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# NECESSARY OPTIMALITY CONDITIONS FOR SPECIAL INTEGRAL PROCESSES

This paper deals with control processes described by nonlinear integral equations of type (2). By means of varied controls necessary optimality conditions are deduced. Volterra integral processes are included, some other processes are considered. A result of J. Pelczewski is given in an equivalent form.

## 1. INTRODUCTION

For control processes described by integral equations the whole control function  $u(\cdot)$  is effective for the value of the trajectory  $x(\cdot)$  at the time  $s \in [0, T]$ :

$$x(s) = h(s) + \int_0^T f(s, t, x(t), u(t))dt, \quad 0 \leq s \leq T \quad (0)$$

J. Pelczewski considered in [5] the following interesting problem

$$\int_0^T g(t, x(t), u(t))dt = \min! \quad (1)$$

under the conditions

$$x(s) = \int_0^T f(s, t, x(t), u(s))dt, \quad 0 \leq s \leq T \quad (2)$$

$$u(s) \in U \text{ for } s \in [0, T] \text{ a.e.}$$

where  $u(\cdot)$  and  $x(\cdot)$  are  $L_2$ -functions with values in  $R^n$ .

If the functional  $g$  and the function  $f$  are quadratic in  $x$  and  $u$  and fulfil some regularity assumptions a minimum principle with an adjoint equation can be derived.

In this paper an analogous necessary optimality condition for non-linear processes governed by Fredholm or Volterra integral

equations will be proved. We will take piecewise continuous functions (and also continuous functions) with values in certain Banach-spaces for the control and the state-functions. Then it is possible to make use of considerations in [6-8]. J. Pelczewski proved the minimum principle using properties of some cones (see S. Walczak [9]), whereas we will do it by means of an abstract model of L. Bittner [1, 2].

## 2. THE PROBLEM

Let be  $E$  and  $S$  Banach-spaces,  $U \subseteq S$  a convex set and  $U_0$  open with  $U \subseteq U_0 \subseteq S$ ,  $I = [0, T]$  a fixed time-interval. Let be defined a continuous functional  $g$  on  $I \times E \times U_0$  and continuous functions  $f: I \times I \times E \times U_0 \rightarrow E$  and  $h: I \rightarrow E$ . We assume the existence of continuous partial Fréchet derivations  $g_e, f_e, g_u, f_u$  with respect to  $e \in E, u \in S$ .

Denote by  $C(I, E)$  the space of all continuous functions  $x(\cdot): I \rightarrow E$  with maximum-norm and by  $PCL(I, E)$  the space of all piecewise continuous and left-hand continuous functions  $x(\cdot)$  (the number of discontinuities is finite and there exist  $x(t+0), x(t-0)$  for  $t \in I$  and it is  $x(t) = x(t-0)$  for  $0 < t \leq T$  and  $x(0+0) = x(0)$ ) with sup-norm. Analogously  $PCL(I, U)$  is the set of all piecewise continuous functions  $u(\cdot): I \rightarrow U$ . Let be  $h(\cdot) \in PCL(I, E)$  and for the first  $f(s, t, e, u) = f^1(s, t, e) + f^2(s, t, u)$  (such process is called separated).

Let be  $\hat{u}(\cdot), \hat{x}(\cdot)$  an optimal solution for the problem: minimize (1) subject to the process equation

$$x(s) = h(s) + \int_0^T f(s, t, x(t), u(s)) dt, \quad s \in I \quad (3)$$

and

$$x(\cdot) \in PCL(I, E), \quad u(\cdot) \in PCL(I, U). \quad (4)$$

We shall write  $\hat{f}(s, t)$  for  $f(s, t, \hat{x}(t), \hat{u}(s))$ ,  $\hat{g}(t)$  for  $g(t, \hat{x}(t), \hat{u}(t))$  and also  $\hat{f}_e^1(s, t) = \hat{f}_e^1(s, t, \hat{x}(t))$ ,  $\hat{f}_u^2(s, t) = \hat{f}_u^2(s, t, \hat{u}(s))$ ,  $\hat{g}_e(t) = g_e(t, \hat{x}(t), \hat{u}(t))$ ,  $\hat{g}_u(t) = g_u(t, \hat{x}(t), \hat{u}(t))$ .

We assume (instead of condition 6 in [5] for linear integral equations in  $L_2$ )

$$\max_{s \in I} \int_0^T ||f_e^1(s, t)|| dt < 1.$$

In order to apply the theory of Bittner define

$$F(x, w) = \int_0^T g(t, x(t), u(t)) dt$$

$$G(x, w)(s) = x(s) - h(s) - \int_0^T f^1(s, t, x(t)) dt - \int_0^T f^2(s, t, u(s)) dt$$

$$w = u(\cdot), x = x(\cdot), \tilde{X} = PCL(I, E), \tilde{W} = PCL(I, U).$$

Then problem (1), (3), (4) is equivalent to:

$$\text{Minimize } F(x, w) \text{ subject to } G(x, w) = 0, x \in \tilde{X}, w \in \tilde{W} \quad (4a)$$

But we can not apply the results of [2], as  $\tilde{X}$  is not a Banach-space. Note that there are  $m$  points  $\sigma_1, \dots, \sigma_m$  ( $m \geq 0$ ) such that  $\hat{u}(\cdot)$  and also  $h(\cdot)$  are continuous on  $I \setminus \{\sigma_1, \dots, \sigma_m\}$ . We choose  $\tau \in I$  and then  $\mu > 0$  such that  $\sigma_i \notin ]\tau - \mu, \tau + \mu]$  for  $i = 1, \dots, m$ . Denote  $\sigma_{m+1} = \max\{\tau - \mu, 0\}$ ,  $\sigma_{m+2} = \min\{\tau + \mu, T\}$ ,  $\Sigma = \{\sigma_1, \dots, \sigma_{m+2}\}$ . We renumber these fixed time points such that  $s_i \in \Sigma$ ,  $s_1 < \dots < s_{m+2}$ ,  $s_0 = 0$ ,  $s_{m+3} = T$ .

Choose an arbitrary (but now fixed) vector  $v \in U$  and define

$$u_\varepsilon(t) = \begin{cases} \hat{u}(t) & \text{for } t \in I \setminus ]\tau - \mu, \tau + \mu] \\ \hat{u}(t) + \varepsilon(v - \hat{u}(t)), & t \in I \cap ]\tau - \mu, \tau + \mu] \end{cases}$$

$$W_{\Sigma, v} = \{u_\varepsilon(\cdot) \mid 0 \leq \varepsilon \leq 1\},$$

$$X_\Sigma = \{x(\cdot) \in \tilde{X} \mid x(\cdot) \text{ continuous on } I \setminus \Sigma\} \quad (5)$$

Then  $u_\varepsilon(\cdot) \in \tilde{W}$  because of the convexity of  $U$ ,  $X_\Sigma$  is a Banach-space with  $||x|| = \sup_{t \in I} ||x(t)||_E$  since  $\Sigma$  is fixed. Note that  $\hat{w} = \hat{u}(\cdot)$ ,  $\hat{x} = \hat{x}(\cdot)$  is also optimal for  $F(x, w) = \min$  with respect to  $G(x, w) = 0$ ,  $x \in X_\Sigma$ ,  $w \in W_{\Sigma, v}$ .

### 3. NECESSARY OPTIMALITY CONDITIONS

In [6] we have proved:

LEMMA 1. Let be  $Z$  a Banach-space,  $\Omega_1, \Omega_2$  compact intervals and  $k: \Omega_1 \times \Omega_2 \times E \rightarrow Z$  a continuous function,  $y(\cdot) \in C(\Omega_2, E)$ . Then we have

$$\lim_{r \rightarrow 0} \sup \{ ||k(s, t, y(t) + e) - k(s, t, y(t))|| \mid s \in \Omega_1, t \in \Omega_2, ||e|| \leq r \} = 0.$$

In the following we are going to verify that the assumptions of Bittner's model are fulfilled:

LEMMA 2. The partial Fréchet derivations  $F_x(x, w)$ ,  $G_x(x, w)$  with respect to  $x$  exist for all  $x \in X_\Sigma$ ,  $w \in W_{\Sigma, V}$  and it holds

$$F_x(x, w)z = \int_0^T g_e(t, x(t), u(t))z(t)dt$$

$$\begin{aligned} (G_x(x, w)z)(s) &= z(s) - \int_0^T f_e(s, t, x(t), u(s))z(t)dt = \\ &= z(s) - \int_0^T f_e^1(s, t, x(t))z(t)dt; \quad z \in X_\Sigma. \end{aligned}$$

P r o o f. Lemma 2 follows from Lemma 1 with  $k(s, t, e) = f_e(s, t, e, u(t))$ ,  $Z = \mathcal{L}(E)$  and  $k(s, t, e, u) = g_e(t, e, u(t))$ ,  $Z = E^*$  respectively, restricting  $x(\cdot)$ , on the intervals  $\Omega_1 = [s_i, s_{i+1}]$ ,  $y(t) = x(t)$  for  $t \in ]s_i, s_{i+1}]$ ,  $y(s_i) = x(s_i + 0)$ ,  $i = 0, \dots, m + 2$ .

LEMMA 3.  $W_{\Sigma, V}$  is a set of admissible variations for  $\hat{u}(\cdot)$ , that means

$$\lim_{r \rightarrow 0} \sup \{ ||G_x(x, w) - G_x(\hat{x}, w)|| \mid w \in W_{\Sigma, V}, ||x - \hat{x}|| \leq r, x \in X_\Sigma \} = 0.$$

P r o o f. We must show

$$\lim_{r \rightarrow 0} \sup_{||\Delta x|| \leq r} \sup_{t \in I} \left\| \int_0^T [f_e^1(s, t, \hat{x}(t) + \Delta x(t)) - \hat{f}_e^1(s, t)] dt \right\| = 0.$$

By interchanging norm and integral the lemma follows from Lemma 1 where  $k = f_e^1$ ,  $Z = \mathcal{L}(E)$ .

LEMMA 4.  $G_x(\hat{x}, \hat{w})$  is a one-to-one mapping of  $X_\Sigma$  onto itself. Equivalent is: the linear integral equation

$$z(s) = \int_0^T \hat{f}_e^1(s, t)z(t)dt + b(s), \quad s \in I$$

has a unique solution in  $X_\Sigma$  for every  $b(\cdot) \in X_\Sigma$ .

P r o o f. Define  $\bar{T}: X_\Sigma \rightarrow X_\Sigma$  by the formula

$$(\bar{T}z)(s) = \int_0^T \hat{f}_e^1(s, t) z(t) dt, \quad s \in I.$$

Then it is

$$\|\bar{T}z\|_{X_\Sigma} \leq \sup_{s \in I} \int_0^T \|\hat{f}_e^1(s, t)\| dt \cdot \|z\|_{X_\Sigma} \quad \text{and therefore } \|\bar{T}\| <$$

< 1. The lemma now follows from Banach's fixed point theorem.

REMARK 1. For  $u_\varepsilon(\cdot) \in W_{\Sigma, V}$  and  $\sup_{s \in I} \left\| \int_0^T [f^2(s, t, u_\varepsilon(s)) - \hat{f}^2(s, t)] dt \right\| \ll 1$  a corresponding solution of (3) in  $X_\Sigma$  exists! This is a consequence of [1].

LEMMA 5. The functional  $F$  has the following property:

$$\lim_{r \rightarrow 0} \sup \{ \|F_X(x, w) - F_X(\hat{x}, w)\| \mid x \in X_\Sigma, \|x - \hat{x}\| \leq r, w \in W_{\Sigma, V} \} = 0$$

which means

$$\lim_{r \rightarrow 0} \sup_{\substack{\|\Delta x\| \leq r \\ 0 \leq t \leq 1}} \left\| \int_0^T [g_e(t, \hat{x}(t) + \Delta x(t), u_\varepsilon(t)) - g_e(t, \hat{x}(t), u_\varepsilon(t))] dt \right\| = 0.$$

Proof. It holds

$$\left\| \int_0^T [g_e(t, \hat{x}(t) + \Delta x(t), u_\varepsilon(t)) - g_e(t, \hat{x}(t), u_\varepsilon(t))] dt \right\| \leq \int_{I(\tau, \mu)} \|g_e(t, \hat{x}(t) + \Delta x(t), \hat{u}(t) + v_\varepsilon(t)) - g_e(t, \hat{x}(t), \hat{u}(t) + v_\varepsilon(t))\| dt$$

where  $v_\varepsilon(t) = \varepsilon(v - \hat{u}(t))$ ,  $t \in I(\tau, \mu)$  and  $I(\tau, \mu) = I \cap [\tau - \mu, \tau + \mu]$ .

The relation stated in Lemma 5 now follows from Lemma 1 with  $k(\varepsilon, t, e) = g_e(t, e, \hat{u}(t) + \varepsilon(v - \hat{u}(t)))$ ,  $Z = \mathbb{R}^n$ ,  $\Omega_1 = [0, 1]$ ,  $\Omega_2 = I(\tau, \mu)$ , by replacing  $u_\varepsilon(\tau - \mu) = u_\varepsilon(\tau - \mu + 0)$ , therefore  $u_\varepsilon(\cdot)$  and  $\hat{x}(\cdot)$  are continuous in  $I(\tau, \mu)$ .

Let be  $\varepsilon_k$  a zero-sequence of positive numbers, and let  $w_k = w_{\varepsilon_k}$ ,  $\gamma_k = \varepsilon_k$ . We calculate

$$\delta G = \lim_{k \rightarrow \infty} (-\gamma_k^{-1}) [G(\hat{x}, w_k) - G(\hat{x}, \hat{w})] \quad \text{and}$$

$\delta F = \lim_{k \rightarrow \infty} \gamma_k^{-1} [F(\hat{x}, w_k) - F(\hat{x}, \hat{w})]$   
 in  $X_\Sigma$  and  $R$  respectively and call them common directional limits (CDL).

LEMMA 6. For  $w_k \in W_{\Sigma, v}$  it is

$$\delta G(s) = \begin{cases} 0 & \text{for } s \in I \setminus I(\tau, \mu) \\ \int_0^T \hat{f}_u^2(s, t) dt (v - \hat{u}(s)), & s \in I(\tau, \mu) \end{cases}$$

$$\delta F = \int_{I(\tau, \mu)} \hat{g}_u(t) (v - \hat{u}(t)) dt.$$

P r o o f. We find the estimation

$$\|G(\hat{x}, w_k) - G(\hat{x}, \hat{w}) - \varepsilon_k \delta G\| = \sup_{s \in I(\tau, \mu)} \left\| \int_0^T [f^2(s, t, v_{\varepsilon_k}(s)) - \hat{f}^2(s, t) - \varepsilon_k \hat{f}_u^2(s, t) (v_{\varepsilon_k}(s) - \hat{u}(s))] dt \right\| \leq \sup_s \int_0^T \|\dots\| dt.$$

For  $s \in I(\tau, \mu)$  we apply the mean-value-theorem in the Banach-space  $E$  (it yields for the "interval"  $[\hat{u}(s), v] \subseteq U_0$ ):

$$\|\dots\| \leq \|f_u^2(s, t, \hat{u}(s) + \vartheta_k(v - \hat{u}(s)) - \hat{f}_u^2(s, t)\| \cdot \|v - \hat{u}(s)\|$$

where  $\vartheta = \vartheta(s) \in ]0, 1[$ ,  $\|v - \hat{u}(s)\|$  is uniformly bounded.

Lemma 1 with  $y(t) = \hat{u}(t)$ ,  $k(s, t, y) = f_u^2(s, t, y)$ ,  $Z = \mathcal{L}(I, S)$  implies

$$\lim_{\varepsilon \rightarrow 0} \sup \{ \|f_u^2(s, t, \hat{u}(s) + \Delta u) - \hat{f}_u^2(s, t)\| \mid t \in I, s \in I(\tau, \mu),$$

$$\|\Delta u\| \leq \varepsilon \} = 0$$

and therefore  $\|G(\hat{x}, w_k) - G(\hat{x}, \hat{w}) - \varepsilon_k \delta G\| \rightarrow 0$  if  $k$  tends to  $\infty$ .

The second part can be estimated in an analogous manner:

$$|F(\hat{x}, w_k) - F(\hat{x}, \hat{w}) - \varepsilon_k \delta F| = \left| \int_{I(\tau, \mu)} [g(t, x(t), u(t) + v_{\varepsilon_k}(t)) - \hat{g}(t) - \hat{g}_u(t) v_{\varepsilon_k}(t)] dt \right| \leq \int_{I(\tau, \mu)} |\dots| dt \leq$$

$$\leq \int_{I(\tau, \mu)} \|g_u(t, \hat{x}(t), \hat{u}(t) + \vartheta_k(v - \hat{u}(t)) - \hat{g}_u(t)\| \cdot$$

$$\cdot \|v - \hat{u}(t)\| dt.$$

According to Lemma 1 the right-hand side of the last inequality tends to 0 as  $k \rightarrow \infty$ .



A necessary condition for  $\hat{w} = \hat{u}(\cdot), \hat{x}(\cdot)$  to be an optimal solution of the abstract problem is given in [1]:

$$F_x(\hat{x}, \hat{w})G_x(\hat{x}, \hat{w})^{-1} \delta G + \delta F \geq 0 \quad \text{for all CDL } \delta F, \delta G \quad (6)$$

We shall rewrite it for (1), (2), (4). Put  $z = G_x(x, w)^{-1} \delta G \in X_\Sigma$ , then  $z$  is the (unique) solution of

$$z = \bar{T}z + \delta G, \quad z \in X_\Sigma$$

$\bar{T}$  defines an integral operator on  $C(I, E)$  with kernel  $A(s, t) = \hat{f}_e^1(s, t), A(s, \cdot) \in PCL(I, \mathcal{L}(E))$  for all  $s \in I, A(\cdot, t) \in C(I, \mathcal{L}(E))$  for  $t \in I$ . The solution of the integral equation (7) in  $C(I, E)$  can be written in terms of the resolvent operator  $R(\cdot, \cdot)$ , which satisfies the relations

$$R(s, t) = A(s, t) + \int_0^T A(s, w)R(w, t)dw \quad (8)$$

$$R(s, t) = A(s, t) + \int_0^T R(s, w)A(w, t)dw$$

(see [6]). We extend  $\bar{T}: X_\Sigma \rightarrow C(I, E)$ . According to [8] we are able to show that the solution of (7) can be represented with the same resolvent operator as

$$z(s) = \delta G(s) + \int_0^T R(s, t)\delta G(t)dt, \quad t \in I.$$

We obtain from (5)

$$\delta F + \int_0^T \hat{g}_e(s)\delta G(s)ds + \int_0^T \hat{g}_e(s) \int_0^T R(s, t)\delta G(t)dtds \geq 0.$$

Applying Fubini's theorem we find

$$\delta F + \int_0^T \left[ \hat{g}_e(s) + \int_0^T \hat{g}_e(t)R(t, s)dt \right] \delta G(s)ds \geq 0. \quad (9)$$

We rewrite it in the form

$$\delta F - \int_0^T \hat{\psi}(s)\delta G(s)ds \geq 0 \quad \text{with } \hat{\psi}(s) = -\hat{g}_e(s) - \int_0^T \hat{g}_e(t)R(t, s)dt.$$

In [6] we proved, that  $\hat{\psi}(\cdot)$  is the unique solution of the adjoint equation

$$\psi(t) = \int_0^T \psi(s)\hat{f}_e^1(s, t)ds - \hat{g}_e(t), \quad t \in I \quad (10)$$

in  $PCL(I, E^*)$ .

By substituting the CDL  $\delta F$ ,  $\delta G$  into the optimality condition we get

$$\int_{I(\tau, \mu)} [\hat{g}_u(s) - \hat{\psi}(s) \int_0^T \hat{f}_u^2(s, t) dt] [v - \hat{u}(s)] ds \geq 0. \quad (11)$$

This inequality was obtained for fixed  $v$ ,  $\tau$ ,  $\mu$ . As it is true for every  $\mu > 0$  and  $\hat{x}(\cdot)$ ,  $\hat{u}(\cdot)$  are continuous at  $\tau$  we obtain

$$[\hat{g}_u(\tau) - \hat{\psi}(\tau) \int_0^T \hat{f}_u^2(\tau, t) dt] [v - \hat{u}(\tau)] \geq 0.$$

Considering other vectors  $v \in U$  and points  $\tau$  of continuity and therefore  $W_{\Sigma, v}$  the consideration is valid for all  $v \in U$  and a.e.  $\tau \in I$ .

**THEOREM 1.** If  $\hat{x}(\cdot)$ ,  $\hat{u}(\cdot)$  is an optimal solution of problem (1), (3), (4) under the assumptions mentioned above the minimum principle

$$[\hat{g}_u(\tau) - \hat{\psi}(\tau) \int_0^T \hat{f}_u^2(\tau, t) dt] \hat{u}(\tau) \leq [\hat{g}_u(\tau) - \hat{\psi}(\tau) \int_0^T \hat{f}_u^2(\tau, t) dt] v \quad (12)$$

holds for a.e.  $\tau \in [0, T]$  and all  $v \in U$ , where  $\hat{\psi}(\cdot)$  is the (piecewise continuous) solution of the adjoint equation (10).

**REMARK 2.** Define a function  $H: [0, T] \times E \times U \times E^*$  by

$$H(\tau, e, u, \psi) = -g(\tau, e, u) + \psi \int_0^T f^2(\tau, t, u) dt \quad (13)$$

and denote

$$H_u(\tau, e, u, \psi) = -g_u(\tau, e, u) + \psi \int_0^T f_u^2(\tau, t, u) dt.$$

Then the optimality condition can be written as

$$H_u(\tau, \hat{x}(\tau), \hat{u}(\tau), \hat{\psi}(\tau)) (v - \hat{u}(\tau)) \leq 0 \text{ for a.e. } \tau \in [0, T].$$

By setting  $\hat{H}_u(\tau) = H_u(\tau, \hat{x}(\tau), \hat{u}(\tau), \hat{\psi})$  we have the maximum principle

$$\hat{H}_u(\tau) \cdot \hat{u}(\tau) = \max_{v \in U} \hat{H}_u(\tau) v \text{ for a.e. } \tau \in [0, T]. \quad (14)$$

**REMARK 3.** In our considerations  $\mu$  depends on  $\tau$  and the Banach-space  $X_\tau$  also depends on  $\tau$  and  $\mu$ . It is possible to take  $\tilde{X}$  as a Banach-space independent of  $\tau$  and  $\mu$  by defining varied controls



$$u_{\varepsilon}(t) = \begin{cases} \hat{u}(t) & \text{for } t \in I \setminus [\tau - \mu, \tau + \mu] \\ \hat{u}(t) + \frac{\varepsilon}{\mu} (t - \tau + \mu)(v - \hat{u}(\tau)), & t \in I \cap [\tau - \mu, \tau] \\ \hat{u}(t) + \frac{\varepsilon}{\mu} (t - \tau - \mu)(\hat{u}(\tau) - v), & t \in I \cap [\tau, \tau + \mu] \end{cases} \quad (15)$$

and  $W_{v, \tau, \mu} = \{u_{\varepsilon}(\cdot) \mid 0 \leq \varepsilon \leq 1\}$ . All these varied controls have the form  $u_{\varepsilon}(\cdot) = \hat{u}(\cdot) + v_{\varepsilon}(\cdot)$ , where  $v_{\varepsilon}(\cdot)$  are of the type (Figure 1):

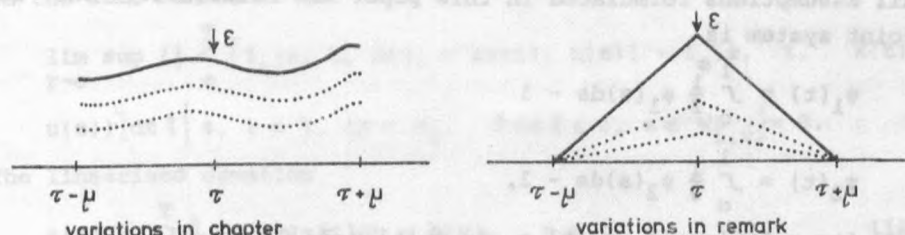


Fig. 1

In all the cases  $\mu$  is (at first) fixed. If we use varied controls of the second type with fixed  $\mu > 0$  for all  $\tau \in I \setminus \{\sigma_1, \dots, \sigma_m\}$ ,  $u_{\varepsilon}(\cdot)$  is not necessarily continuous in  $[\tau - \mu, \tau + \mu]$  and we must take this fact into considerations when deriving the lemmas. Note that in this case the CDL are

$$\delta G(s) = \begin{cases} 0 & s \in I(\tau, \mu) \\ \frac{s - \tau + \mu}{\mu} \int_0^T \hat{f}_u^2(s, t) dt [v - \hat{u}(\tau)] & s \in I \cap [\tau - \mu, \tau] \\ \frac{\tau - s + \mu}{\mu} \int_0^T \hat{f}_u^2(s, t) dt [v - \hat{u}(\tau)] & s \in I \cap [\tau, \tau + \mu] \end{cases} \quad (16)$$

$$\delta F = \left[ \int_{I \cap [\tau - \mu, \tau]} \hat{g}_u(t) \frac{t - \tau + \mu}{\mu} dt + \int_{I \cap [\tau, \tau + \mu]} \hat{g}_u(t) \frac{\tau - t + \mu}{\mu} dt \right] [v - \hat{u}(\tau)] \quad (17)$$

Theorem 1 can be proved in the same way as above.

**Examples.** In [5] J. Pelczewski considered

$$\int_0^1 [3x_1(t) + 2x_2(t) + u_1(t) + 2u_2(t)] dt = \min!$$

under the conditions

$$x_1(s) = \int_0^1 \frac{1}{2} x_1(t) dt + 2u_1(s)$$

$$x_2(s) = \int_0^1 \frac{1}{3} x_2(t) dt + u_2(s) \quad s \in [0, 1]$$

$$|u_i(s)| \leq \alpha_i, \quad i = 1, 2.$$

All assumptions formulated in this paper are fulfilled. The adjoint system is

$$\psi_1(t) = \int_0^1 \frac{1}{2} \psi_1(s) ds - 3$$

$$\psi_2(t) = \int_0^1 \frac{1}{3} \psi_2(s) ds - 2,$$

and its solution is  $\psi_1(t) = -6$ ,  $\psi_2(t) = -3$ ,  $0 \leq t \leq 1$ .

According to Theorem 1 we find  $u(\tau)$  from the optimization problem

$$[(1, 2) - (\psi_1(\tau), \psi_2(\tau)) \int_0^1 \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} dt] \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \min!$$

subject to  $|v_1| \leq \alpha_1$ ,  $|v_2| \leq \alpha_2$ . Rewriting the function we have

$$(1 - 6)v_1 + (2 - 3)v_2 = \min! \quad |v_i| \leq \alpha_i, \quad i = 1, 2.$$

It differs from the function in [5], but it has the same solution  $\hat{u}_1(t) = \alpha_1$ ,  $u_2(t) = \alpha_2$  for all  $t \in [0, 1]$ .

#### 4. NONSEPARATED FREDHOLM INTEGRAL PROCESSES

Which difficulties can arise if it is impossible to split up  $f(s, t, e, u)$  into  $f^1(s, t, e) + f^2(s, t, u)$ ? Let be  $\hat{u}(\cdot)$ ,  $\hat{x}(\cdot)$  optimal for (1), (3), (4) and  $\sigma_1, \dots, \sigma_m$  points of discontinuity of  $\hat{u}(\cdot)$  and  $h(\cdot)$  and let  $\Sigma, W_{\Sigma, V}, X_{\Sigma}$  be defined in the previous sense. We assume  $U$  to be convex and

$$\sup_{s \in I} \int_0^T \|\hat{f}_e(s, t)\| dt < 1.$$

Applying Lemma 1 for  $\Omega_1 = [s_i, s_{i+1}]$ ,  $\Omega_2 = [s_j, s_{j+1}]$  and replacing  $\hat{u}(s_i)$ ,  $\hat{x}(s_j)$  by their right-hand limits ( $i, j = 0, \dots$ ,

$m + 2$ ) the existence of Fréchet derivations  $F_x(x, w)$ ,  $G_x(x, w)$  can easily be shown for  $x \in X_\Sigma$ ,  $w \in W_{\Sigma, v}$  and it is

$$F_x(x, w)z = \int_0^T g_e(t, x(t), u(t))z(t)dt$$

$$(G_x(x, w)z)s = z(s) - \int_0^T f_e(s, t, x(t), u(s))z(t)dt.$$

In the same manner we get

$$\limsup_{r \rightarrow 0} \left\{ \left\| \int_0^T [f_e(s, t, \hat{x}(t) + \Delta x(t), u(s)) - f_e(s, t, \hat{x}(t), u(s))] dt \right\| \mid s, t \in I, \Delta x \in X_\Sigma, \|\Delta x\| \leq r, u \in W_{\Sigma, v} \right\} = 0.$$

The linearized equation

$$z(s) = \int_0^T \hat{f}_e(s, t)z(t)dt + b(s), \quad s \in I \quad (18)$$

has an unique solution for all  $b \in X_\Sigma$  in  $X_\Sigma$ .

We calculate the CDL:

$$\delta G(s) = \begin{cases} 0 & s \in I \setminus I(\tau, \mu) \\ \int_0^T \hat{f}_u(s, t)dt[v - \hat{u}(s)] & s \in I(\tau, \mu) \end{cases}$$

and

$$\delta F = \int_{I(\tau, \mu)} g_u(t)(v - \hat{u}(t))dt.$$

In formula (14) the kernel  $A(s, t) = \hat{f}_e(s, t)$  has the property  $A(s, \cdot) \in \text{PCL}(I, \mathcal{L}(E))$  and  $A(\cdot, t) \in \text{PCL}(I, \mathcal{L}(E))$  for  $s, t \in I$ . We shall show the existence and some properties of resolvent operators also for this extended class  $\mathcal{F}$  of kernel-functions. (14) can be solved by the iterative method:  $z^0(s) = b(s)$   
 $z^{n+1}(s) = \int_0^T A(s, t)z^n(t)dt + b(s), \quad s \in I, n = 0, \dots$  By induction

we can show the existence of operators

$$R^n(\cdot, \cdot): I \times I \rightarrow \mathcal{L}(E) \text{ with}$$

$$z^n(s) = b(s) + \int_0^T R^n(s, t)b(t)dt, \quad s \in I.$$

All  $R^n(\cdot, \cdot)$  are continuous for  $s_i < s \leq s_{i+1}$ ,  $s_j < t \leq s_{j+1}$ ; they converge uniformly on each rectangle  $s_i \leq s \leq s_{i+1}$ ,  $s_j \leq t \leq s_{j+1}$ ;  $i, j = 0, \dots, m+1$  to certain  $R(\cdot, \cdot)$ , if we replace

$\hat{u}(s_j)$  and  $\hat{x}(s_j)$  by their right-hand limits. Therefore  $R^n(\cdot, \cdot)$  converges to  $R(\cdot, \cdot)$  in  $\mathcal{F}$  with respect to the sup-norm  $\|R(\cdot, \cdot)\| = \sup_{s, t} \|R(s, t)\|_{\mathcal{L}(E)}$ . It is

$$R^{n+1}(s, t) = A(s, t) + \int_0^T A(s, w) R^n(w, t) dw$$

$$R^{n+1}(s, t) = A(s, t) + \int_0^T R^n(s, w) A(w, t) dw$$

and (7), (8) follow for  $n \rightarrow \infty$ . For a more detailed discussion we refer to [6]. Using the technique of resolvent operators we can prove a minimum principle:

**THEOREM 2.** A necessary condition for  $\hat{u}(\cdot)$ ,  $\hat{x}(\cdot)$  to be an optimal solution of process (1), (3), (4) is the minimum principle

$$[\hat{g}_u(\tau) - \hat{\psi}(\tau) \int_0^T \hat{f}_u(\tau, t) dt] \hat{u}(\tau) =$$

$$= \min_{v \in U} [\hat{g}_u(\tau) - \hat{\psi}(\tau) \int_0^T \hat{f}_u(\tau, t) dt] v,$$

for a.e.  $\tau \in I$ , where  $\hat{\psi}(\cdot)$  is the solution of the adjoint equation

$$\dot{\psi}(t) = -\hat{g}_e(t) + \int_0^T \psi(s) \hat{f}_e(s, t) ds, \quad t \in I.$$

## 5. CONTINUOUS CONTROLS

Sometimes the optimal control is a continuous function and for certain practical problems only such controls are considered. Therefore we deal with the integral problem:

minimize (1) subject to (3) and

$$u(\cdot) \in C(I, U), \quad x(\cdot) \in C(I, E) \quad (19)$$

where  $h \in C(I, E)$ ,  $\max_{s \in I} \int_0^T \|\hat{f}_e(s, t)\| dt < 1$ ,  $U$  convex and all

assumptions mentioned above are valid. Let  $\hat{u}(\cdot)$ ,  $\hat{x}(\cdot)$  be an optimal pair. The abstract space we considered in (4a) shall be  $X = C(I, E)$  with maximum-norm. Let be  $\mu > 0$ ,  $\tau \in [0, 1]$  and  $v \in U$ . Define  $u_{\varepsilon}(\cdot)$  to be the continuous function of (15) and  $W = W_{\tau, \mu, v} = \{u_{\varepsilon}(\cdot) \mid 0 \leq \varepsilon \leq 1\}$ . Then

$$\delta G(s) = \begin{cases} 0 & s \in I \setminus I(\tau, \mu) \\ \frac{s - \tau + \mu}{\mu} \int_0^T \hat{f}_u(s, t) dt [v - \hat{u}(\tau)] & s \in I \cap [\tau - \mu, \tau] \\ \frac{\tau - s + \mu}{\mu} \int_0^T \hat{f}_u(s, t) dt [v - \hat{u}(\tau)] & s \in I \cap [\tau, \tau + \mu] \end{cases}$$

and  $\delta F$  is calculated in (17).

In equation (18) it is  $\Lambda(\cdot, \cdot) = \hat{f}_e(\cdot, \cdot) \in C(I \times I, \mathcal{X}(E))$  and it follows  $R(\cdot, \cdot) \in C(I \times I, \mathcal{X}(E))$ , too. With the adjoint equation

$$\psi(t) = \int_0^T \psi(s) \hat{f}_e(s, t) ds - \hat{g}_e(t) \quad t \in I \quad (20)$$

in  $C(J, E^*)$  the optimality condition is

$$\delta F - \int_0^T \hat{\psi}(s) \delta G(s) ds \geq 0.$$

Hence

$$\left\{ \int_{I \cap [\tau - \mu, \tau]} \frac{s - \tau + \mu}{\mu} [\hat{g}_u(s) - \hat{\psi}(s) \int_0^T \hat{f}_u(s, t) dt] ds + \right. \\ \left. + \int_{I \cap [\tau, \tau + \mu]} \frac{\tau - s + \mu}{\mu} [\hat{g}_u(s) - \hat{\psi}(s) \int_0^T \hat{f}_u(s, t) dt] ds \right\} (v - \hat{u}(\tau)) \geq 0.$$

Since  $\mu > 0$  is arbitrary it follows

**THEOREM 3.** If  $\hat{u}(\cdot)$ ,  $\hat{x}(\cdot)$  are optimal for (1), (3), (19) then the maximum principle

$$\hat{H}_u(\tau) \cdot \hat{u}(\tau) = \max_{v \in U} \hat{H}_u(\tau) \cdot v \quad \text{for all } \tau \in I,$$

where  $\hat{H}_u(\tau) = -\hat{g}_u(\tau) + \hat{\psi}(\tau) \int_0^T \hat{f}_u(\tau, t) dt$ ,  $\tau \in I$ , and  $\hat{\psi}(\cdot)$  is the solution of the adjoint equation (20).

## 6. VOLTERRA INTEGRAL PROCESSES

What differences appear if the process equation (3) is a Volterra integral equation

$$x(s) = h(s) + \int_0^s f(s, t, x(t), u(s)) dt, \quad s \in I \quad (21)$$

and  $h(\cdot) \in PCL(I, E)$ ? In this case the state function  $x(\cdot)$  is also piecewise continuous for every piecewise continuous control  $u(\cdot)$ . Put

$$G(x, w)(s) = x(s) - h(s) - \int_0^s f(s, t, x(t), u(s))dt, \\ s \in I, \quad X = PCL(I, E), \quad x \in X \quad (22)$$

Let be  $\hat{u}(\cdot), \hat{x}(\cdot)$  an optimal solution of (1), (21) and  $u(\cdot) \in PCL(I, U)$ . Choose  $\tau \in ]0, T[$ ,  $v \in U$ ,  $1 \gg \mu > 0$  and denote  $\Sigma$ ,  $X_\Sigma \leq PCL(I, U)$ ,  $W_{\Sigma, v}$  as above. Then  $\hat{u}(\cdot), \hat{x}(\cdot)$  is optimal for the process

$F(x, w) = \min!$  subject to  $G(x, w) = 0$ ,  $u(\cdot) \in W_{\Sigma, v}$ ,  $x(\cdot) \in X_\Sigma$ . Under the previous assumptions of Fréchet-differentiability and continuity for  $f$  and  $g$  we get

$$(G_x(x, w)z)(s) = z(s) - \int_0^s f_e(s, t, x(t), u(s))z(t)dt, \\ s \in I, \quad z \in X_\Sigma \quad (23)$$

The linearized process equation corresponding to the (optimal) solution  $\hat{u}(\cdot), \hat{x}(\cdot)$

$$z(s) = \int_0^s \hat{f}_e(s, t)z(t)dt + b(s), \quad s \in I$$

is uniquely solvable for every  $b \in X_\Sigma$ , as the spectral radius of a Volterra integral operator is 0. Let be  $U$  also convex. For controls  $u_\epsilon(\cdot)$  defined by (15)  $W_{\Sigma, v} = \{u_\epsilon(\cdot) \mid 0 \leq \epsilon \leq 1\}$  is a set of varied controls in the sense of [1].

We can prove Lemmas 3, 5 repeating all estimations. We obtain

$$\delta G(s) = \begin{cases} 0 & s \notin I(\tau, \mu) \\ \frac{s - \tau + \mu}{\mu} \int_0^s \hat{f}_u(s, t)dt[v - \hat{u}(\tau)] & \tau - \mu \leq s \leq \tau \\ \frac{\tau - s + \mu}{\mu} \int_0^s \hat{f}_u(s, t)dt[v - \hat{u}(\tau)] & \tau \leq s \leq \tau + \mu. \end{cases}$$

The resolvent operator  $R(s, t)$  corresponding to  $A(s, t) = \hat{f}_e(s, t)$  exists and it holds

$$R(s, t) = A(s, t) + \int_t^T R(s, w)A(w, t)dw \\ R(s, t) = A(s, t) + \int_0^s A(s, w)R(w, t)dw, \text{ consequently } R(s, t) = \\ = 0 \text{ for } t > s \quad (8a)$$



From (9) we find

$$\delta F - \int_0^T \hat{\psi}(s) \delta G(s) ds \geq 0 \quad \text{where} \quad \hat{\psi}(s) = -\hat{g}_e(s) - \int_0^T \hat{g}_e(t) R(t, s) dt$$

Hence

$$\begin{aligned} \hat{\psi}(t) &= -\hat{g}_e(t) - \int_t^T \hat{g}_e(s) A(s, t) ds - \int_t^T \int_t^T \hat{g}_e(s) R(s, w) A(w, t) dw ds \\ &= -\hat{g}_e(t) + \int_t^T [-\hat{g}_e(s) - \int_t^T \hat{g}_e(w) R(w, s) dw] A(s, t) ds = \\ &= -\hat{g}_e(t) + \int_t^T \hat{\psi}(s) \hat{f}_e(s, t) ds. \end{aligned}$$

We rewrite the optimality condition in the form

$$\begin{aligned} &\int_{\tau-\mu}^{\tau} \frac{s-\tau+\mu}{\mu} [\hat{g}_u(s) - \hat{\psi}(s) \int_0^s \hat{f}_u(s, t) dt] ds [v - \hat{u}(\tau)] + \\ &+ \int_{\tau}^{\tau+\mu} \frac{\tau-s+\mu}{\mu} [\hat{g}_u(s) - \hat{\psi}(s) \int_0^s \hat{f}_u(s, t) dt] ds [v - \hat{u}(\tau)] \geq 0. \end{aligned}$$

The integrand is continuous for a.e.  $\tau \in [0, 1]$  and  $\mu$  can be chosen sufficiently small, consequently

$$[\hat{g}_u(\tau) - \hat{\psi}(\tau) \int_0^{\tau} \hat{f}_u(\tau, t) dt] [v - \hat{u}(\tau)] \geq 0 \quad \text{for } 0 < \tau < T$$

and it is also true for  $\tau = 0$ ,  $\tau = T$ . Thus we have proved.

**THEOREM 4.** A necessary condition for  $\hat{u}(\cdot)$ ,  $\hat{x}(\cdot)$  to be an optimal solution of (1), (21) in the class of piecewise continuous controls is

$$[\hat{g}_u(\tau) - \hat{\psi}(\tau) \int_0^{\tau} \hat{f}_u(\tau, t) dt] \hat{u}(\tau) = \min_{v \in U} [\hat{g}_u(\tau) - \hat{\psi}(\tau) \int_0^{\tau} \hat{f}_u(\tau, t) dt] v \quad \text{for a.e. } \tau \in [0, T] \quad (24)$$

where  $\hat{\psi}(\cdot) \in PCL(I, E^*)$  is the solution of the adjoint equation

$$\dot{\psi}(t) = -\hat{g}_e(t) + \int_t^T \hat{\psi}(s) \hat{f}_e(s, t) ds, \quad t \in I. \quad (25)$$

**COROLLARY.** If  $h(\cdot) \in C(I, E)$  and controls  $u(\cdot)$  are continuous, we set  $X = C(I, E)$  and  $W_{\tau, \mu, v} \subseteq C(I, U)$  and get the minimum principle of Theorem 4 for every  $\tau \in I$ .

**THEOREM 5.** Let be  $\hat{u}(\cdot)$ ,  $\hat{x}(\cdot)$  an optimal solution of: minimize  
(3) subject to

$$x(s) = h(s) + \int_0^s f^1(s, t, x(t))dt + \int_0^T f^2(s, t, u(s))dt$$

$$x(\cdot) \in PCL(I, E), \quad u(\cdot) \in PCL(I, U). \quad (26)$$

We assume that  $U$  is convex,  $h \in PCL(I, E)$ ,  $f_e^1, f_u^2, g_e, g_u$  exist (and are continuous) for all admissible arguments. Then

$$[\hat{g}_u(\tau) - \hat{\psi}(\tau) \int_0^T \hat{f}_u^2(\tau, t)dt] \hat{u}(\tau) = \min_{v \in U} [\hat{g}_u(\tau) - \hat{\psi}(\tau) \int_0^T \hat{f}_u^2(\tau, t)dt]v \quad \text{for a.e. } \tau \in I \quad (27)$$

and  $\psi(\cdot)$  is the solution of

$$\psi(t) = -g_e(t) + \int_t^T \psi(s) f_e^1(s, t)ds \quad (28)$$

**Proof.** This result is obtained with

$$F(x, w) = \int_0^T g(t, x(t), u(t))dt, \quad G(x, w)(s) = x(s) - h(s) - \int_0^s f^1(s, t, x(t))dt - \int_0^T f^2(s, t, u(s))dt, \quad X = PCL(I, E),$$

$$W = PCL(I, U)$$

when for fixed  $\tau \in I$ ,  $1 \gg \mu > 0$ ,  $v \in U$  the problem is restricted to

$$F(x, w) = \min \text{ subject to } G(x, w) = 0, \quad x \in X_\Sigma, \quad w \in W_{\Sigma, v},$$

and where  $\Sigma$  is the finite set of points of discontinuity of  $\hat{u}(\cdot)$  or  $h(\cdot)$ .

**REMARK 4.** A corresponding condition holds for

1)  $h \in C(I, E)$ ,  $x(\cdot) \in C(I, E)$ ,  $u(\cdot) \in C(I, U)$  and then for all  $\tau \in I$ .

2)  $h \in PCL(I, E)$ ,  $x(\cdot) \in PCL(I, E)$ ,  $u(\cdot) \in C(I, U)$  for a.e.  $\tau \in I$

3) varied controls can be combined, f.i. (Figure 2).

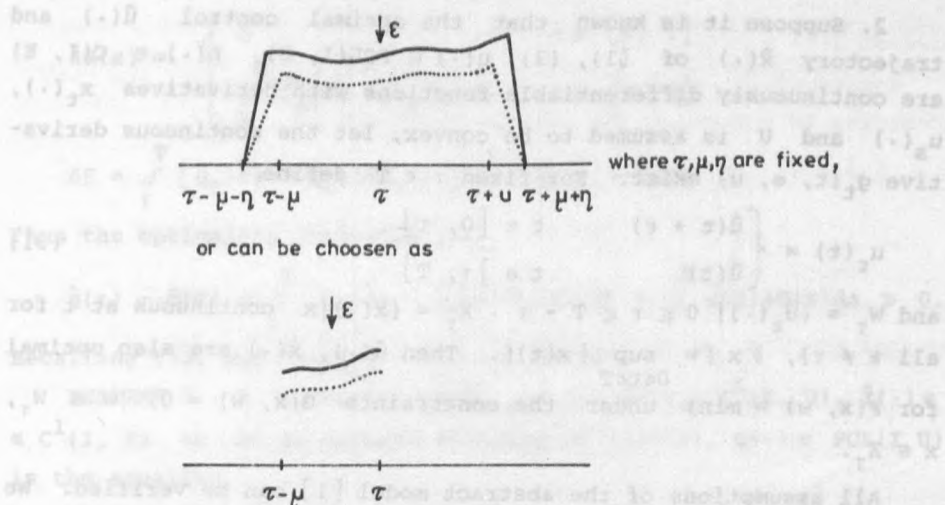


Fig. 2

## 7. OUTLOOK

1. It is possible to derive any optimality condition, if  $U$  is not convex?

Let  $\hat{u}(\cdot)$ ,  $\hat{x}(\cdot)$  be admissible for (1), (3), (4). Define for every  $\tau \in I$  the directional cone

$$K(\tau, U) = \{v \in U \mid \exists \mu > 0, \varepsilon(\tau, \mu) > 0: \hat{u}(t) + \varepsilon v \in U \text{ for } t \in I(\tau, \mu), 0 \leq \varepsilon \leq \varepsilon(\tau, \mu)\}.$$

For fixed  $\tau$ ,  $1 \gg \mu > 0$ ,  $v \in K(\tau, U)$  define varied controls

$$u_\varepsilon(t) = \begin{cases} \hat{u}(t) & t \notin I(\tau, \mu) \\ \hat{u}(t) + \varepsilon v & t \in I(\tau, \mu) \end{cases} \quad (29)$$

$X = X_\Sigma$ ,  $W = \{u_\varepsilon(\cdot) \mid 0 \leq \varepsilon \leq \varepsilon(\tau, \mu)\}$  in the sense used above.

Then we can prove in the same fashion:

**THEOREM 6.** Necessary for the optimality of  $\hat{u}(\cdot)$ ,  $\hat{x}(\cdot)$  with respect to (1), (3), (4) is

$$[\hat{g}_u(\tau) - \hat{\psi}(\tau) \int_0^T \hat{f}_u(\tau, t) dt] v \geq 0 \text{ for all } v \in K(\tau, U), \text{ and for a.e. } \tau \in I \quad (30)$$

where  $\hat{\psi}(\cdot)$  is the adjoint function.

2. Suppose it is known that the optimal control  $\hat{u}(\cdot)$  and trajectory  $\hat{x}(\cdot)$  of (1), (3)  $u(\cdot) \in \text{PCL}(I, U)$ ,  $h(\cdot) \in C(I, E)$  are continuously differentiable functions with derivatives  $x_t(\cdot)$ ,  $u_s(\cdot)$  and  $U$  is assumed to be convex, let the continuous derivative  $g_t(t, e, u)$  exist. For fixed  $\tau < T$  define

$$u_\varepsilon(t) = \begin{cases} \hat{u}(t + \varepsilon) & t \in [0, \tau] \\ \hat{u}(t) & t \in ]\tau, T] \end{cases} \quad (31)$$

and  $W_\tau = \{u_\varepsilon(\cdot) \mid 0 \leq \varepsilon \leq T - \tau\}$ .  $X_\tau = \{x(\cdot) \mid x \text{ continuous at } t \text{ for all } t \neq \tau\}$ ,  $\|x\| = \sup_{0 \leq t \leq T} \|x(t)\|$ . Then  $\hat{u}(\cdot)$ ,  $\hat{x}(\cdot)$  are also optimal for  $F(x, w) = \min!$  under the constraints  $G(x, w) = 0$ ,  $w \in W_\tau$ ,  $x \in X_\tau$ .

All assumptions of the abstract model [1] can be verified. We calculate

$$\delta G(s) = \begin{cases} \int_0^T \hat{f}_u(s, t) \hat{u}_s(s) dt & s \in [0, \tau] \\ 0 & s \in ]\tau, T] \end{cases}$$

and

$$\delta F = - \int_0^\tau [\hat{g}_t(t) + \hat{g}_e(t) \hat{x}_t(t)] dt + \hat{g}(\tau) - \hat{g}(0).$$

From (9) the following optimality condition can be obtained:

$$- \int_0^\tau [\hat{g}_t(t) + \hat{g}_e(t) \hat{x}_t(t)] dt + \hat{g}(\tau) - \hat{g}(0) - \int_0^\tau \hat{\psi}(s) \delta G(s) ds \geq 0.$$

Hence if  $\tau$  tends to  $T$

$$\begin{aligned} & \hat{g}(T) - \hat{g}(0) - \int_0^T [\hat{g}_t(t) + \hat{g}_e(t) \hat{x}_t(t)] dt - \\ & - \int_0^T \hat{\psi}(s) \int_0^T \hat{f}_u(s, t) \hat{u}_s(s) dt ds \geq 0 \end{aligned} \quad (32)$$

Replacing  $u_\varepsilon(\cdot)$  by

$$\tilde{u}_\varepsilon(t) = \begin{cases} u(t) & t \in [0, \tau] \\ u(t - \varepsilon) & t \in ]\tau, T] \end{cases}$$

and putting a new set  $\tilde{W}_\tau = \{\tilde{u}_\varepsilon(\cdot) \mid 0 \leq \varepsilon \leq T - \tau\}$  and a new problem  $F(x, w) = \min!$  subject to  $G(x, w) = 0$ ,  $w \in \tilde{W}_\tau$ ,  $x \in X_\tau$ , we obtain

$$\delta G(s) = \begin{cases} 0 & t \in [0, \tau] \\ -\int_0^T \hat{f}_u(s, t) \hat{u}_s(s) dt & t \in ]\tau, T] \end{cases}$$

$$\delta F = \int_{\tau}^T [\hat{g}_t(t) + \hat{g}_e(t) \hat{x}_t(t)] dt + \hat{g}(\tau) - \hat{g}(T).$$

Thus the optimality condition is

$$\hat{g}(\tau) - \hat{g}(T) + \int_{\tau}^T [\hat{g}_t(t) - \hat{g}_e(t) \hat{x}_t(t)] dt - \int_{\tau}^T \hat{\psi}(s) \delta G(s) ds \geq 0.$$

Recalling (32) and letting  $\tau \rightarrow 0$  we obtain

**THEOREM 7.** A necessary condition for  $\hat{u}(\cdot) \in C^1(I, U)$ ,  $\hat{x}(\cdot) \in C^1(I, E)$  to be an optimal solution of (1), (3),  $u(\cdot) \in PCL(I, U)$  is the equality

$$\begin{aligned} & \hat{g}(0) + \int_0^T [\hat{g}_t(t) + \hat{g}_e(t) \hat{x}_t(t)] dt + \\ & + \int_0^T \hat{\psi}(s) \int_0^T \hat{f}_u(s, t) u_s(s) dt ds = \hat{g}(T) \end{aligned} \quad (33)$$

where  $\hat{\psi}(\cdot)$  is the solution of (20).

**Example.** Consider

$$\int_0^1 [tx(t) - \frac{1}{2}u(t)] dt = \min! \quad \text{subject to}$$

$$x(s) = -\int_0^1 su(s)^2 dt + \frac{1}{2} \int_0^1 x(t) dt, \quad |u| \leq 1.$$

To the controls  $u_1(s) = 1$  and  $u_2(s) = -1$  corresponds the state function  $x(s) = -s - \frac{1}{2}$ ,  $s \in [0, 1]$ , and it is easily shown that (33) is valid. Also for  $u_3(\cdot) = 0$  (33) is true. Condition (33) is not a powerful tool to find optimal controls!

If  $u(s) = s$  then  $x(s) = -s^3 - \frac{1}{4}$ . The adjoint equation is

$$\psi(t) = -t + \int_0^1 \psi(s) \frac{1}{2} ds \quad (\text{independent on } u(\cdot)) \quad \text{and has the so-}$$

lution  $\psi(t) = -t - \frac{1}{2}$ . Then condition (33) is

$$\begin{aligned} g(1) &= 0 + \int_0^1 [-t^3 - \frac{1}{4} - t \cdot 3t^2] dt + \int_0^1 [-s - \frac{1}{2}] \int_0^1 (-2s^2) 1 dt ds \\ &= -\frac{5}{12}, \end{aligned}$$

however  $g(1) = -\frac{7}{4}$ .

Consequently  $u(s) = s$ ,  $s \in I$  is not optimal. By the way, according to theorem 1 the minimum principle is

$$\begin{aligned} & \left[ -\frac{1}{2} + \left( \tau + \frac{1}{2} \right) (-2\tau) \cdot \hat{u}(\tau) \right] \hat{u}(\cdot) = \\ & = \min_{-1 \leq v \leq 1} \left[ -\frac{1}{2} + \left( \tau + \frac{1}{2} \right) (-2\tau) \hat{u}(\tau) \right] \cdot v, \quad 0 \leq \tau \leq 1. \end{aligned}$$

This condition is valid for  $u(\tau) = +1$ . We see that for  $\tau \ll 1$  the optimal control has to be the value +1 and  $\hat{u}(\tau) = 1$  too, if

$$\left[ -\frac{1}{2} - 2\tau \left( \tau + \frac{1}{2} \right) \hat{u}(\tau) \right] < 0.$$

3. Sometimes the set of all CDL is a convex cone. Then it is possible to take side conditions into considerations. We give an example of such a process. Let be  $E_T$  a given convex set in  $E$  and  $\text{int } E_T \neq \emptyset$ .

The problem under consideration is:

minimize  $\int_0^T g(t, x(t), u(t)) dt$  under the conditions

$$\begin{aligned} x(s) &= h(s) + \int_0^s f(s, t, x(t), u(t)) dt, \quad s \in I, \\ x(\cdot) &\in \text{PCL}(I, E), \quad u(\cdot) \in \text{PCL}(I, U), \\ x(T) &\in E_T \end{aligned} \quad (34)$$

We assume:  $f_e, f_u, g_e, g_u$  exist and are continuous.

Let  $\hat{u}(\cdot), \hat{x}(\cdot)$  be an optimal solution. There exist numbers  $0 = s_0 < s_1 < \dots < s_m < s_{m+1} = T$  such that  $\hat{x}(\cdot), \hat{u}(\cdot), h(\cdot)$  are continuous on every interval  $I_i = ]s_i, s_{i+1}]$ ;  $i = 0, \dots, m$ . We suppose the existence of convex cones  $K_i \subseteq S$  and positive numbers  $\delta_i$  with the property  $\hat{u}(t) + \varepsilon \frac{v}{\|v\|} \in U$  for all  $v \in K_i$ ,  $0 \leq \varepsilon \leq \delta_i$  and all  $t \in I_i$ ,  $i = 0, \dots, m$ . Denoting  $\Sigma = \{s_1, \dots, s_m\}$  we set  $X_\Sigma, W = \text{PCL}(I, U)$ ,  $f(x, w), G(x, w)$  in the previous sense and  $M = \{x(\cdot) \in X_\Sigma | x(T) \in E_T\}$ .

Then  $\hat{u}(\cdot), \hat{x}(\cdot)$  is also an optimal solution of the problem: minimize  $F(x, w)$  subject to  $G(x, w) = 0$ ,  $x \in X_\Sigma, w \in W_\Sigma$ .  $M$  is



a convex subset of  $X_\Sigma$ . Necessary optimality conditions for this abstract problem are given in [1]. Fredholm integral processes of type (0), (1),  $x(\cdot) \in C(I, E)$ ,  $u(\cdot) \in PCL(I, U)$ ,  $x(T) \in E_T$  are considered in [7].

For every  $a(\cdot) \in PCL(I, S)$  with  $a|_{I_k} \in C(I_k, S_k)$ ,  $k = 0, \dots, m$ , we define a set of varied controls by

$$W_0 = \{u_\varepsilon(\cdot) | u_\varepsilon(\cdot) = \hat{u}(\cdot) + \varepsilon \frac{a(\cdot)}{\|a(\cdot)\|}, \quad 0 \leq \varepsilon \leq \min_{i=0, m} \delta_i\}.$$

Without verifying the Lemmas 3-5 we calculate for every such  $a(\cdot)$ ,

$$\varepsilon_k \searrow 0, \quad \gamma_k = \varepsilon_k \|a(\cdot)\|^{-1} \quad \text{the CDL and obtain}$$

$$\delta G(s) = \int_0^s \hat{f}_u(s, t) dt a(s), \quad s \in I$$

$$\delta F = \int_0^T \hat{g}_u(t) a(t) dt$$

The set of all these CDL is a convex cone. According to [1] a necessary optimality condition holds:

LEMMA 7. There exist a nonnegative number  $\delta$  and a linear functional

$$e^* \in E^*, \quad \delta + \|e^*\| > 0, \quad \text{such that}$$

$$e^* \delta G(T) + \rho \delta F - \int_0^T \hat{\psi}(s) \delta G(s) ds \geq 0 \quad \text{for all CDL } \delta F, \delta G \quad (35)$$

and the transversality condition

$$e^* \hat{x}(T) \geq e^* e \quad \text{for all } e \in E_T \quad \text{holds} \quad (36)$$

$\hat{\psi}(\cdot)$  is a solution of the adjoint equation

$$\psi(t) = \int_t^T \psi(s) \hat{f}_e(s, t) ds - \rho \hat{g}_e(t) - e^* \hat{f}_e(T, t), \quad t \in I \quad (37)$$

Proof: (35) follows from the general condition (see [1])

$$x^*([G_x(\hat{x}, \hat{w})^{-1} \delta G](T)) + \rho F_x(\hat{x}, \hat{w}) G_x(\hat{x}, \hat{w})^{-1} \delta G + \rho \delta F \geq 0,$$

where  $x^* \in [PCL(I, E)]^*$ ,  $\|x^*\| + \rho > 0$ ,  $x^* \hat{x}(\cdot) \geq x^* x(\cdot)$  for all  $x(\cdot) \in M$ , by substituting  $z = G_x(\hat{x}, \hat{w})^{-1} \delta G$ . By means of the resolvent operator we have  $z(s) = \delta G(s) + \int_0^s R(s, t) \delta G(t) dt$ ,  $s \in I$ .

The functional  $x^*$  can be represented by a Stieltjes integral. From  $x^*\hat{x}(\cdot) \geq x^*x(\cdot)$  for  $x(\cdot) \in M$  it follows the existence of  $e^* \in E^*$ , such that  $x^*x(\cdot) = e^*x(T)$ . Hence

$$\begin{aligned} e^*\delta G(T) + e^*\int_0^T R(T, t)\delta G(t)dt + \rho \int_0^T \hat{g}_e(t)\delta G(t)dt + \\ + \rho \int_0^T \hat{g}_e(t) \int_0^t R(t, w)\delta G(w)dw dt + \rho\delta F \geq 0 \end{aligned} \quad (38)$$

and therefore

$$\begin{aligned} e^*\delta G(T) + \rho\delta F + \int_0^T [e^*R(T, t) + \rho \hat{g}_e(t) + \\ + \rho \int_t^T \hat{g}_e(w)R(w, t)]\delta G(t)dt \geq 0. \end{aligned}$$

Applying (8) we obtain that

$$\hat{\psi}(t) = -e^*R(T, t) - \rho \hat{g}_e(t) - \rho \int_t^T \hat{g}_e(w)R(w, t)dw \text{ is the uni-}$$

que solution of (37) in  $PCL(I, E^*)$ .

Considering for  $v \in K_k$ ,  $\tau \in ]s_k, s_{k+1}]$ ,  $0 < \mu < 1$ ,  $k = 0, \dots, m$  special functions

$$a(t) = \begin{cases} 0 & t \notin I(\tau, \mu) \\ \frac{t - \tau + \mu}{\mu} v & t \in [\tau - \mu, \tau] \\ \frac{-t + \tau + \mu}{\mu} v & t \in [\tau, \tau + \mu] \end{cases}$$

from (38) we obtain with the help of the corresponding CDL  $\delta G$ ,  $\delta F$

$$\begin{aligned} \int_{\tau-\mu}^{\tau} \{ [\rho \hat{g}_u(s) - \hat{\psi}(s) \int_0^s \hat{f}_u(s, t)dt] \frac{s - \tau + \mu}{\mu} ds \cdot v - \\ - \int_{\tau}^{\tau+\mu} [\rho \hat{g}_u(s) - \hat{\psi}(s) \int_0^s \hat{f}_u(s, t)dt] \frac{s - \tau - \mu}{\mu} ds \cdot v \geq 0. \end{aligned}$$

It follows

$$[\rho \hat{g}_u(\tau) - \hat{\psi}(\tau) \int_0^{\tau} \hat{f}_u(\tau, t)dt] v \geq 0 \text{ for all } v \in K_k, s_k < \tau \leq s_{k+1}, k = 0, \dots, m.$$

This inequality is valid for  $\tau = s_0, \dots, s_{m+1}$ , too, as all terms are continuous in  $]s_k, s_{k+1}]$ . Put now for  $0 < \mu < 1$  and  $v \in K_m$

$$a(t) = \begin{cases} 0 & t \in [0, T - \mu] \\ \frac{t - T + \mu}{\mu} v & t \in [T - \mu, T]. \end{cases}$$

(38) can be written as

$$\left[ e^* \int_0^T \hat{f}_u(T, t) dt + \int_{T-\mu}^T (\rho \hat{g}_u(s) - \hat{\psi}(s) \int_0^s \hat{f}_u(s, t) dt) \right.$$

$$\left. \frac{s - T + \mu}{\mu} ds \right] v \geq 0.$$

Thus we have proved:

THEOREM 8. If  $\hat{u}(\cdot)$ ,  $\hat{x}(\cdot)$  are optimal with respect to (1), (21), (34) under the assumptions mentioned above there exist a number  $\rho \geq 0$  and a functional  $e^* \in E^*$ ,  $\rho + \|e^*\| > 0$ , such that

$$\left[ \rho \hat{g}_u(s) - \hat{\psi}(s) \int_0^s \hat{f}_u(s, t) dt \right] v > 0 \text{ for all } v \in K_k, s_k < s \leq s_{k+1}, k = 0, \dots, m,$$

$$e^* \int_0^T \hat{f}_u(T, t) dt v \geq 0 \text{ for all } v \in K_m,$$

$$e^* \hat{x}(T) = \max_{e \in E_T} e^* e$$

and  $\hat{\psi}(\cdot)$  is the solution of (37).

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#### WARUNKI KONIECZNE OPTYMALIZACJI DLA PROCESÓW OPISANYCH SPECJALNYMI UKŁADAMI CAŁKOWYMI

Prezentowany artykuł przedstawia procesy sterowania opisane przez nieliniowe równania całkowe typu (2). Zostały udowodnione pewne konieczne warunki optymalności. Rozważa się różne równania całkowe, w tym typu Volterra. Rezultat J. Pełczewskiego przedstawiono w równoważnej formie.