}$. Let $f_k(x) = \chi_{E_k}(x)$ for each $k \in N$. The sequence $\{f_k\}_{k\in N}$ is convergent $\mathcal{N}$-a.e. to the function $f = 0$, but it is not $(\mathcal{N}_n)$-uniformly convergent. Otherwise, there would exist a set $A_{i_o} \in \mathcal{N}_{i_o}$ such that the sequence $\{f_{k|X-A_{i_o}}\}_{k\in N}$ converges uniformly to $f_{|X-A_{i_o}}$. This implies the existence of $k_o \in N$ such that, for any $k \in N$, $k \geq k_o$, $\chi_{E_k}(x) = 0$ for $x \notin A_{i_o}$. Hence $A_{i_o} \supset \bigcup_{k=k_o}^{\infty} E_k$ and $E_k \in \mathcal{N}_{i_o}$ for $k \geq k_o$. This contradicts the fact that $E_k \notin \mathcal{N}_{i_o}$ for any $k \in N$.

REMARK. Considering a small system $\{\mathcal{N}_n\}_{n\in N}$ over the infinite space X, such that $\mathcal{N}_n = \{\emptyset\}$ for each $n \in N$, we see that any convergent sequence of finite $\mathcal{S}$-measurable real functions which is not uniformly convergent is not uniformly convergent in the sense of Egoroff with respect to the small system $\{\mathcal{N}_n\}_{n\in N}$. Of course, this small system $\{\mathcal{N}_n\}_{n\in N}$ is not upper semicontinuous.

It is possible to characterize $(\mathcal{N}_n)$-uniform convergence by the convergence of some sequence of $\mathcal{S}$-measurable functions with respect to the small system $\{\mathcal{N}_n\}_{n\in N}$ or by the following condition of the vanishing restriction.

DEFINITION 5. We shall say that a sequence $\{f_n\} \subset M[\mathcal{S}, \mathcal{J}]$ converges to a function $f \in M[\mathcal{S}, \mathcal{N}]$ with respect to the small system $\{\mathcal{N}_n\}_{n\in N}$ if

$$\forall_{\varepsilon>0} \ \forall_{n\in N} \ \exists_{n_o\in N} \ \forall_{k\geq n_o} \ \{x: |f_k(x) - f(x)| > \varepsilon\} \in \mathcal{N}_n.$$

DEFINITION 6. We shall say that a sequence $\{f_n\} \subset M[\mathcal{S}, \mathcal{N}]$ satisfies the vanishing restriction for a function $f \in M[\mathcal{S}, \mathcal{N}]$ with respect to the small system $\{\mathcal{N}_n\}_{n\in N}$ if

$$\forall_{\alpha>0} \ \forall_{n\in N} \ \exists_{n_o\in N} \ \bigcup_{k=n_o}^{\infty} \{x: |f_k(x) - f(x)| > \alpha\} \in \mathcal{N}_n.$$

THEOREM 2. Let $\{f_n\}_{n\in N}$ be an arbitrary sequence of functions from $M[\mathcal{S}, \mathcal{N}]$ then the following conditions are equivalent:

(i) the sequence $\{f_n\}_{n\in N}$ is $(\mathcal{N}_n)$-uniformly convergent to a function $f \in M[\mathcal{S}, \mathcal{N}]$;

(ii) the sequence $\{f_n\}_{n\in N}$ satisfies the vanishing restriction for the function $f \in M[\mathcal{S}, \mathcal{N}]$ with respect to the small system $\{\mathcal{N}_n\}_{n\in N}$;

(iii) the sequence $\{\phi_n\}_{n\in N} \subset M[\mathcal{S}, \mathcal{N}]$ where $\phi_n(x) = \sup\{|f_i(x) - f(x)|, i \geq n\}$ converges to the function $\phi = 0$ with respect to the small system $\{\mathcal{N}_n\}_{n\in N}$.

P r o o f. Since

$$\bigcup_{k=n}^{\infty} \{x: |f_k(x) - f(x)| > \alpha\} = \{x: |\phi_n(x)| > \alpha\},$$

we see that conditions (ii) and (iii) are equivalent. Now, we assume that (i) is satisfied. Let $\{f_n\}_{n\in N}$ be a sequence of functions from $M[\mathcal{S}, \mathcal{N}]$ which is $(\mathcal{N}_n)$-uniformly convergent to a function $f \in M[\mathcal{S}, \mathcal{N}]$ Let α be an arbitrary positive real number and let $n_o \subset N$. There exists a set $A_{n_o} \in \mathcal{N}_{n_o}$ such that the sequence $\{f_{k|X-A_{n_o}}\}_{k\in N}$ converges uniformly to the function $\{f_{|X-A_{n_o}}\}$ Thus there exists $k_o \in N$ such that

$$X - A_{n_o} \subset \bigcap_{k=k_o}^{\infty} \{x: |f_k(x) - f(x)| \leq \alpha\}.$$

This implies that

$$\bigcup_{k=k_o}^{\infty} \{x: |f_k(x) - f(x)| > \alpha\} \in \mathcal{N}_{n_o}.$$

Hence the sequence $\{f_n\}_{n\in N}$ satisfies the vanishing restriction with respect to the small system $\{\mathcal{N}_n\}_{n\in N}$ for the function f.

Sufficiency. Let us suppose that a sequence $\{f_n\}_{n\in N} \subset M[\mathcal{S},\mathcal{N}]$ satisfies the vanishing restriction for a function $f \in M[\mathcal{S},\mathcal{N}]$ with respect to the small system $\{\mathcal{N}_n\}_{n\in N}$. Let $n_o \in N$. By condition (ii) of Definition 1, there exists a sequence $\{l_i\}_{i\in N}$ of positive integers such that if $E_i \in \mathcal{N}_{l_i}$, then $\bigcup_{i=1}^{\infty} E_i \in \mathcal{N}_{n_o}$. By the property of the vanishing restriction, there exists a sequence $\{n_i\}$ of positive integers such that

$$\bigcup_{k=n_i}^{\infty} \{x: |f_k(x) - f(x)| \geq \frac{1}{i}\} \in \mathcal{N}_{l_i},$$

Hence

$$A_{n_o} = \bigcup_{i=1}^{\infty} \bigcup_{k=n_i}^{\infty} \{x: |f_k(x) - f(x)| > \frac{1}{i}\} \in \mathcal{N}_{n_o}.$$

Let us observe that the sequence $\{f_k\}_{k\in N}$ converges uniformly on the set $X - A_{n_o}$. Let α be an arbitrary positive real number and let $i_o \in N$ be such that $\alpha > \frac{1}{i_o}$. Let $n(\alpha) = n_{i_o}$, then

$$X - A_{n_o} \subset \bigcap_{k=n_{i_o}}^{\infty} \{x: |f_k(x) - f(x)| < \alpha\}.$$

We conclude that the sequence $\{f_n\}_{n\in N}$ is $(\mathcal{N}_n)$-uniformly convergent.

Let us observe that convergence with respect to the small system with M-convergence implies $(\mathcal{N}_n)$-uniform convergence.

DEFINITION 7 (cf. [1]). A sequence $\{f_n\}_{n\in N} \subset M[\mathcal{S},\mathcal{N}]$ is M-convergent to a function $f \in M[\mathcal{S},\mathcal{N}]$ if, for an arbitrary positive number $\alpha > 0$,

$$\{x \in X: |f_i(x) - f(x)| > \alpha\} \subset \{x \in X: |f_j(x) - f(x)| > \alpha\}$$

for $i \geq j$, $i, j \in N$.

THEOREM 3. If a sequence $\{f_n\}_{n\in N} \subset M[\mathcal{S},\mathcal{N}]$ is M-convergent to a function $f \in M[\mathcal{S},\mathcal{N}]$, then the sequence $\{f_n\}_{n\in N}$ is convergent with respect to the small system $\{\mathcal{N}_n\}_{n\in N}$ to the function f if and only if the sequence $\{f_n\}_{n\in N}$ is $(\mathcal{N}_n)$-uniformly convergent.

P r o o f. Necessity. Let us suppose that the sequence $\{f_n\}_{n\in N}$ is not $(\mathcal{N}_n)$-uniformly convergent. This means, by Theorem 2, that $\{f_n\}_{n\in N}$ does not satisfy the vanishing restriction for the function f. Hence there exist $n_o \in N$ and a positive number α such that, for each $k \in N$,

$$\bigcup_{i=k}^{\infty} \{x: |f_i(x) - f(x)| > \alpha\} \notin \mathcal{N}_{n_o}.$$

But

$$\bigcup_{i=k}^{\infty} \{x: |f_i(x) - f(x)| > \alpha\} = \{x: |f_k(x) - f(x)| > \alpha\}.$$

We have that, for each $k \in N$, the set $\{x: |f_k(x) - f(x)| > \alpha\} \in \mathcal{N}_{n_o}$, which contradicts the fact of the convergence of the sequence $\{f_n\}_{n\in N}$ to the function f with respect to the small system $\{\mathcal{N}_n\}_{n\in N}$.

Sufficiency. A sequence $\{f_n\}_{n\in N} \in M[\mathcal{S},\mathcal{N}]$ which is $(\mathcal{N}_n)$-uniformly convergent to a function f is $\mathcal{N}$-a.e. convergent to the function f. Thus it is convergent with respect to the small system $\{\mathcal{N}_n\}_{n\in N}$ to the function f.

In [5], the rotation of the uniform convergence in the sense of Egoroff with respect to an arbitrary σ-ideal $\mathcal{J}$ of subsets of X was introduced.

DEFINITION 8. We shall say that a sequence $\{f_n\}_{n\in N} \subset M[\mathcal{S},\mathcal{N}]$

converges uniformly in the sense of Egoroff with respect to the σ-ideal $\mathcal{T}$ (abbreviation $\mathcal{T}$-uniformly convergent) if there exists a sequence $\{E_m\}_{m\in N}$ of $\mathcal{S}$-measurable sets such that $X - \bigcup_{m=1}^{\infty} E_m \in \mathcal{T}$ and the sequence $\{f_{n|E_m}\}_{n\in N}$ converges uniformly to the function $f_{|E_m}$ for each $m \in N$.

THEOREM 4. $(\mathcal{N})$-uniform convergent is equivalent to $(\mathcal{N}_n)$-uniform convergence if and only if the small system $\{\mathcal{N}_n\}_{n\in N}$ is upper semicontinuous.

P r o o f. Necessity. Let us suppose that the small system $\{\mathcal{N}_n\}_{n\in N}$ is not upper semicontinuous. Let $\{f_n\}_{n\in N}$ be the sequence of $\mathcal{S}$-measurable functions considered in the proof of Theorem 1. As we know, this sequence is not $(\mathcal{N}_n)$-uniformly convergent, but it is easy to see the sequence $\{f_{n|X-E_m}\}_{n\in N}$ is uniformly convergent to 0 on the set $X - E_m$. Thus we conclude that the sequence $\{f_n\}_{n\in N}$ is $\mathcal{N}$-uniformly convergent to the function $f \equiv 0$. This contradiction ends the proof of necessity.

Sufficiency. We see that any sequence $\{f_n\}_{n\in N} \subset M[\mathcal{S},\mathcal{N}]$ which is $(\mathcal{N}_n)$-uniformly convergent to a function $f \in M[\mathcal{S},\mathcal{N}]$ is $\mathcal{N}$-uniformly convergent to f. Let us suppose that the small system $\{\mathcal{N}_n\}_{n\in N}$ is upper semicontinuous. Let $\{f_n\}_{n\in N} \subset M[\mathcal{S},\mathcal{N}]$ be an arbitrary sequence $\mathcal{N}$-uniformly convergent to a function $f \in M[\mathcal{S},\mathcal{N}]$ and let us suppose that the sequence $\{f_n\}_{n\in N}$ is not $(\mathcal{N}_n)$-uniformly convergent to the function f. By Theorem 2, the sequence $\{f_n\}_{n\in N}$ does not fulfil the vanishing restriction for the function f with respect to the small system $\{\mathcal{N}_n\}_{n\in N}$. This means that

$$\underset{\alpha>0}{\exists}\ \underset{n_o\in N}{\exists}\ \underset{n\in N}{\forall}\ \bigcup_{k=n}^{\infty} \{x: |f_k(x) - f(x)| > \alpha\} \notin \mathcal{N}_{n_o}.$$

Let

$$A = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} \{x: |f_k(x) - f(x)| > \alpha\}.$$

By the condition of the upper semicontinuity of the small system

$\{\mathcal{N}_n\}_{n\in N}$, the set A does not belong to $\mathcal{N}$. Let us suppose that the sequence $\{f_n\}_{n\in N}$ is $\mathcal{N}$-uniformly convergent to the function f. Then there exists a sequence $\{E_n\}_{n\in N}$ of $\mathcal{S}$-measurable sets such that $X - \bigcup_{n=1}^{\infty} E_n \in \mathcal{N}$ and the sequence $\{f_{k|E_n}\}_{k\in N}$ is uniformly convergent to the function $f_{|E_n}$. If $A \notin \mathcal{N}$, then there exists an $n_o \in \mathcal{N}$ such that $E_{n_o} \cap A \notin \mathcal{N}$. Hence $E_{n_o} \cap A \neq \emptyset$. But the sequence $\{f_k\}_{k\in N}$ does not converge to the function f on the set $E_{n_o} \cap A$. The contradiction obtained completes the proof.

Let us recall the notation of the vanishing restriction with respect to an arbitrary σ-ideal $\mathcal{T}$ of subsets of X.

DEFINITION 9 (cf. [5]). We shall say that a sequence $\{f_n\}_{n\in N}$ of $\mathcal{S}$-measurable real functions satisfies the vanishing restriction for an $\mathcal{S}$-measurable function f with respect to the σ-ideal $\mathcal{T}$ if $\bigcap_{n=1}^{\infty} E_n(\alpha) \in \mathcal{T}$ for all $\alpha > 0$, where

$$E_n(\alpha) = \bigcup_{i=n}^{\infty} \{x \in X: |f_i(x) - f(x)| > \alpha\}.$$

THEOREM A. A sequence $\{f_n\}_{n\in N} \subset M[\mathcal{S}, \mathcal{T}]$ satisfies the vanishing restriction for an $\mathcal{S}$-measurable function f with respect to the σ-ideal $\mathcal{T}$ if and only if the sequence $\{f_n\}_{n\in N}$ is convergent $\mathcal{T}$-a.e.

Let us recall two more definitions.

DEFINITION 10 (cf. [6]). We shall say that the pair $(\mathcal{S}, \mathcal{T})$ fulfils the countable chain condition (abbreviation c.c.c.) if each subfamily pairwise disjont sets of $\mathcal{S} \setminus \mathcal{T}$ is at most countable.

DEFINITION 11 (cf. [6]). We shall say that the pair $(\mathcal{S}, \mathcal{T})$ fulfils the condition (E) if, for each double sequence $\{B_{j,n}\}_{j,n\in N}$ of $\mathcal{S}$-measurable sets and each $\mathcal{S}$-measurable set B not belonging to $\mathcal{T}$ such that

1° $B_{j,n} \subset B_{j,n+1}$ for any $j, n \in N$,

2° $\bigcup_{n=1}^{\infty} B_{j,n} = B$ for each $j \in N$, there exists an increasing sequence $\{n_j\}_{j\in N}$ of positive integers such that $\bigcap_{j=1}^{\infty} B_{j,n_j} \notin \mathcal{J}$.

THEOREM B (cf. [6]). If the pair $(\mathcal{S}, \mathcal{J})$ fulfils c.c.c., then an arbitrary sequence $\{f_n\}_{n\in N} \subset M[\mathcal{S}, \mathcal{J}]$ convergent $\mathcal{J}$-a.e. to a function $f \in M[\mathcal{S}, \mathcal{J}]$ is $\mathcal{J}$-uniformly convergent if and only if the pair $(\mathcal{S}, \mathcal{J})$ fulfils the condition (E).

Let $\{\mathcal{N}_n\}_{n\in N}$ be an upper semicontinuous small system on $\mathcal{S}$. Then the pair $(\mathcal{S}, \mathcal{N})$ fulfils both c.c.c. (cf. [3]) and the condition (E) (cf. [7]).

Thus, by Theorem 1, 2, 4 and Theorems A, B, we obtain the following theorem:

THEOREM 5. Let $\{\mathcal{N}_n\}_{n\in N}$ be an upper semicontinuous small system on $\mathcal{S}$. Let $\{f_n\}_{n\in N}$ be an arbitrary sequence of functions from $M[\mathcal{S}, \mathcal{N}]$ and $\{\phi_n\}_{n\in N}$ a sequence such that $\phi_n(x) = \sup\{x \in X: |f_i(x) - f(x)|, i \geq n\}$. Then the following conditions are equivalent:

(i) the sequence $\{f_n\}_{n\in N}$ is $(\mathcal{N}_n)$-uniformly convergent to f;

(ii) the sequence $\{f_n\}_{n\in N}$ fulfils the vanishing restriction for f with respect to the small system $\{\mathcal{N}_n\}_{n\in N}$;

(iii) the sequence $\{\phi_n\}_{n\in N}$ converges to f with respect to the small system $\{\mathcal{N}_n\}_{n\in N}$;

(iv) the sequence $\{f_n\}_{n\in N}$ is convergent $\mathcal{N}$-a.e. to f;

(v) the sequence $\{f_n\}_{n\in N}$ is $\mathcal{N}$-uniformly convergent to f;

(vi) the sequence $\{f_n\}_{n\in N}$ fulfils the vanishing restriction for f with respect to the σ-ideal

REFERENCES

[1] R. G. Bartle, An extension of Egoroff's theorem, Amer. Math. Month., 8 (1980).

[2] T. Neubrunn, B. Riečan, Miera a integral, Bratislava 1981.

[3] J. N i e w i a r o w s k i, Convergence of sequence of real functions with respect to small system, Math. Slov., 4 (1988).

[4] W. S i e r p i ń s k i, Remarque sur le théoreme de M. Egoroff, Compt. Rend. des sciences de la Soc. des Sciences et des Lettres de Varsovie, 20 (1928), 84-87.

[5] E. W a g n e r-B o j a k o w s k a, Remarks on convergence of sequences of measurable functions, Acta Univ. Lodz., Folia math. (in print).

[6] E. W a g n e r, W. W i l c z y ń s k i, Convergence almost everywhere of sequences of measurable functions, Colloq. Math., 45 (1981), 119-124.

[7] E. W a j c h, Small systems - on approximation of compact sets of measurable functions to compact subsets of $C_{c-o}(x)$ (in print).

[8] W. W i l c z y ń s k i, Remark on the theorem of Egoroff, Casopis pro pest. mat., 102 (1971), 228-229.

Institute of Mathematics
University of Łódź

Jacek Hejduk

PEWNE UWAGI DOTYCZĄCE TWIERDZENIA JEGOROWA

W artykule rozważa się zbieżność jednostajną względem małego systemu $\{\mathcal{N}_n\}_{n\in N}$. Udowodnione są warunki równoważne jednostajnej zbieżności względem małego systemu $\{\mathcal{N}_n\}_{n\in N}$, przy założeniu półciągłości małego systemu.