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REFLECTIONS ON WALSH EQUICONVERGENCE AND EQUISUMMABILITY

To Professor Lech Włodarski on His 80th birthday

We present here a brief survey of some recent results on Walsh equiconvergence and its extension to equisummability by R. Brück. We also show how the results of Brück on equisummability can be further extended for a general class of linear operators on functions in A_R. A simple case of $(0,2)$ interpolation is given as an example.

1. Introduction

The theorem of Walsh which is the source for the type of problems to be considered here was proved about 60 years ago and is very simple to prove and easy to state. In its simplest form the theorem of Walsh deals with functions of class A_R $(R > 1)$ which are analytic in a disc $D_R = \{z : |z| < R\}$ but not in $\overline{D_R}$. The theorem asserts that

$$\lim_{n\to\infty} \Delta_n(z; f) = 0 \quad \text{in the disc} \quad D_{R^2}$$

where $\Delta_n(z; f)$ is the difference between the Lagrange interpolant to f in the n^{th} roots of unity and the Taylor polynomial of degree $n-1$ for f about the origin. It is known that both the Lagrange interpolant

$L_{n-1}(f;z)$ and the Taylor section $S_{n-1}(f;z)$ of f converge to f in the disc D_R, so it is surprising to see that the difference tends to zero in a larger region.

A straight forward extension of this theorem is given by

Theorem 1. *Let $f(z) = \sum_{k=0}^{\infty} a_k z^k \in A_R$. If for any integer $l \geq 1$, we set the difference*

$$\Delta_{l,n-1}(f;z) := L_{n-1}(f;z) - Q_{n-1,l}(f;z) \tag{1}$$

where

$$Q_{n-1,l}(f;z) = \sum_{j=0}^{l-1} \sum_{k=0}^{n-1} a_{k+jn}\, z^k,$$

then

$$\lim_{n\to\infty} \Delta_{l,n-1}(f;z) = 0 \quad \text{for} \quad |z| < R^{1+l}. \tag{2}$$

The convergence in (2) is uniform and geometric in any compact subset of the disc $D_{R^{1+l}}$. Moreover the result is best possible in the sense that for any point z_0 with $|z_0| = R^{1+l}$, there is a function $f(z) \in A_R$ for which (2) fails when $z = z_0$.

Many extensions of this result have been given and analogous result have been obtained for Hermite and lacunary interpolation. Extensions of this theorem for meromorphic functions with ν poles have been given [3]. Here $M_R(\nu)$ denotes the class of functions $F(z)$ which have the representation $f(z)/B_\nu(z)$, where $f(z) \in A_R$ and $B_\nu(z)$ is a monic polynomial of degree ν with exactly ν zeros in D_R. Let $P(z)/Q(z)$ be a (n,ν) rational function which interpolates $F(z)$ in the $n+\nu+1$ roots of unity. Let $R(z)/S(z)$ be the (n,ν) Hermite-Padé interpolant to $F(z)$ at the origin.

We suppose that $B_\nu(z)$ has no zero at the origin and on the unit circle. The extension of Walsh theorem for function in $M_R(\nu)$ states that if

$$\Delta_{n,\nu}(f;z) := \frac{P(z)}{Q(z)} - \frac{R(z)}{S(z)}, \quad \text{then} \tag{3}$$

$$\lim_{n\to\infty} \Delta_{n,\nu}(f;z) = 0 \quad \text{for} \quad z \in D_{R^2} \setminus \{z_j\}_1^\nu, \tag{4}$$

where $z_1, \ldots, z_\nu$ are the poles of $F(z)$. Again, the convergence is uniform and geometric in the region D_{R^2} with the poles deleted.

A cursory examination of these results leads one to ask if the region of convergence can be extended by using a suitable methods of summability. This problem was raised by R. Brück and is resolved by him in his dissertation [1]. He applied methods of summability to the theorem of Walsh and some of its extensions and has determined the regions where the operators are summable by a suitable process of summability. He calls it the region of equisummability as compared to the region of equiconvergence. Brück shows that the region of equiconvergence is always contained in the region of equisummability.

In 1993, we considered a general class of linear operators on functions in A_R and under fairly general conditions determined the Walsh radius of equiconvergence and Walsh region of equisummability. Here we discuss briefly the class of $(0,2)$ interpolation on the n^{th} roots of unity and show how to find the region of equiconvergence in the case of $(0, m_1, \ldots, m_q)$ interpolation on the n^{th} roots unity. The regions of equiconvergence in this general case is already known, but the region of equisummability is not known. The case of lacunary interpolation was not treated by Brück.

2. Summability methods

We shall consider summability methods of the following form. Let $X \subset \mathbb{R}$, $x^* \in \mathbb{R}$ be an accumulation point of X and let $\{a_n(x)\}$ be a sequence of functions on X. If (s_n) is a sequence of complex numbers, we put formally

$$\sigma(x) = \sum_{n=0}^{\infty} a_n(x) s_n, \quad x \in X.$$

We say that the sequence (s_n) is summable A to the value $s \in \mathbb{C}$ and we write $A - \lim_{n\to\infty} s_n = s$, if the above sum converges for all $x \in X$ and $\sigma(x) \to s$ as $x \to x^*$. We make the following assumptions on the summability method A. For each $x \in X$, let the power series $\varphi(x,z) = \sum_{n=0}^{\infty} a_n(x) z^n$ be an entire function. Furthermore, let $G_A \subset \mathbb{C}$ be a region such that the unit disc $D \subset G_A$, $1 \notin G_A$, G_A being star-shaped and

$$\lim_{x \to x^*} \varphi(x,z) = 0 \quad \forall z \in G_A,$$

the convergence being uniform on compact (closed and bounded) subsets of G_A, i.e. in particular $A - \lim_{n\to\infty} z^n = 0$, $\forall z \in G_A$. Finally, we require that $\lim_{x\to x^*} \varphi(x,1) = 1$.

Let $S \equiv S_f$ be the Mittag Leffler star domain of function $f \in A_R$. Let $G \equiv G_A$ be the summability domain associated with the summability method $(a_n(x))$. We shall also use the notation that if $\varphi_\nu(z) = z^\nu$ for $\nu \in \mathbb{N}$, then φ_ν^{-1} means the preimage.

3. Representation of the kernel $K_n(z,t)$

In the case of $(0,2)$ interpolation on the n^{th} roots of unity the difference of the interpolant and the Taylor section of degree $2n-1$ can be written in the form of an integral. Thus we get

$$I_n(z;f) - S_{2n}(z,f) = \frac{1}{2\pi i} \int\limits_{|t|=r_0} f(t) K_n(z,t)dt, \quad |z| > |t| = r_0 > 1$$

where

$$K_n(z,t) = L_n^{(1)}(z,t) + L_n^{(2)}(z,t) + L_n^{(3)}(z,t) - L_n^{(4)}(z,t)$$

and

$$(1) \quad \begin{cases} L_n^{(1)}(z,t) = \frac{t^n - z^n}{(t^n-1)(t-z)}, & L_n^{(2)}(z,t) = \frac{(z^n-1)(t^n-z^n)}{(t^n-1)^2(t-z)} \\ L_n^{(3)}(z,t) = \frac{2n(z^n-1)}{(t^n-1)^3} \sum\limits_{j=0}^{n-1} \frac{t^{n-j-1}z^j}{2j+n-1}, & L_n^{(4)}(z,t) = \frac{t^{2n}-z^{2n}}{t^{2n}(t-z)}. \end{cases}$$

It is easy to see that

$$\text{(1a)}\quad (t-z)\left(L_n^{(1)}(z,t)+L_n^{(2)}(z,t)\right) = -\sum_{k=0}^{\infty}\frac{k-1}{t^{kn}}+\left(\frac{z}{t}\right)^n\sum_{k=0}^{\infty}\frac{2k}{t^{kn}}-\left(\frac{z}{t}\right)^{2n}\sum_{k=0}^{\infty}\frac{k+1}{t^{kn}}.$$

In order to put $L_n^{(3)}(z,t)$ in a suitable form, we denote by γ any rectifiable Jordan curve beginning at 0 and ending at 1. Then, since the functions we are dealing with are entire, we have

$$\sum_{j=0}^{n-1}\frac{1}{2j+n-1}\left(\frac{z}{t}\right)^j = \int_\gamma \sum_{j=0}^{n-1}\left(\frac{z}{t}\right)^j u^{2j+n-2}\,du$$

$$\text{(2)}\qquad = \int_\gamma \frac{\left(\frac{z}{t}\right)^n u^{3n}-u^n}{u^2\left(\frac{z}{t}u^2-1\right)}du.$$

We now assume that $1 < r_0 = |t| < |z|$, $\left|\frac{t}{z}\right|^{1/2} < \eta < 1$ and that θ is a number satisfying $0 < \theta < \min(1, \frac{1}{\eta}-1)$. Then there is a rectifiable Jordan curve $\gamma_{z,t}(\theta)$ beginning at 0 and ending at 1 with the following properties:

(i) For each $u \in \gamma_{z,t}(\theta)$, we have $|u| \le 1$.
(ii) $L(\gamma_{z,t}(\theta)) \le 1 + 2\pi(1-\eta) < 2\pi + 1$ where $L(\gamma)$ denotes the length of γ.
(iii) For each $u \in \gamma_{z,t}(\theta)$, $\left|\frac{z}{t}u^2 - 1\right| \ge \theta^2$.
(iv) For each $u \in \gamma_{z,t}(\theta)$, $|\arg u| \le \pi\theta$.

(For details we refer to [4].) Choose γ to be $\gamma_{z,t}(\theta)$ in (2) and we can write

$$I_n^{(3)}(z;f) := \frac{1}{2\pi i}\int_{|t|=r_0} f(t)L_n^{(3)}(z,t)dt$$

$$= \frac{1}{2\pi i}\int_{|t|=r_0} f(t)dt \int_{\gamma_{z,t}(\theta)} \frac{du}{\left(\frac{z}{t}u^2-1\right)}\, 2n\left(\sum_{j=1}^{3} S_n^{(j)}(z,t)+S_n^{(4)}(t)\right),$$

where $S_n^{(j)}(z,t)$ are infinite series in powers of z/t, while $S_n^{(4)}(t)$ is a power series in $1/t$, as given below:

$$S_n^{(1)}(z,t) = \sum_{k=0}^{\infty} \binom{k+2}{k} \frac{u}{t^{k+2}} \left(\frac{z}{t}\right)^2 \left(\left(\frac{z}{t}\right)^2 \frac{u^3}{t^{k+1}}\right)^{n-1}$$

$$S_n^{(2)}(z,t) = -\sum_{k=0}^{\infty} \binom{k+2}{k} \frac{1}{t^{2(k+1)+1}} \left(\frac{z}{t}\right)^2 \left(\frac{z}{t}\frac{u}{t^{k+1}}\right)^{n-2}$$

$$S_n^{(3)}(z,t) = -\sum_{k=0}^{\infty} \binom{k+2}{k} \frac{u}{t^{k+3}} \left(\frac{z}{t}\right) \left(\frac{z}{t}\frac{u^3}{t^{k+2}}\right)^{n-1}$$

$$S_n^{(4)}(z,t) = \sum_{k=0}^{\infty} \binom{k+2}{k} \frac{u^2}{t^{2(k+2)+1}} \left(\frac{u}{t^{k+2}}\right)^{n-2}.$$

Thus

$$I_n^{(3)}(z,f) = \sum_{j=1}^{3} A_n^{(j)}(z,f) + B_n^{(1)}(f) \tag{3}$$

where

$$A_n^{(j)}(z;f) = \frac{1}{2\pi i} \int_{|t|=r_0} f(t) S_n^{(j)}(z,f), \quad j = 1,2,3$$

and $B_n^{(1)}(f)$ is similarly defined by the kernel $S_n^{(4)}(t)$.

4. Walsh radius of equiconvergence

From (1) and (1a), we see that

$$(t-z)\big(L_n^{(1)}(z,f) + L_n^{(2)}(z,f) - L_n^{(4)}(z,f)\big)$$
$$= -\sum_{k=2}^{\infty} \frac{k-1}{t^{kn}} + \left(\frac{z}{t}\right)^n \sum_{k=1}^{\infty} \frac{2k}{t^{kn}} - \left(\frac{z}{t}\right)^{2n} \sum_{k=1}^{\infty} \frac{k+1}{t^{kn}}.$$

From these we can see that the Walsh radius determined by these is

$$R_{1,2}(r) = r^{3/2}.$$

The Walsh radius from $I_n^{(3)}(z, f)$ depends upon its components in (3) and it can be seen that

$$R_3(r) := \min\left(\inf_{k\geq 0}\left(r^{k+1}r^2\right)^{1/2}, \inf_{k\geq 0}\left(r^{k+1}\cdot r\right), \inf_{k\geq 0}\left(r^{k+2}\cdot r\right)\right).$$

Since the function $R_3(r)$ is strictly increasing and continuous for $r > 0$ and $R_3(r) > r$ when $r > 1$, we choose r_1 such that $R < r_1 < R_3(r)$. Since $R_3(r)$ is strictly increasing to $R_3(R)$ as $r \uparrow R$, there exists an r_0, $1 < r_0 < R$, such that $r_1 < R(r_0)$. In particular, we have $1 < r_0 < R < r_1 < R_3(r_0)$. We choose now $|t| = r_0$ and $|z| = r_1$. Then $\left|\frac{t}{z}\right| = \frac{r_0}{r_1} = \eta^2 < 1$. We shall assume that $0 < \theta < \min\left(1, \left(\frac{r_1}{r_0}\right)^{1/2} - 1\right)$. Then from the definition of $R_3(r)$, we get for $k = 0, 1, \ldots$ and $0 \leq |u| \leq 1$,

$$\frac{1}{\left(R_3(r_0)\right)^2} \geq \frac{1}{r_0^{k+1}r^2} = \frac{1}{|t|^{k+1}}\frac{1}{t^2} = \frac{1}{t^{k+1}}\left|\frac{z}{t}\right|^2 \frac{1}{|z|^2}$$

$$\geq \left|\frac{u^3}{t^{k+1}}\left(\frac{z}{t}\right)^2\right|\frac{1}{|z|^2}$$

i.e. $\frac{|z|^2}{(R_3(r_0))^2} \geq \left|\frac{u^3}{t^{k+1}} \cdot \left(\frac{z}{t}\right)^2\right|$.

Therefore we have

$$\left(A_n^{(1)}(z,t)\right) \leq \frac{1}{2\pi}2\pi r_0\left(\max_{|t|=r_0}|f(t)|\right)\cdot$$

$$(2\pi+1)\frac{1}{\theta^2}2n\left|\frac{z}{t}\right|^2\sum_{k=0}^{\infty}\binom{k+2}{k}\frac{1}{r_0^{k+1}}\left(\frac{|z|^2}{R_3(r_0)}\right)^{n-1}$$

$$\leq c_1 n\left(\frac{r_1}{R_3(r_0)}\right)^n\left(\frac{r_1}{R_3(r_0)}\right)^n = c_2\left(\frac{r_1}{R_3(r_0)}\right)^n.$$

A similar reasoning gives

$$\left|A_n^{(2)}(z,f)\right| \le c_2 \left(\left(\frac{r_1}{R_3(r_0)} \right)^{1/2} \right)^n$$
$$\left|A_n^{(3)}(z,f)\right| \le c_2 \left(\left(\frac{r_1}{R_3(r_0)} \right)^{1/2} \right)^n$$

and

$$|B_n^{(1)}(z,f)| \le c_1 \left(\frac{1}{r_0^2} \right)^n \to 0 \quad \text{as} \quad n \to \infty.$$

This proves that

$$R_3(r) = r^{3/2}.$$

5. Region of equisummability

If φ_ν maps z into z^ν, then φ_ν^{-1} is the preimage. With this convention, we define the following five sets $\mathcal{A}_i$ $(i = 1, \dots, 5)$ where

$$A_1 := \bigcap_{k\ge 0} \bigcap_{c \notin S} \varphi_1^{-1}(c \cdot c^{k+1} G_A),$$
$$A_2 := \bigcap_{k\ge 0} \bigcap_{c \notin S} \varphi_2^{-1}(c^2 \cdot c^{k+1} G_A),$$
$$A_3 := \bigcap_{k\ge 0} \bigcap_{c \notin S} \varphi_2^{-1}(c^2 \cdot c^{k+1} G_A),$$
$$A_4 := \bigcap_{k\ge 0} \bigcap_{c \notin S} \varphi_1^{-1}(c \cdot c^{k+1} G_A),$$
$$A_5 := \bigcap_{k\ge 0} \bigcap_{c \notin S} \varphi_1^{-1}(c \cdot c^{k+2} G_A), \quad \mathcal{E} := \bigcap_{j=1}^{5} A_j.$$

It can be shown in the case of $(0,2)$ interpolation on the n roots of unity the region of equisummability is given by $\mathcal{E}$.

For the details of the proof we refer the reader to [4] where a diagram is given which shows the region of equisummability when $f(z)$ has only one singularity at 2.

It will be interesting to see how the above method can be modified to deal with cases of $(0,3)$ or $(0,m)$ interpolation. We also wonder if in the case of the results in [3] for functions $\in M_R(\nu)$, a region of equisummability can be obtained.

References

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REFLEKSJE NAD RÓWNO-ZBIEŻNOŚCIĄ I RÓWNO-SUMOWALNOŚCIĄ W SENSIE WALSH'A

W tej pracy podajemy krótki przegląd najnowszych wyników dotyczących równo-zbieżności w sensie Walsh'a i ich rozszerzenia do równo-sumowalności Brück'a. Pokazujemy również jak wyniki Brück'a dotyczące równo-sumowalności można również rozszerzyć do ogólnej

klasy operatorów liniowych na funkcjach z A_R. Prosty przypadek $(0,2)$ interpolacji jest podany jako przykład.

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