ACTA UNIVERSITATIS LODZIENSIS
FOLIA MATHEMATICA 5, 1992

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ON AN ESTIMATE OF SOME FUNCTIONAL IN THE CLASS OF ODD BOUNDED UNIVALENT FUNCTIONS

Let us denote by $S(M)$, $M > 1$, the family of functions of the form

$$F(z) = z + A_2 z^2 + \ldots + A_n z^n + \ldots,$$

univalent and holomorphic in the disc $E = \{z: |z| < 1\}$ and satisfying in it the condition $|F(z)| < M$, $M > 1$. Denote by $S^{(2)}(\sqrt{M})$ the class of odd univalent functions of the form

$$H(z) = z + C_3 z^3 + C_5 z^5 + \ldots + C_{2n+1} z^{2n+1} + \ldots,$$

satisfying in E the condition $|H(z)| < \sqrt{M}$, $M > 1$.

Of course, for each function $F \in S(M)$, the function $H(z) = \sqrt{F(z^2)}$ belongs to $S^{(2)}(\sqrt{M})$, and vice versa.

In the paper, it is proved that the following theorem takes place.

THEOREM. If H is any function of the class $S^{(2)}(\sqrt{M})$, then the following estimates

$$|C_3|^2 + |C_5|^2 \le \begin{cases} (1 - \frac{1}{M})^2 + [(1 - \frac{1}{M})(1 - \frac{2}{M})]^2 & \text{when } 1 < M \le 6, \\ [(\nu_o + 1)e^{-\nu_o} - \frac{1}{M}]^2 + \frac{1}{4}[(3\nu_o^2 + 2\nu_o + 1)e^{-2\nu_o} - \frac{6}{M}(\nu_o + 1)e^{-\nu_o} + \frac{4}{M^2} + 1]^2 & \text{when } M > 6 \end{cases}$$

hold, where $\nu_o \in (0, \log M)$ is the root of the equation

$$2[(\nu + 1)e^{-\nu} - \frac{1}{M}] + [(3\nu - 1)e^{-\nu} - \frac{3}{M}]$$

$$\cdot\,[(3\nu^2 + 2\nu + 1)e^{-2\nu} - \frac{6}{M}(\nu + 1)e^{-\nu} + \frac{4}{M^2} + 1] = 0.$$

For each $M > 1$, there exist functions of the class $S^{(2)}(\sqrt{M})$ for which the equality sign in the above estimate takes place.

1. Let us denote by S the family of functions of the form

$$F(z) = z + A_2z^2 + A_3z^3 + \dots + A_nz^n + \dots,$$

univalent and holomorphic in the disc $E = \{z: |z| < 1\}$.

Let $S^{(2)}$ stand for the class of odd univalent functions having in E the expansion

$$H(z) = z + C_3z^3 + C_5z^5 + \dots + C_{2n+1}z^{2n+1} + \dots \tag{1}$$

It is known that $H \in S^{(2)}$ if and only if there exists a function $F \in S$ such that

$$H(z) = \sqrt{F(z^2)}, \quad z \in E. \tag{2}$$

Let $S(M)$, $M > 1$, be a subclass of S of functions satisfying in the disc E the condition $|F(z)| < M$. Denote by $S^{(2)}(\sqrt{M})$ the class of univalent functions of form (1), bounded by $\sqrt{M}$, that is, $|H(z)| < \sqrt{M}$, $z \in E$. Of course, for any function $F \in S(M)$, the function H defined by relationship (2) belongs to the class $S^{(2)}(\sqrt{M})$, and vice versa.

Making use of this relationship, we get

$$C_3 = \tfrac{1}{2}A_2, \quad C_5 = \tfrac{1}{2}(A_3 - \tfrac{1}{4}A_2^2). \tag{3}$$

From the well-known estimate of the modulus of the coefficient A_2 in the class $S(M)$ ([3]) one knows that

$$|C_3| \le 1 - \frac{1}{M}, \quad M > 1, \tag{4}$$

with the equality in (4) holding only for the Pick function $w = P(z, M)$, $P(0, M) = 0$, given by the equation

$$\frac{M^2w}{(M + \varepsilon w)^2} = \frac{z}{(1 + \varepsilon z)^2}, \quad z \in E, \quad |\varepsilon| = 1. \tag{5}$$

One also knows the estimate of the functional $|A_3 - \alpha A_2^2|$, for any real α, in the class $S(M)$ ([1], [1']); in the case $\alpha = \frac{1}{4}$, the maximum of this functional is not attained for the Pick function.

The aim of our paper is to determine the maximum of the functional

(6) $\mathcal{F}(H) = |C_3|^2 + |C_5|^2$

in the classes $S^{(2)}(\sqrt{M})$ for $M > 1$.

In the full class $S^{(2)}$, functional (6) was estimated by M. S. Robertson [4].

In paper [5] we obtained a partial result, namely, an estimate of the maximum of the functional $\mathcal{F}(H)$ in the classes $S^{(2)}(\sqrt{M})$ for $M \geq 3$. The method applied there brought about difficulties in the investigation of this functional for the remaining M, that is, $M \in (1, 3)$.

In the present paper we obtain a final result, i.e. an estimate of functional (6) from above for all $M > 1$; of course, for $M \geq 3$, the result is the same as that in [5].

In the proof, use is made again of some general lemmas proved in [1], special corollaries following from them and the properties of the functional considered itself. The basic modification of the procedure from [5], arisen, among other things, after many discussions with Z. J. Jakubowski, consists mainly in a skilful use of the above-mentioned lemmas and other estimates of some well-known functionals. On account of the method applied, our reasoning is carried out for all $M > 1$; therefore, unfortunately, it turns out to be indispensable to repeat some fragments of paper [5].

2. Note that (3) and the properties of the classes $S(M)$ imply that the determination of the maximum of functional (6) is equivalent to the determination of the maximum of the functional

(7) $G(F) = \frac{1}{4}|A_2|^2 + [\mathrm{Re}\ \frac{1}{2}(A_3 - \frac{1}{4}A_2^2)]^2, \quad F \in S(M), \quad M > 1.$

Evidently, for the purpose, it is sufficient to determine the upper bound of the functional $G(F)$ in the subclass $S^*(M)$ of $S(M)$ of functions of the form (cf. [2])

$$F(z) = \lim_{t \to m} e^t f(z, t), \quad m = \log M,$$

where $f(z, t)$ is a holomorphic function of the variable z in the disc E, $|f(z, t)| < 1$ for $z \in E$, $f(0, t) = 0$ and $f'_z(0, t) > 0$, and $f(z, t)$ is, for $0 \leq t \leq m$, a solution of the Löwner equation

$$\frac{\partial f}{\partial t} = -f\frac{1 + kf}{1 - kf},$$

satisfying the initial condition $f(z, 0) = z$. The function $k = k(t)$, $|k(t)| = 1$, is any function continuous in the interval $\langle 0, m\rangle$ except a finite number of points of discontinuity of the first kind.

Since the coefficients A_2 and A_3 of functions of the class $S^*(M)$ are expressed by the formulae ([2], [1]):

$$A_2 = -2\int_o^m e^{-\tau}k(\tau)d\tau,$$

$$A_3 = -2\int_o^m e^{-2\tau}k^2(\tau)d\tau + 4\left(\int_o^m e^{-\tau}k(\tau)d\tau\right)^2, \quad m = \log M,$$

therefore it follows from (7) that we ought to determine the maximum of the expression

$$G(F) = \left(\int_o^m e^{-\tau}\cos\theta(\tau)d\tau\right)^2 + \left(\int_o^m e^{-\tau}\sin\theta(\tau)d\tau\right)^2 \tag{8}$$

$$+ \frac{1}{4}\left\{3\left(\int_o^m e^{-\tau}\cos\theta(\tau)d\tau\right)^2 - 3\left(\int_o^m e^{-\tau}\sin\theta(\tau)d\tau\right)^2\right.$$

$$\left. - 4\int_o^m e^{-2\tau}\cos^2\theta(\tau)d\tau + 1 - e^{-2m}\right\}^2$$

where $\theta(\tau) = \arg k(\tau)$, $\theta(\tau) \in \langle 0, 2\pi\rangle$, over all possible functions $k(\tau)$ satisfying the assumptions of the Löwner theorem.

In the further part of the paper, we shall make use of the lemmas from [1], mentioned of in the introduction.

LEMMA A. If: 1° λ is any real function of a real variable τ, defined and continuous in the interval $\langle 0, m\rangle$ except a finite number of points of discontinuity of the first kind, 2° $|\lambda(\tau)| \leq e^{-\tau}$ for $\tau \in \langle 0, m\rangle$ and 3°

$$\int_o^m \lambda^2(\tau)d\tau \leq me^{-2m}, \tag{A.1}$$

then

$$\left(\int_o^m \lambda(\tau)d\tau\right)^2 \leq m(me^{-2m} - \nu e^{-2\nu}) \tag{A.2}$$

where ν, $0 \leq \nu \leq m$, is the root of the equation

(A.3) $\int_0^m \lambda^2(\tau)d\tau = me^{-2m} - \nu e^{-2\nu}$.

For each $\nu \in \langle 0, m\rangle$, there exists a constant function $\lambda(\tau) = c$ such that in (A. 2) the equality holds. Then the relation $mc^2 = me^{-2m} - \nu e^{-2\nu}$ should take place.

LEMMA B. If a function λ satisfies assumptions 1^o and 2^o of Lemma A and the condition

(B.1) $\int_0^m \lambda^2(\tau)d\tau \geq me^{-2m}$,

then

(B.2) $\left|\int_0^m \lambda(\tau)d\tau\right| \leq (\nu + 1)e^{-\nu} - e^{-m}$

where ν, $0 \leq \nu \leq m$, is the root of the equation

(B.3) $\int_0^m \lambda^2(\tau)d\tau = (\nu + \frac{1}{2})e^{-2\nu} - \frac{1}{2}e^{-2m}$.

Estimate (B.2) is sharp for every ν and the equality sign occurs only if $\lambda(\tau) = \pm\varkappa(\tau)$ where

$$\varkappa(\tau) = \begin{cases} e^{-\nu} & \text{for } 0 \leq \tau \leq \nu, \\ e^{-\tau} & \text{for } \nu \leq \tau \leq m. \end{cases}$$

Put $A_2 = -2(x + iy)$, that is,

$$x = \int_0^m \lambda_1(\tau)d\tau, \qquad y = \int_0^m \lambda_2(\tau)d\tau,$$

(9)

$$\lambda_1(\tau) = e^{-\tau}\cos\Theta(\tau), \quad \lambda_2(\tau) = e^{-\tau}\sin\Theta(\tau).$$

From the properties of the function $k(\tau)$, the definition of the function $\Theta(\tau)$ and from (9) it follows that the functions $\lambda_1(\tau)$, $\lambda_2(\tau)$ satisfy assumptions 1^o-2^o of Lemma A and, moreover, either (A.1) or (B.1).

Let $\nu = \nu(\Theta)$ be the root of the equation

(10) $\int_0^m \lambda_1^2(\tau)d\tau = \Omega_A(\nu)$

where

(11) $\Omega_A(\nu) = me^{-2m} - \nu e^{-2\nu}, \quad 0 \le \nu \le \nu^*,$

with that $\nu^* = m$ when $0 < m \le \frac{1}{2}$ or $me^{-2m} - \nu^* e^{-2\nu^*} = 0$ when $m > \frac{1}{2}$, or the root of the equation

(12) $\int_0^m \lambda_1^2(\tau)d\tau = \Omega_B(\nu)$

where

(13) $\Omega_B(\nu) = (\nu + \frac{1}{2})e^{-2\nu} - \frac{1}{2}e^{-2m}, \quad 0 \le \nu \le m.$

Evidently, the function $\Omega_A(\nu)$ satisfies condition (A.1) of Lemma A, whereas $\Omega_B(\nu)$ - condition (B.1) of Lemma B.

Analogously, let $\mu = \mu(\theta)$ be the root of the equation

(10′) $\int_0^m \lambda_2^2(\tau)d\tau = \Omega_A(\mu), \quad 0 \le \mu \le \nu^*,$

or of the equation

(12′) $\int_0^m \lambda_2^2(\tau)d\tau = \Omega_B(\mu), \quad 0 \le \mu \le m,$

where Ω_A, Ω_B are defined by the formulae (11), (13), respectively

Of course, for all admissible $\theta(\tau)$,

(14) $\int_0^m e^{-2\tau}\sin^2\theta(\tau)d\tau = \frac{1}{2}(1 - e^{-2m}) - \int_0^m e^{-2\tau}\cos^2\theta(\tau)d\tau.$

Note that if $m \in (0, \hat{m}\rangle$ where $\hat{m}$ is the root of the equation

(15) $\frac{1}{2}(1 - e^{-2m}) = 2me^{-2m},$

then the equation

(11′) $\Omega_A(\nu) = \frac{1}{2}(1 - e^{-2m}) - me^{-2m}$

possesses exactly one root $\hat{\nu}_A \in (0, \nu^*)$.

If $m \in \langle\hat{m}, +\infty)$, $\hat{m}$ is defined by (15), then the equation

(13′) $\Omega_B(\nu) = \frac{1}{2}(1 - e^{-2m}) - me^{-2m}$

possesses exactly one root $\hat{\nu}_B \in (0, m)$.

Examining the functions $\Omega_A(\nu)$, $\Omega_B(\nu)$, $\frac{1}{2}(1 - e^{-2m}) - \Omega_A(\nu)$,

$\frac{1}{2}(1 - e^{-2m}) - \Omega_B(\nu)$ and making use of (14), we shall obtain the relations below:

if $0 < m \leq \hat{m}$,

then

$$(16) \qquad \mu = \begin{cases} \Omega_A^{-1}[\frac{1}{2}(1 - e^{-2m}) - \Omega_B(\nu)] & \text{where } 0 \leq \nu \leq m, \\ & \hat{\nu}_A \leq \mu \leq \nu^*, \\ \Omega_A^{-1}[\frac{1}{2}(1 - e^{-2m}) - \Omega_A(\nu)] & \text{where } 0 \leq \nu \leq \hat{\nu}_A, \\ & 0 \leq \mu \leq \hat{\nu}_A, \\ \Omega_B^{-1}[\frac{1}{2}(1 - e^{-2m}) - \Omega_A(\nu)] & \text{where } \hat{\nu}_A \leq \nu \leq \nu^*, \\ & 0 \leq \mu \leq m; \end{cases}$$

if $m \geq \hat{m}$, then

$$(17) \qquad \mu = \begin{cases} \Omega_A^{-1}[\frac{1}{2}(1 - e^{-2m}) - \Omega_B(\nu)] & \text{where } 0 \leq \nu \leq \hat{\nu}_B, \\ & 0 \leq \mu \leq \nu^*, \\ \Omega_B^{-1}[\frac{1}{2}(1 - e^{-2m}) - \Omega_B(\nu)] & \text{where } \hat{\nu}_B \leq \nu \leq m, \\ & \hat{\nu}_B \leq \mu \leq m, \\ \Omega_B^{-1}[\frac{1}{2}(1 - e^{-2m}) - \Omega_A(\nu)] & \text{where } 0 \leq \nu \leq \nu^*, \\ & 0 \leq \mu \leq \hat{\nu}_B, \end{cases}$$

$\hat{m}$, $\hat{\nu}_A$, $\hat{\nu}_B$ being defined by equations (15), (11´), (13´), respectively.

If we use Lemmas A, B as well as (9), we shall get an estimate for $x^2 = \frac{1}{4}(\text{Re } A_2)^2$. Moreover, taking account of the above properties of the functions $\Omega_A(\mu)$, $\Omega_B(\mu)$ and equality (14), we shall also get the respective estimate for $y^2 = \frac{1}{4}(\text{Im } A_2)^2$.

Consequently, if condition (A.1) holds, then, in virtue of (A.3), (A.2) and (9), we have

$$0 \leq x^2 \leq X_A(\nu)$$

where

$$(18) \qquad X_A(\nu) = m(me^{-2m} - \nu e^{-2\nu}).$$

The function $X_A(\nu)$ is decreasing in the interval $\langle 0, \nu^*\rangle$, and let us recall that $\nu^* = m$ when $0 < m \leq \frac{1}{2}$ or $me^{-2m} - \nu^* e^{-2\nu^*} = 0$ when $m > \frac{1}{2}$. Besides, $0 \leq X_A(\nu) \leq m^2 e^{-2m}$.

If condition (B.1) holds, then, in virtue of (B.3), (B.2) and (9), we have

$$0 \le x^2 \le X_B(\nu)$$

where

(19) $$X_B(\nu) = [(\nu + 1)e^{-\nu} - e^{-m}]^2.$$

The function $X_B(\nu)$ is decreasing in the interval $\langle 0, m\rangle$. Besides $m^2e^{-2m} \le X_B(\nu) \le (1 - e^{-m})^2$.

From (16) or (17), for fixed m and ν, we can determine the value μ corresponding to them; using again Lemma A or Lemma B, respectively, we shall obtain - in consequence - that, for fixed m and ν,

$$0 \le y^2 \le X_A(\mu) \quad \text{or} \quad 0 \le y^2 \le X_B(\mu);$$

X_A, X_B are defined by formulae (18), (19).

The above estimates of the quantities $x^2 = \frac{1}{4}(\mathrm{Re}\, A_2)^2$ and $y^2 = \frac{1}{4}(\mathrm{Im}\, A_2)^2$, being consequences of Lemmas A and B, will be made use of in the next section of the paper.

3. The assumptions of Lemmas A and B as well as (9) imply that the function $\lambda_1(\tau)$ satisfies either condition (A.1) or (B.1). Since $\lambda_1(\tau) = e^{-\tau} \cos \theta(\tau)$, therefore, using the appropriate lemma, we consider some subset of functions $\theta(\tau)$, thus some subset of functions $k(\tau)$ $(\theta(\tau) = \arg k(\tau))$, and in consequence, some subclass of the family $S(M)$.

From (9) it follows that expression (8) takes the form

(20) $$G(F) = x^2 + y^2 + \frac{1}{4}[3x^2 - 3y^2 - 4\int_0^m e^{-2\tau} \cos^2 \theta(\tau)d\tau + 1 - e^{-2m}]^2, \quad m = \log M.$$

From (9) and estimate (4) we have

(21) $$x^2 + y^2 \le (1 - e^{-m})^2, \quad m > 0.$$

By using Lemma A or Lemma B and taking account of inequality (21), the problem of determining the maximum of $G(F)$ will be reduced to the investigation of the maxima of some functions of the variable ν where ν is defined by (10) or (12).

Denote by $G(x^2, y^2; \nu)$ the right-hand side of (20), i.e.

$$(20') \quad G(x^2, y^2; \nu) \equiv x^2 + y^2 + \frac{1}{4}[3x^2 - 3y^2 - 4\int_0^m e^{-2\tau}\cos^2\theta(\tau)d\tau + 1 - e^{-2m}]^2.$$

Note first that, for a fixed $\nu = \nu(\Theta)$, $G(x^2, y^2; \nu)$ is a convex function of the variables x^2, y^2 and, as such, does not attain its maximum inside the set of variability of x^2, y^2. Taking account of the properties obtained in section 2 as well as (21), we shall consider six cases in which we determine all possible values of x^2 and y^2 for which the function G can attain its maximum.

a. Let $0 < m \le \hat{m}$ where $\hat{m}$ is the root of equation (15). Consider the case when $\nu = \nu(\Theta)$ is the root of equation (10), i.e. $\int_0^m e^{-2\tau}\cos^2\theta(\tau)d\tau = \Omega_A(\nu)$, whereas $\mu = \mu(\Theta)$ - the root of equation (10´), i.e. $\int_0^m e^{-2\tau}\sin^2\theta(\tau)d\tau = \Omega_A(\mu)$, where Ω_A is given by formula (11). Then (16) implies that $0 \le \nu \le \hat{\nu}_A$ and $0 \le \mu \le \hat{\nu}_A$, where $\hat{\nu}_A$ is the root of equation (11´). From Lemma A we have

$$0 \le x^2 \le X_A(\nu) \quad \text{and} \quad 0 \le y^2 \le X_A(\mu),$$

where X_A is given by (18). It can be verified that $X_A(\nu) + X_A(\mu \ge \ge (1 - e^{-m})^2$ when $0 \le \nu \le \hat{\nu}_A$ and $0 \le \mu \le \hat{\nu}_A$. In consequence, the maximum of $G(x^2, y^2; \nu)$ can be attained only in the cases when:

1° $x^2 = 0$ and $y^2 = 0$,

2° $x^2 = X_A(\nu)$ and $y^2 = 0$,

3° $x^2 = X_A(\nu)$ and $y^2 = (1 - e^{-m})^2 - X_A(\nu)$,

4° $x^2 = (1 - e^{-m})^2 - X_A(\mu)$ and $y^2 = X_A(\mu)$,

5° $x^2 = 0$ and $y^2 = X_A(\mu)$,

with that $0 \le \nu \le \hat{\nu}_A$ and $0 \le \mu \le \hat{\nu}_A$.

b. Let, as above, $0 < m \le \hat{m}$. Consider the case when $\nu = \nu(\Theta)$ is the root of equation (10), whereas $\mu = \mu(\Theta)$ - the root of equation (12´). Then it follows from (16) that $\hat{\nu}_A \le \nu \le \nu^*$ and $0 \le \mu \le m$. From Lemmas A and B we have, respectively,

$$0 \le x^2 \le X_A(\nu) \quad \text{and} \quad 0 \le y^2 \le X_B(\mu),$$

where X_A, X_B are defined by formulae (18), (19). It can be shown that $X_A(\nu) + X_B(\mu) \ge (1 - e^{-m})^2$ when $\hat{\nu}_A \le \nu \le \nu^*$ and $0 \le \mu \le m$. Consequently, the maximum of $G(x^2, y^2; \nu)$ can be attained only if

1° $x^2 = 0$ and $y^2 = 0$,
2° $x^2 = X_A(\nu)$ and $y^2 = 0$,
3° $x^2 = X_A(\nu)$ and $y^2 = (1 - e^{-m})^2 - X_A(\nu)$,
4° $x^2 = (1 - e^{-m})^2 - X_B(\mu)$ and $y^2 = X_B(\mu)$,
5° $x^2 = 0$ and $y^2 = X_B(\mu)$,

with that $\hat{\nu}_A \le \nu \le \nu^*$ and $0 \le \mu \le m$.

c. Let $0 < m \le \hat{m}$ and let $\nu = \nu(\Theta)$ be the root of equation (12), whereas $\mu = \mu(\Theta)$ - the root of equation (10´). Then from (16) we have $0 \le \nu \le m$ and $\hat{\nu}_A \le \mu \le \nu^*$, and from Lemmas B and A it follows, respectively, that

$$0 \le x^2 \le X_B(\nu) \quad \text{and} \quad 0 \le y^2 \le X_A(\mu).$$

Also in this case, $X_B(\nu) + X_A(\mu) \ge (1 - e^{-m})^2$ when $0 \le \nu \le m$ and $\hat{\nu}_A \le \mu \le \nu^*$. Hence the maximum of $G(x^2, y^2; \nu)$ can be attained only if:

1° $x^2 = 0$ and $y^2 = 0$,
2° $x^2 = X_B(\nu)$ and $y^2 = 0$,
3° $x^2 = X_B(\nu)$ and $y^2 = (1 - e^{-m})^2 - X_B(\nu)$,
4° $x^2 = (1 - e^{-m})^2 - X_A(\mu)$ and $y^2 = X_A(\mu)$,
5° $x^2 = 0$ and $y^2 = X_A(\mu)$,

with that $0 \le \nu \le m$ and $\hat{\nu}_A \le \mu \le \nu^*$.

d. Let $m \ge \hat{m}$ where $\hat{m}$ is the root of equation (15). Consider now the case when $\nu = \nu(\Theta)$ is the root of equation (10), whereas $\mu = \mu(\Theta)$ - the root of equation (12´). In this case, from (17) we have $0 \le \nu \le \nu^*$ and $0 \le \mu \le \hat{\nu}_B$ where $\hat{\nu}_B$ is the root of equation (13´). From Lemmas A and B we have, respectively,

$$0 \le x^2 \le X_A(\nu) \quad \text{and} \quad 0 \le y^2 \le X_B(\mu).$$

It can be checked that $X_A(\nu) + X_B(\mu) \ge (1 - e^{-m})^2$ when $0 \le \nu \le \nu^*$ and $0 \le \mu \le \hat{\nu}_B$. Thus the maximum of $G(x^2, y^2; \nu)$ can be attained only if:

1^o $x^2 = 0$ and $y^2 = 0$,

2^o $x^2 = X_A(\nu)$ and $y^2 = 0$,

3^o $x^2 = X_A(\nu)$ and $y^2 = (1 - e^{-m})^2 - X_A(\nu)$,

4^o $x^2 = (1 - e^{-m})^2 - X_B(\mu)$ and $y^2 = X_B(\mu)$,

5^o $x^2 = 0$ and $y^2 = X_B(\mu)$,

with that $0 \le \nu \le \nu^*$ and $0 \le \mu \le \hat{\nu}_B$.

It can be seen that, in relation to case (b), only the intervals of variability of ν and μ have changed.

e. Let, as before, $m \ge \hat{m}$. Consider the case when $\nu = \nu(\Theta)$ is the root of equation (12), whereas $\mu = \mu(\Theta)$ - the root of equation (10´). Then from (17) we have $0 \le \nu \le \hat{\nu}_B$ and $0 \le \mu \le \nu^*$, and Lemmas B and A imply that

$$0 \le x^2 \le X_B(\nu) \quad \text{and} \quad 0 \le y^2 \le X_A(\mu).$$

Also in this case, $X_B(\nu) + X_A(\mu) \ge (1 - e^{-m})^2$ when $0 \le \nu \le \hat{\nu}_B$ and $0 \le \mu \le \nu^*$. Hence the maximum of $G(x^2, y^2; \nu)$ can be attained only if:

1^o $x^2 = 0$ and $y^2 = 0$,

2^o $x^2 = X_B(\nu)$ and $y^2 = 0$,

3^o $x^2 = X_B(\nu)$ and $y^2 = (1 - e^{-m})^2 - X_B(\nu)$,

4^o $x^2 = (1 - e^{-m})^2 - X_A(\mu)$ and $y^2 = X_A(\mu)$,

5^o $x^2 = 0$ and $y^2 = X_A(\mu)$,

with that $0 \le \nu \le \hat{\nu}_B$ and $0 \le \mu \le \nu^*$.

It is evident that, in relation to case (c), only the intervals of variability of ν and μ have changed.

f. Let $m \ge \hat{m}$. Finally, consider the case when $\nu = \nu(\Theta)$ is the root of equation (12), whereas $\mu = \mu(\Theta)$ - the root of equation (12´). Then from (17) we have $\hat{\nu}_B \le \nu \le m$ and $\hat{\nu}_B \le \mu \le m$, and from Lemma B it follows that

$$0 \le x^2 \le X_B(\nu) \quad \text{and} \quad 0 \le y^2 \le X_B(\mu).$$

It can be demonstrated that $X_B(\nu) + X_B(\mu) \ge (1 - e^{-m})^2$ when $\hat{\nu}_B \le \le \nu \le m$ and $\hat{\nu}_B \le \mu \le m$. So, the maximum of $G(x^2, y^2; \nu)$ can be attained only if:

1° $x^2 = 0$ and $y^2 = 0$,

2° $x^2 = X_B(\nu)$ and $y^2 = 0$,

3° $x^2 = X_B(\nu)$ and $y^2 = (1 - e^{-m})^2 - X_B(\nu)$,

4° $x^2 = (1 - e^{-m})^2 - X_B(\mu)$ and $y^2 = X_B(\mu)$,

5° $x^2 = 0$ and $y^2 = X_B(\mu)$,

with that $\hat{\nu}_B \le \nu \le m$ and $\hat{\nu}_B \le \mu \le m$.

Summing up cases a-f, we shall next obtain the suitable functions of the variable ν, mentioned of earlier, whose maxima can realize the sought-for maximum of the functional G(F).

From cases a.1°, b.1° and d.1° as well as from (11) and (20´) we have, for $m > 0$,

$$G(x^2, y^2; \nu) \le \mathcal{A}_1(\nu)$$

where

(22) $$\mathcal{A}_1(\nu) = \tfrac{1}{4}[4(me^{-2m} - \nu e^{-2\nu}) - (1 - e^{-2m})]^2, \quad 0 \le \nu \le \nu^*.$$

From cases c.1°, e.1° and f.1° as well as from (13) and (20´) we obtain, for $m > 0$,

$$G(x^2, y^2; \nu) \le \mathcal{B}_1(\nu)$$

where

(23) $$\mathcal{B}_1(\nu) = \tfrac{1}{4}[2(2\nu + 1)e^{-2\nu} - (1 + e^{-2m})]^2, \quad 0 \le \nu \le m.$$

Cases a.2°, b.2° and d.2° as well as (11), (18) and (20´) yield, for $m > 0$,

$$G(x^2, y^2; \nu) \le \mathcal{A}_2(\nu)$$

where

(24) $$\mathcal{A}_2(\nu) = m(me^{-2m} - \nu e^{-2\nu}) + \tfrac{1}{4}[(3m - 4)(me^{-2m} - \nu e^{-2\nu}) + 1 - e^{-2m}]^2, \quad 0 \le \nu \le \nu^*.$$

Cases c.2°, e.2° and f.2° as well as (13), (19) and (20´) give, for $m > 0$,

$$G(x^2, y^2; \nu) \le \mathcal{B}_2(\nu)$$

where

$$(25)\qquad \mathcal{B}_2(\nu) = [(\nu + 1)e^{-\nu} - e^{-m}]^2 + \tfrac{1}{4}[(3\nu^2 + 2\nu + 1)e^{-2\nu} - 6(\nu + 1)e^{-\nu}e^{-m} + 4e^{-2m} + 1]^2, \quad 0 \le \nu \le m.$$

From cases a.3°, b.3° and d.3° as well as from (11), (18) and (20´) we get, for $m > 0$,

$$G(x^2, y^2; \nu) \le \mathcal{A}_3(\nu)$$

where

$$(26)\qquad \mathcal{A}_3(\nu) = (1 - e^{-m})^2 + [(3m - 2)(me^{-2m} - \nu e^{-2\nu}) - (1 - e^{-m})(1 - 2e^{-m})]^2, \qquad 0 \le \nu \le \nu^*.$$

From cases c.3°, e.3° and f.3° as well as from (13), (19) and (20´) we have, for $m > 0$,

$$G(x^2, y^2; \nu) \le \mathcal{B}_3(\nu)$$

where

$$(27)\qquad \mathcal{B}_3(\nu) = (1 - e^{-m})^2 + [(3\nu^2 + 4\nu + 2)e^{-2\nu} - 6(\nu + 1)e^{-\nu}e^{-m} + 4e^{-2m} - (1 - e^{-m})(1 - 2e^{-m})]^2, \qquad 0 \le \nu \le m.$$

By taking account of relation (14), it is not difficult to check that:

from cases a.4°, c.4° and e.4° we shall obtain, for $m > 0$,

$$G(x^2, y^2; \mu) \le \mathcal{A}_3(\mu), \qquad 0 \le \mu \le \nu^*,$$

where $\mathcal{A}_3$ is defined by formula (26);

cases b.4°, d.4° and f.4° will yield, for $m > 0$,

$$G(x^2, y^2; \mu) \le \mathcal{B}_3(\mu), \qquad 0 \le \mu \le m,$$

$\mathcal{B}_3$ being defined by (27);

from cases a.5°, c.5° and e.5° we shall get, for $m > 0$,

$$G(x^2, y^2; \mu) \le \mathcal{A}_2(\mu), \qquad 0 \le \mu \le \nu^*,$$

$\mathcal{A}_2$ being defined by (24);

cases b.5°, d.5° and f.5° will give, for $m > 0$,

$$G(x^2, y^2; \mu) \le \mathcal{B}_2(\mu), \qquad 0 \le \mu \le m,$$

with $\mathcal{B}_2$ defined by (25).

In consequence, the above considerations imply that, for a fixed $m > 0$,

$$G(x^2, y^2; \nu) \le \max\{\mathcal{A}_k(\nu), \mathcal{B}_k(\nu), \ k = 1, 2, 3\} \quad \text{if } \nu \in \langle 0, \nu^* \rangle,$$

whereas

$$G(x^2, y^2; \nu) \le \max\{\mathcal{B}_k(\nu), \ k = 1, 2, 3\} \quad \text{if } \nu \in (\nu^*, m\rangle.$$

4. In this section we shall occupy ourselves with the examination of the functions $\mathcal{A}_k$, $\mathcal{B}_k$, $k = 1, 2, 3$; namely, we shall determine the maxima of the functions $\mathcal{A}_k(\nu)$, $0 \le \nu \le \nu^*$ and $\mathcal{B}_k(\nu)$, $0 \le \nu \le m$, for any fixed $m > 0$.

In an easy way, from (22) and (23) one obtains, for $m > 0$,

$$(28) \qquad \begin{aligned} \mathcal{A}_1(\nu) &\le \mathcal{A}_1(\nu^*) \equiv \mathcal{A}^{(1)}(m), & \nu &\in \langle 0, \nu^* \rangle, \\ \mathcal{B}_1(\nu) &\le \mathcal{B}_1(0) \equiv \mathcal{A}^{(1)}(m), & \nu &\in \langle 0, m \rangle, \end{aligned}$$

where

$$(29) \qquad \mathcal{A}^{(1)}(m) = \tfrac{1}{4}(1 - e^{-2m})^2, \quad m > 0.$$

Examining the function $\mathcal{A}_2(\nu)$ given by (24), for $0 \le \nu \le \nu^*$, we get (cf. [5])

$$(30) \qquad \begin{aligned} \mathcal{A}_2(\nu) &\le \mathcal{A}_2(0) \equiv \mathcal{A}^{(2)}(m) & &\text{when } 0 < m \le m_1, \\ \mathcal{A}_2(\nu) &\le \mathcal{A}_2(\nu^*) = \mathcal{A}^{(1)}(m) & &\text{when } m_1 < m \le m_2, \\ \mathcal{A}_2(\nu) &\le \mathcal{A}_2(0) \equiv \mathcal{A}^{(2)}(m) & &\text{when } m > m_2, \end{aligned}$$

where $\mathcal{A}^{(1)}(m)$ is given by (29), whereas

$$(31) \qquad \mathcal{A}^{(2)}(m) = m^2 e^{-2m} + \tfrac{1}{4}[(3m^2 - 4m - 1)e^{-2m} + 1]^2,$$

and m_1, m_2 are the roots of the equation $\mathcal{A}^{(2)}(m) - \mathcal{A}^{(1)}(m) = 0$, $m_1 \in (0{,}28;\ 0{,}3)$, $m_2 \in (0{,}5;\ 0{,}54)$.

The investigation of the function $\mathcal{B}_2(\nu)$ given by formula (25) is very arduous. Proceeding similarly as in paper [5], one can prove that $\mathcal{B}_2$ is a decreasing function of the variable $\nu \in \langle 0, m\rangle$ if $0 < m \le \log 6$; whereas if $m > \log 6$, then $\mathcal{B}_2(\nu)$ has a local maximum at a point ν_o where ν_o, $\nu_o \in (0, m)$, is the only root of the equation $\mathcal{B}_2'(\nu) = 0$. Consequently,

$$(32)\qquad \begin{aligned} &\mathcal{B}_2(\nu) \le \mathcal{B}_2(0) \equiv \mathcal{B}^{(1)}(m) \quad \text{when} \quad 0 < m \le \log 6,\\ &\mathcal{B}_2(\nu) \le \mathcal{B}_2(\nu_o) \qquad\qquad\quad \text{when} \quad m > \log 6, \end{aligned}$$

where

$$(33)\qquad \mathcal{B}^{(1)}(m) = (1 - e^{-m})^2 + [(1 - e^{-m})(1 - 2e^{-m})]^2,$$

$$(34)\qquad \begin{aligned} \mathcal{B}_2(\nu_o) = {} & [(\nu_o + 1)e^{-\nu_o} - e^{-m}]^2 \\ & + \tfrac{1}{4}[(3\nu_o^2 + 2\nu_o + 1)e^{-2\nu_o} \\ & - 6(\nu_o + 1)e^{-\nu_o}e^{-m} + 4e^{-2m} + 1]^2, \end{aligned}$$

while ν_o, $\nu_o \in (0, m)$, is the root of the equation

$$(35)\qquad \begin{aligned} & 2[(\nu_o + 1)e^{-\nu_o} - e^{-m}] + [(3\nu_o - 1)e^{-\nu_o} - 3e^{-m}] \\ & \cdot [(3\nu_o^2 + 2\nu_o + 1)e^{-2\nu_o} - 6(\nu_o + 1)e^{-\nu_o}e^{-m} \\ & + 4e^{-2m} + 1] = 0. \end{aligned}$$

In turn, examining the function $\mathcal{A}_3(\nu)$ given by (26), for $0 \le \nu \le \nu^*$, we obtain

$$(36)\qquad \begin{aligned} &\mathcal{A}_3(\nu) \le \mathcal{A}_3(\nu^*) = \mathcal{B}^{(1)}(m) \quad \text{when} \quad o < m \le \tfrac{2}{3},\\ &\mathcal{A}_3(\nu) \le \mathcal{A}_3(0) \quad \equiv \mathcal{A}^{(3)}(m) \quad \text{when} \quad \tfrac{2}{3} < m \le m_3,\\ &\mathcal{A}_3(\nu) \le \mathcal{A}_3(\nu^*) = \mathcal{B}^{(1)}(m) \quad \text{when} \quad m > m_3, \end{aligned}$$

where $\mathcal{B}^{(1)}(m)$ is defined by (33),

$$\begin{aligned} \mathcal{A}^{(3)}(m) = {} & (1 - e^{-m})^2 + [(3m - 2)me^{-2m} \\ & - (1 - e^{-m})(1 - 2e^{-m})]^2, \end{aligned}$$

while m_3 is the root of the equation $\mathcal{A}^{(3)}(m) - \mathcal{B}^{(1)}(m) = 0$, $m_3 \in (0{,}7;\ 0{,}8)$.

To finish with, let us examine the function $\mathcal{B}_3(\nu)$ given by formula (27), for $0 \le \nu \le m$, $m > 0$. In paper [5], only its partial examination was carried out, namely, with a fixed $m \geq \log 3$

We have

$$\mathcal{B}_3'(\nu) = -4\nu e^{-\nu} g(\nu) h(\nu)$$

where

$$g(\nu) = (3\nu + 1)e^{-\nu} - 3e^{-m},$$

$$h(\nu) = (3\nu^2 + 4\nu + 2)e^{-2\nu} - 6(\nu + 1)e^{-\nu}e^{-m}$$
$$+ 4e^{-2m} - (1 - e^{-m})(1 - 2e^{-m}), \quad 0 \le \nu \le m.$$

Note that if $0 < m \le \frac{2}{3}$, then $g(\nu) \le 0$, $0 \le \nu \le m$. If $m \ge \log 3$, then $g(\nu) \ge 0$, $0 \le \nu \le m$. If $\frac{2}{3} < m < \log 3$, then the function $g(\nu)$ has exactly one zero $\nu_1 \in (0, \frac{2}{3})$. Since $h'(\nu) = -2\nu e^{-\nu} g(\nu)$, it suffices to examine the values of $h(0)$ and $h(m)$. It can be shown that $h(0) \le 0$ when $0 < m \le \log 2$ and $h(0) > 0$ when $m > \log 2$. Whereas $h(m) \le 0$ when $0 < m \le m_4$ and $m \ge m_5$ and $h(m) > 0$ when $m_4 < m < m_5$, with m_4, m_5 being the roots of the equation $h(m) = 0$, $m_4 < \frac{2}{3}$, $m_5 \in (\log 2, \log 3)$. Making use of the form of the derivative of the function $\mathcal{B}_3(\nu)$, we shall obtain that:

- if $0 < m \le m_4$, then $\mathcal{B}_3(\nu)$ is a decreasing function of the variable ν;

- if $m_4 < m \le \frac{2}{3}$, then $\mathcal{B}_3(\nu)$ has a local minimum at the point ν_2 where $h(\nu_2) = 0$, $\nu_2 \in (0, m)$;

- if $\frac{2}{3} < m \le \log 2$, then $\mathcal{B}_3(\nu)$ has a local minimum at the point ν_2, $h(\nu_2) = 0$, and $\mathcal{B}_3(\nu)$ has a local maximum at the point ν_1 where $g(\nu_1) = 0$, $\nu_2 < \nu_1$;

- if $\log 2 < m \le m_5$, then $\mathcal{B}_3(\nu)$ has a local maximum at the point ν_1, $g(\nu_1) = 0$;

- if $m_5 < m < \log 3$, then $\mathcal{B}_3(\nu)$ has a local maximum at the point ν_1, $g(\nu_1) = 0$, and $\mathcal{B}_3(\nu)$ has a local minimum at the point ν_2 where $h(\nu_2) = 0$, $\nu_1 < \nu_2$;

- if $m \geq \log 3$, then $\mathcal{B}_3(\nu)$ has a local minimum at the point ν_2 where $h(\nu_2) = 0$.

Hence and from the examination of the values of the function $\mathcal{B}_3(\nu)$ at the points $\nu = 0$ and $\nu = m$ we shall finally get

$$\mathcal{B}_3(\nu) \leq \mathcal{B}_3(0) = \mathcal{B}^{(1)}(m) \quad \text{when} \quad 0 < m \leq \tfrac{2}{3},$$

(37)
$$\mathcal{B}_3(\nu) \leq \mathcal{B}_3(\nu_1) \quad \text{when} \quad \tfrac{2}{3} < m < \log 3,$$

$$\mathcal{B}_3(\nu) \leq \mathcal{B}_3(0) = \mathcal{B}^{(1)}(m) \quad \text{when} \quad m \geq \log 3,$$

where $\mathcal{B}^{(1)}(m)$ is given by formula (33), while

$$\mathcal{B}_3(\nu_1) = (1 - e^{-m})^2 + [(3\nu_1^2 + 4\nu_1 + 2)e^{-2\nu_1} - 6(\nu_1 + 1)e^{-\nu_1}e^{-m} + 4e^{-2m} - (1 - e^{-m})(1 - 2e^{-m})]^2,$$

ν_1 being the only root of the equation $g(\nu) = 0$, i.e.

$$(3\nu_1 + 1)e^{-\nu_1} - 3e^{-m} = 0, \quad \nu_1 \in (0, \tfrac{2}{3}).$$

We have thus determined the maxima of the functions $\mathcal{A}_k$, $\mathcal{B}_k$, $k = 1, 2, 3$, for all values of $m > 0$.

5. We shall next carry out a comparison of the estimates of the functions $\mathcal{A}_k$, $\mathcal{B}_k$ obtained, for suitable values of m. Before we proceed to this, let us observe that the functions $\mathcal{A}_3(\nu)$ and $\mathcal{B}_3(\nu)$ given by formulae (26) and (27), respectively, have been obtained in the case when $x^2 + y^2 = (1 - e^{-m})^2$, $m > 0$ (compare a-f in section 3). It is known from the estimate of the coefficient $A_2 = -2(x + iy)$ in the class S(M), $\log M = m$, that this equality is possible only for the Pick function $w = P(z, M) \equiv P_\varepsilon(z, e^m)$ given by equation (5). The coefficients A_2, A_3 of this function are defined by the formulae

$$A_2 = 2\varepsilon(e^{-m} - 1),$$

$$A_3 = \varepsilon^2(e^{-m} - 1)(5e^{-m} - 3), \quad |\varepsilon| = 1, \quad m = \log M.$$

Putting $F = P_\varepsilon(z, e^m)$, from (7) we shall get

$$G(P_\varepsilon(z, e^m)) = (1 - e^{-m})^2 + [(1 - e^{-m})(1 - 2e^{-m})\cos 2\phi]^2,$$

$$\varepsilon = e^{i\phi}, \quad 0 \le \phi \le 2\pi.$$

It is easily verified that

$$(38)\qquad G(P_\varepsilon(z, e^m)) \le \max_{|\varepsilon|=1} G(P_\varepsilon(z, e^m)) = (1 - e^{-m})^2 + [(1 - e^{-m})(1 - 2e^{-m})]^2,$$

the last equality holding for $\phi = l\frac{\pi}{2}$, $l = 0, 1, 2, 3$.

On the other hand, as we have mentioned above, for any $\nu \in \langle 0, \nu^*\rangle$ and $m > 0$, there should exist an ε_1, $|\varepsilon_1| = 1$, such that $\mathcal{A}_3(\nu) = G(P_{\varepsilon_1}(z, e^m))$. Thus (38) implies that we may take into consideration only those ν and m for which

$$(39)\qquad \mathcal{A}_3(\nu) \le \max_{|\varepsilon_1|=1} G(P_{\varepsilon_1}(z, e^m)) = (1 - e^{-m})^2 + [(1 - e^{-m})(1 - 2e^{-m})]^2.$$

In consequence, in the case $\frac{2}{3} < m \le m_3$, estimate (36) contradicts (39) because it can be checked that $\mathcal{A}^{(3)}(m) \ge (1 - e^{-m})^2 + [(1 - e^{-m})(1 - 2e^{-m})]^2$ for $\frac{2}{3} < m \le m_3$. Since $\mathcal{B}^{(1)}(m) = (1 - e^{-m})^2 + [(1 - e^{-m})(1 - 2e^{-m})]^2$, therefore, of course, the remaining two estimates in (36) satisfy condition (39).

Analogously, for any $\nu \in \langle 0, m\rangle$ and $m > 0$, there should exist an ε_2, $|\varepsilon_2| = 1$, such that $\mathcal{B}^{(3)}(\nu) = G(P_{\varepsilon_2}(z, e^m))$. Consequently, (38) implies that we may take into account only those ν and m for which

$$(40)\qquad \mathcal{B}_3(\nu) \le \max_{|\varepsilon_2|=1} G(P_{\varepsilon_2}(z, e^m)) = (1 - e^{-m})^2 + [(1 - e^{-m})(1 - 2e^{-m})]^2.$$

If $\frac{2}{3} < m < \log 3$, then from the examination of the function $\mathcal{B}_3(\nu)$ it follows that, in (37), also $\mathcal{B}_3(\nu_1) > (1 - e^{-m})^2 + [(1 - e^{-m})(1 - 2e^{-m})]^2$, which, in virtue of (40), is impossible. The remaining two estimates in (37) evidently satisfy condition (40).

The above remarks do not concern, of course, the remaining functions, i.e. $\mathcal{A}_1(\nu)$, $\mathcal{A}_2(\nu)$, $\mathcal{B}_1(\nu)$, $\mathcal{B}_2(\nu)$, given by formulae (22), (24), (23), (25), respectively (cf. a-f). So, taking account of estimates (28), (30) and (32) obtained for them and of the above conclusions concerning estimates (36) and (37), we shall get that, for any function $F \in S(M)$, $M = e^m$,

$$G(F) \le \max\{\mathcal{A}^{(1)}(m), \mathcal{A}^{(2)}(m), \mathcal{B}^{(1)}(m)\}$$

when $m \in (0, m_1\rangle \cup (m_2, \log 6\rangle$,

$$G(F) \le \max\{\mathcal{A}^{(1)}(m), \mathcal{B}^{(1)}(m)\}$$

when $m \in (m_1, m_2\rangle$,

$$G(F) \le \max\{\mathcal{A}^{(1)}(m), \mathcal{A}^{(2)}(m), \mathcal{B}^{(1)}(m), \mathcal{B}_2(\nu_o)\}$$

when $m \in (\log 6, +\infty)$.

Let us first notice that, for each $m > 0$, the inequalities

$$\mathcal{A}^{(1)}(m) \le \mathcal{A}^{(2)}(m) < \mathcal{B}^{(1)}(m)$$

hold. Whereas from the examination of the function $\mathcal{B}_2(\nu)$ defined by (25), carried out in section 4, it follows that

$$\mathcal{B}^{(1)}(m) < \mathcal{B}_2(\nu_o) \quad \text{when} \quad m > \log 6.$$

So, we have finally obtained that, for each function $F \in S(M)$, $M = e^m > 1$, the following estimate of functional (7) takes place:

$$(41) \qquad G(F) \le \begin{cases} \mathcal{B}^{(1)}(m) & \text{when} \quad 0 < m \le \log 6, \\ \mathcal{B}_2(\nu_o) & \text{when} \quad m > \log 6, \end{cases}$$

where $\mathcal{B}^{(1)}(m)$, $\mathcal{B}_2(\nu_o)$ are defined by formulae (33), (34), respectively, with ν_o being the only root of equation (35).

It still remains to prove that estimate (41) we have obtained is sharp for each $m > 0$. If $m \in (0, \log 6\rangle$, the equality in (41) takes place for the Pick function defined by equation (5) for $\varepsilon = \pm 1$, $\varepsilon = \pm i$, $m = \log M$.

In order to show that, also for $m > \log 6$, estimate (41) in the class $S(M)$ is sharp, it is enough to prove, in view of c.2^o, e.2°, f.2° from section 3 and on account of Lemma B, that there exists a function $\Theta_*(\tau)$, $0 \le \tau \le m$, for which $y^2 = 0$, i.e.

$$(42) \qquad \int_0^m e^{-\tau} \sin \Theta_*(\tau) d\tau = 0$$

and $|\lambda_1(\tau)| = \varkappa(\tau)$.

Let ν_o, $\nu_o \in (0, m)$, be a solution of equation (35), whereas $\Theta_*(\tau)$ a function defined by the formulae

$$\cos \Theta_*(\tau) = \begin{cases} e^{\tau-\nu_o} & \text{for } 0 \le \tau \le \nu_o, \\ 1 & \text{for } \nu_o \le \tau \le m. \end{cases}$$

Then

$$\sin \Theta_*(\tau) = \begin{cases} \pm\sqrt{1-e^{2(\tau-\nu_o)}} & \text{for } 0 \le \tau \le \nu_o, \\ 0 & \text{for } \nu_o \le \tau \le m, \end{cases}$$

whence one can easily obtain the formulae for the function $k_*(\tau) = e^{i\Theta_*(\tau)}$ and, in consequence, determine the respective solution $F_* \in S(M)$ of the Löwner equation. Of course, $\lambda_*(\tau) = e^{-\tau} \cos \Theta_*(\tau) = \varkappa(\tau)$.

By choosing different signs in portions of the interval $\langle 0, \nu_o \rangle$, one can make condition (42) be satisfied. Indeed, let us consider, for instance, the function

$$\phi(x) = \int_0^x e^{-\tau}\sqrt{1-e^{2(\tau-\nu_o)}}\, d\tau - \int_x^{\nu_o} e^{-\tau}\sqrt{1-e^{2(\tau-\nu_o)}}\, d\tau,$$

$$x \in \langle 0, \nu_o \rangle.$$

It is continuous in the interval $\langle 0, \nu_o \rangle$, $\phi(0) < 0$, $\phi(\nu_o) > 0$, thus there exists a point $x_o \in (0, \nu_o)$ such that $\phi(x_o) = 0$. Putting then

$$\sin \Theta_*(\tau) = \begin{cases} \sqrt{1-e^{2(\tau-\nu_o)}} & \text{for } 0 \le \tau \le x_o, \\ -\sqrt{1-e^{2(\tau-\nu_o)}} & \text{for } x_o \le \tau \le \nu_o, \\ 0 & \text{for } \nu_o \le \tau \le m, \end{cases}$$

we finally obtain condition (42).

We have thus shown that, for each $M > 1$, there exist functions of the classes $S(M)$ realizing the equality of estimate (41), with that $m = \log M$. Thereby, (41), (7) and (3) imply the following

THEOREM. If H is any function of form (1) from the class $S^{(2)}(\sqrt{M})$, then the following estimates hold:

$$(43)\qquad |c_3|^2 + |c_5|^2 \le \begin{cases} (1-\frac{1}{M})^2 + [(1-\frac{1}{M})(1-\frac{2}{M})]^2 \\ \text{when } 1 < M \le 6, \\ [(\nu_o + 1)e^{-\nu_o} - \frac{1}{M}]^2 + \frac{1}{4}[(3\nu_o^2 + 2\nu_o + 1)e^{-2\nu_o} \\ - \frac{6}{M}(\nu_o + 1)e^{-\nu_o} + \frac{4}{M^2} + 1]^2 \\ \text{when } M > 6, \end{cases}$$

where $\nu_o \in (0, \log M)$ is the root of the equation

$$2[(\nu + 1)e^{-\nu} - \frac{1}{M}] + [(3\nu - 1)e^{-\nu} - \frac{3}{M}][(3\nu^2 + 2\nu + 1)e^{-2\nu} - \frac{6}{M}(\nu + 1)e^{-\nu} + \frac{4}{M^2} + 1] = 0.$$

For each $M > 1$, there exist functions of the class $S^{(2)}(\sqrt{M})$ for which the equality sign in (43) takes place.

REMARK. It can be shown that if $M \to \infty$, then the root ν_o of equation (35), tends to zero. Consequently, from (43) it follows that, for each function $H \in S^{(2)}$ ([4]),

$$|c_3|^2 + |c_5|^2 \le 2.$$

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O OSZACOWANIU PEWNEGO FUNKCJONAŁU W KLASIE OGRANICZONYCH NIEPARZYSTYCH FUNKCJI JEDNOLISTNYCH

Oznaczmy przez $S(M)$, $M > 1$, rodzinę funkcji jednolistnych, holomorficznych w kole $E = \{z: |z| < 1\}$ postaci

$$F(z) = z + A_2z^2 + \dots + A_nz^n + \dots,$$

spełniających w kole E warunek $|F(z)| < M$, $M > 1$. Przez $S^{(2)}(\sqrt{M})$ oznaczmy klasę funkcji jednolistnych, nieparzystych, postaci

$$H(z) = z + C_3z^3 + C_5z^5 + \dots + C_{2n+1}z^{2n+1} + \dots,$$

spełniających w kole E warunek $|H(z)| < \sqrt{M}$, $M > 1$.

Oczywiście, dla każdej funkcji $F \in S(M)$ funkcja $H(z) = \sqrt{F(z^2)}$ należy do $S^{(2)}(\sqrt{M})$ i na odwrót.

W pracy dowodzi się, że ma miejsce następujące

Twierdzenie. Jeżeli H jest dowolną funkcją klasy $S^{(2)}(\sqrt{M})$, to zachodzą następujące oszacowania

$$|C_3|^2 + |C_5|^2 \leqq \begin{cases} (1 - \frac{1}{M})^2 + \left[(1 - \frac{1}{M})(1 - \frac{2}{M})\right]^2, & \text{gdy } 1 < M \leqq 6, \\ \left[(\nu_o + 1)e^{-\nu_o} - \frac{1}{M}\right]^2 + \frac{1}{4}\left[(3\nu_o^2 + 2\nu_o + 1)e^{-2\nu_o} - \right. \\ \left. - \frac{6}{M}(\nu_o + 1)e^{-\nu_o} + \frac{4}{M^2} + 1\right]^2, & \text{gdy } M > 6, \end{cases}$$

gdzie $\nu_o \in (0, \log M)$ jest pierwiastkiem równania

$$2\left[(\nu+1)e^{-\nu} - \frac{1}{M}\right] + \left[(3\nu-1)e^{-\nu} - \frac{3}{M}\right]\left[(3\nu^2+2\nu+1)e^{-2\nu} - \right.$$

$$\left. - \frac{6}{M}(\nu+1)e^{-\nu} + \frac{4}{M^2} + 1\right] = 0.$$

Dla każdego $M > 1$ istnieją funkcje klasy $S^{(2)}(\sqrt{M})$, dla których ma miejsce znak równości w powyższym oszacowaniu.