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ON THE COEFFICIENTS OF THE UNIVALENT FUNCTIONS OF THE CLASSES N_1 AND N_2

In this paper we solve certain problems for the coefficients of $N_{1,2}$ classes of Nevanlinna analytic functions.

1. Let N_1 denote the class of Nevanlinna analytic functions

$$(1)\qquad f(z) = \int_{-1}^{1} \frac{d\mu(t)}{z - t} = \sum_{n=1}^{\infty} \frac{c_n}{z^n}, \quad z \notin \{z \mid -1 \le z \le 1\},$$

where $\mu(t)$ is a probability measure on $[-1, 1]$ and

$$(2)\qquad c_n = \int_{-1}^{1} t^{n-1}\, d\mu(t), \quad n = 1, 2, \ldots \quad (c_1 = 1).$$

Let N_2 denote the class of associated analytic functions

$$(3)\qquad \phi(z) \equiv f\left(\frac{1}{z}\right) \equiv \int_{-1}^{1} \frac{z\,d\mu(t)}{1-tz} = \sum_{n=1}^{\infty} c_n z^n$$

in the z-plane with the cuts $-\infty \le z \le -1$ and $1 \le z \le +\infty$. Certain properties of the coefficients (2) were examined in [1], where it was noted that the functions (1) and (3) are univalent for $|z| > 1$ and $|z| < 1$, respectively. Now we shall continue the study of the coefficients (2). Further, we shall indicate the class N_2 only.

THEOREM 1. For each arbitrary fixed positive integer $k \ge 1$, we have the sharp inequalities:

$$\text{(i)}\quad c_{2k} - \frac{2m - 2k + 1}{2m} t_1^{2k-1} \le c_{2m+1} \le c_{2k} + 2,$$

$$m = k,\ k+1,\ k+2,\ \ldots,\quad t_1 = \left(\frac{2k-1}{2m}\right)^{1/(2m-2k+1)};$$

$$\text{(ii)}\quad c_{2k} - \frac{2m-2k}{2m-1}\, t_2^{2k-1} \le c_{2m} \le c_{2k} + \frac{2m-2k}{2m-1}\, t_2^{2k-1},$$

$$m = k+1,\ k+2,\ k+3,\ \ldots,\quad t_2 = \left(\frac{2k-1}{2m-1}\right)^{1/(2m-2k)};$$

$$\text{(iii)}\quad c_{2k+1} - \frac{m-k}{m}\, t_3 \le c_{2m+1} \le c_{2k+1},$$

$$m = k+1,\ k+2,\ k+3,\ \ldots,\quad t_3 = \left(\frac{k}{m}\right)^{k/(m-k)};$$

$$\text{(iv)}\quad c_{2k+1} - 2 \le c_{2m} \le c_{2k+1},\quad m = k+1,\ k+2,\ k+3, \ldots$$

The equalities hold in (i)-(iv) only for the following extremal functions:

(a) on the left-hand side of (i), for the function

$$\phi(z) = \frac{z}{1-t_1 z} = \sum_{n=1}^{\infty} t_1^{n-1} z^n \in N_2,$$

and on the right-hand side of (i), for the function

$$\phi(z) = \frac{z}{1+z} = \sum_{n=1}^{\infty} (-1)^{n-1} z^n \in N_2; \tag{4}$$

(b) on the left-hand side of (ii), for the function

$$\phi(z) = \frac{z}{1-t_2 z} = \sum_{n=1}^{\infty} t_2^{n-1} z^n \in N_2,$$

and on the right-hand side of (ii), for the function

$$\phi(z) = \frac{z}{1+t_2 z} = \sum_{n=1}^{\infty} (-1)^{n-1} t_2^{n-1} z^n \in N_2;$$

(c) on the left-hand side of (iii), for the functions

$$\phi(z) = \frac{Az}{1+t_4 z} + \frac{(1-A)z}{1-t_4 z}$$

$$= \sum_{n=1}^{\infty} \left((-1)^{n-1} A t_4^{n-1} + (1-A) t_4^{n-1}\right) z^n \in N_2,$$

$$0 \le A \le 1,\quad t_4 = t_3^{1/2k},$$

and on the right-hand side of (iii), for the functions

$$\phi(z) = \frac{A_1 z}{1+z} + (1 - A_1 - A_2)z + \frac{A_2 z}{1-z}$$
$$= z + \sum_{n=2}^{\infty} ((-1)^{n-1} A_1 + A_2) z^n \in N_2,$$
$$A_{1,2} \geq 0, \quad 0 \leq A_1 + A_2 \leq 1;$$

(d) on the left-hand side of (iv), for the function (4), and on the right-hand side of (iv), for the functions

(5) $$\phi(z) = (1 - A)z + \frac{Az}{1-z} = z + \sum_{n=2}^{\infty} Az^n \in N_2, \quad 0 \leq A \leq 1.$$

P r o o f. The cases (i)-(iv) are proved analogously. For example, we shall prove the case (iii). By aid of (2) we obtain the identity

(6) $$c_{2m+1} - c_{2k+1} = \int_{-1}^{1} G(t)d\mu(t)$$

for a fixed integer $k \geq 1$ and $m = k + 1, k + 2, k + 3, \ldots$, where

(7) $$G(t) = t^{2m} - t^{2k}, \quad -1 \leq t \leq 1.$$

From (7) it follows that

(8) $$-\frac{m-k}{m} t_3 \leq G(t) \leq 0, \quad -1 \leq t \leq 1,$$

where the equalities on the left-hand side and on the right-hand side hold only for $t = \pm t_4$ and $t = 0, \pm 1$, respectively. (The numbers t_3 and t_4 are indicated in (iii) and (c), respectively.) Thus from (8) and (6) we obtain the sharp inequalities

(9) $$-\frac{m-k}{m} t_3 \leq c_{2m+1} - c_{2k+1} \leq 0,$$
$$m = k + 1, k + 2, k + 3, \ldots,$$

where the equality holds on the left-hand side if and only if $\mu(t)$ is a step function with two jumps $A_{1,2} \geq 0$ with sum 1 at the points $t = \pm t_4$; the equality holds on the right-hand side if and only if $\mu(t)$ is a step function with three jumps $A_{1,2,3} \geq 0$ with sum 1 at the points $t = -1$, $t = 0$ and $t = 1$, respectively. Therefore, from (9) and the representation formula (3) we obtain the assertions in (iii) and (c), respectively.

2. Further, for arbitrary $x_1, x_2, \ldots$ we shall use the ordinary Bell polynomials D_{nk} generated by the formal expansions (see Comtet [2], p. 136, the Remark)

$$(10)\qquad \Big(\sum_{m=1}^{\infty} x_m z^m\Big)^k \equiv \sum_{n=k}^{\infty} D_{nk} z^n, \quad k = 1, 2, \ldots .$$

The polynomials $D_{nk} \equiv D_{nk}(x_1, \ldots, x_{n-k+1})$, for $1 \le k \le n, n \ge 1$, have the explicit form (see [3], p. 83)

$$(11)\qquad D_{nk}(x_1, \ldots, x_{n-k+1}) \equiv \sum \frac{k!(x_1)^{\nu_1} \ldots (x_{n-k+1})^{\nu_{n-k+1}}}{\nu_1! \ldots \nu_{n-k+1}!},$$

where the sum is taken over all nonnegative integers $\nu_1, \ldots, \nu_{n-k+1}$ satisfying

$$(12)\qquad \begin{aligned} &\nu_1 + \nu_2 + \ldots + \nu_{n-k+1} = k, \\ &\nu_1 + 2\nu_2 + \ldots + (n - k + 1)\nu_{n-k+1} = n, \end{aligned}$$

and they are easily computed if one uses the recurrence relation (see [3], p. 83)

$$(13)\qquad \begin{aligned} &D_{nk} = \sum_{\mu=1}^{n-k+1} x_\mu D_{n-\mu,k-1}, \\ &1 \le k \le n, \quad n \ge 1, \quad D_{no} = 0, \quad D_{oo} = 1. \end{aligned}$$

The first and the last polynomials are

$$(14)\qquad D_{n1} = x_n, \quad D_{nn} = x_1^n, \quad n \ge 1.$$

For $1 \le n \le 5$ from (13) and (14) we obtain the following short table (see in [2], p. 309, a longer table for $1 \le n \le 10$)

$$(15)\qquad \begin{aligned} &D_{11} = x_1; \quad D_{21} = x_2, \quad D_{22} = x_1^2; \\ &D_{31} = x_3, \quad D_{32} = 2x_1x_2, \quad D_{33} = x_1^3; \\ &D_{41} = x_4, \quad D_{42} = 2x_1x_3 + x_2^2, \quad D_{43} = 3x_1^2x_2, \\ &D_{44} = x_1^4; \quad D_{51} = x_5, \quad D_{52} = 2x_1x_4 + 2x_2x_3, \\ &D_{53} = 3x_1^2x_3 + 3x_1x_2^2, \quad D_{54} = 4x_1^3x_2, \quad D_{55} = x_1^5 \end{aligned}$$

Now let

$$(16)\qquad z = \Psi(w) = \sum_{n=1}^{\infty} b_n w^n, \quad b_1 = 1,$$

denote the inverse of any function $w = \phi(z)$, determined by (3). The largest common region of convergence of all series (16) is the disc $|w| < 1/2$ (see [4], p. 345, Corollary 1).

THEOREM 2. In terms of the coefficients c_n in (3), the coefficients b_n in (16) satisfy the recurrence relation

$$(17)\qquad b_n = -\sum_{k=1}^{n-1} b_k D_{nk}(c_1, \ldots, c_{n-k+1}),$$

$$n = 2, 3, \ldots, \quad b_1 = c_1 = 1,$$

where $D_{nk}(c_1, \ldots, c_{n-k+1})$ are determined by (10)-(12).

P r o o f. From (16), (3) and (10) we obtain the identity

$$(18)\qquad z = \sum_{k=1}^{\infty} b_k \left(\sum_{m=1}^{\infty} c_m z^m\right)^k$$

$$= \sum_{k=1}^{\infty} b_k \sum_{n=k}^{\infty} z^n D_{nk}(c_1, \ldots, c_{n-k+1})$$

$$= \sum_{n=1}^{\infty} z^n \sum_{k=1}^{n} b_k D_{nk}(c_1, \ldots, c_{n-k+1}).$$

From (18) it follows that

$$(19)\qquad \sum_{k=1}^{n} b_k D_{nk}(c_1, \ldots, c_{n-k+1}) = 0, \quad n \geq 2.$$

Thus from (19) and the second relation in (14) we obtain (17), having in mind that $b_1 = c_1 = 1$.

In particular, for $n = 2, 3, 4, 5, \ldots$, from (17) and (15) we obtain the first few coefficients

$$(20)\qquad \begin{aligned} b_2 &= -c_2, \\ b_3 &= -c_3 + 2c_2^2, \\ b_4 &= -c_4 + 5c_2c_3 - 5c_2^3, \\ b_5 &= -c_5 + 6c_2c_4 + 3c_3^2 - 21\, c_2^2 c_3 + 14c_2^4, \end{aligned}$$

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THEOREM 3. The coefficients b_2 and b_3 satisfy the sharp inequalities

$$(21)\qquad -1 \le b_n \le 1, \quad n = 2, 3,$$

where the equalities hold only for the following extremal functions:

(a) if $n = 2$, on the left-hand side of (21), for the function

$$(22)\qquad \Psi(w) = \frac{w}{1+w} = \sum_{n=1}^{\infty} (-1)^{n-1} w^n,$$

inverse of the function (5) with $A = 1$, and on the right-hand side of (21), for the function

$$(23)\qquad \Psi(w) = \frac{w}{1-w} = \sum_{n=1}^{\infty} w^n,$$

inverse of the function (4);

(b) if $n = 3$, on the left-hand side of (21), for the function

$$(24)\qquad \Psi(w) = \frac{1}{2w}\left(-1 + \sqrt{1+4w^2}\right) = \sum_{n=1}^{\infty} \binom{\frac{1}{2}}{n} 2^{2n-1} w^{2n-1},$$

inverse of the function

$$(25)\qquad \phi(z) = \frac{z}{1-z^2} = \sum_{n=1}^{\infty} z^{2n-1} \in N_2,$$

and on the right-hand side of (21), for the functions (22) and (23), inverse of the functions (5) with $A = 1$ and (4), respectively.

P r o o f. If $n = 2$, the sharp inequalities (21) and the extremal functions (22) and (23) follow from the first equation in (20) and Theorem 1 in ([1], p. 152).

If $n = 3$, from (2) and the second equation in (20) we obtain the sharp inequality

$$(26)\qquad 1 + b_3 = \int_{-1}^{1} (1 - t^2)\,d\mu(t) + 2\left(\int_{-1}^{1} t\,d\mu(t)\right)^2 \ge 0,$$

where the equality holds if and only if $\mu(t)$ is a step function with two jumps 1/2 and 1/2 at the points $t = \mp 1$, respectively. Therefore, from (26) and the representation formula (3) we obtain the first sharp inequality in (21) (for $n = 3$) and the extremal function (25), the inverse of which is the function (24).

Further, with the help of the Cauchy inequality and (2) we obtain that

$$(27)\qquad c_2^2 \le \int_{-1}^{1} 1^2 d\mu(t) . \int_{-1}^{1} t^2 d\mu(t) = c_3 .$$

Now from (27), the second equation in (20) and Theorem 1 in ([1], p. 152), we obtain the sharp inequalities

$$(28)\qquad b_3 \le c_3 \le 1$$

with the unique extremal functions (22) and (23), inverse of the functions (5) with $A = 1$ and (4), respectively. Therefore, from (28) we get the second sharp inequality in (21) (for $n = 3$) and the corresponding extremal functions (22) and (23).

This completes the proof of Theorem 3.

REMARK. Let us note that the coefficients b_n in (16) cannot be uniformly bounded over N_2, since if they were it would be possible to replace the convergence disc $|w| < 1/2$ by a larger one, namely $|w| < 1$. Hence, the inequality $|b_n| \le 1$ for all $n = 4, 5, \dots$ is impossible. The problem of finding the sharp lower and upper bounds of b_n for $n = 4, 5, \dots$ is open.

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O WSPÓŁCZYNNIKACH FUNKCJI JEDNOLISTNYCH W KLASIE N_1 i N_2

W pracy rozwiązuje się istotne problemy dotyczące współczynników funkcji analitycznych w klasach N_1 i N_2.