A C T A   U N I V E R S I T A T I S   L O D Z I E N S I S
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THE SYMMETRIC $\mathfrak{I}$-APPROXIMATE DERIVATIVES

In this paper we shall give a definition of a symmetric $\mathfrak{I}$-approximate derivative of a function $f : R \to R$. We shall prove several properties of its $\mathfrak{I}$-approximate derivative.

Throughout this paper $\mathfrak{B}$ will denote the family of all subsets of R (the real line) having the Baire property, $\mathfrak{I}$ will denote sigma ideal of sets of the first category. For two sets $A, B \subset R$, $A \sim B$ will mean that $A \triangle B \in \mathfrak{I}$. For $a \in R$ and $A \subset R$ we denote $a \cdot A = \{a \cdot x : x \in A\}$ and $A - a = \{x - a : x \in A\}$. Recall [4] that 0 is an $\mathfrak{I}$-density point of a set $A \in \mathfrak{B}$ if and only if $\chi(n \cdot A) \cap [-1, 1] \xrightarrow[n \to \infty]{\mathfrak{I}} 1$ i.e. if and only if for every increasing sequence $\{n_m\}_{m \in N}$ of natural numbers there exists a subsequence $\{n_{m_p}\}_{p \in N}$ such that $\chi(n_{m_p} \cdot A) \cap [-1, 1] \xrightarrow[p \to \infty]{} 1$ except on a set belonging to $\mathfrak{I}$ (in abbr. $\mathfrak{I}$-a.e.). A point $x_o \in R$ is an $\mathfrak{I}$-density point of $A \in \mathfrak{B}$ if and only if 0 is an $\mathfrak{I}$-density point of $A - x_o$. The set of all $\mathfrak{I}$-density points of A will be denoted by $\Phi(A)$. Recall that Φ has the following properties: for each $A \in \mathfrak{B}$ $\Phi(A) \sim A$; for each $A, B \in \mathfrak{B}$ if $A \sim B$ then $\Phi(A) = \Phi(B)$; $\Phi(\emptyset) = \emptyset$; $\Phi(R) = R$; for each $A, B \in \mathfrak{B}$ $\Phi(A \cap B) = \Phi(A) \cap \Phi(B)$.

Further the family $\mathcal{T}_{\mathfrak{I}} = \{A \in \mathfrak{B} : A \subset \Phi(A)\}$ is a topology, which we call $\mathfrak{I}$-density topology (see [4]).

Real functions continuous with respect to $\mathcal{T}_{\mathfrak{I}}$ topology we call the $\mathfrak{I}$-approximately continuous functions.

In [1] was introduced the topology τ such that τ is a coarsest topology making all $\mathfrak{I}$-approximately continuous functions continuous.

Throughout this paper cl(A), int(A) will denote closure and interior of the set A with respect to natural topology. Except where a topology is specifically mentioned, all topological notions are with respect to the natural topology.

Definition 1 [1]. For $x \in R$ by $\mathcal{P}(x)$ we will denote the family of all closed intervals $[a, b]$ such that $x \in (a, b)$ and of all interval sets $\bigcup_{n=1}^{\infty} [a_n, b_n] \cup \bigcup_{n=1}^{\infty} [c_n, d_n] \cup \{x\}$ where for all n $b_{n-1} < a_n < b_n < x$ and $x < c_n < d_n < c_{n-1}$ and $x \in \Phi(\bigcup_{n=1}^{\infty} [a_n, b_n] \cup \bigcup_{n=1}^{\infty} [c_n, d_n])$.

Definition 2 [1]. Let τ be the collection of all subsets U of R such that:

1. $U \in \mathcal{T}_{\mathcal{J}}$
2. if $U \neq \emptyset$ and $x \in U$, then there exists a set $P \in \mathcal{P}(x)$ such that $P \subset \operatorname{int} U \cup \{x\}$.

Theorem 1 [1]. τ is topology on R, $\tau \subsetneq \mathcal{T}_{\mathcal{J}}$ and if f is any $\mathcal{J}$-approximately continuous function then f is a continuous function with respect to τ.

In this paper we shall need the following lemmas:

Lemma 1 [4]. If 0 is an $\mathcal{J}$-density point of a set A, then for every natural number n there exists a number $\delta_n > 0$ such that for every h, with $0 < h < \delta_n$ and for every natural k fulfilling the inequality $-n \leq k \leq n-1$ we have

$$A \cap [\frac{k}{n} \cdot h, \frac{k+1}{n} \cdot h] \neq \emptyset$$

Lemma 2 [2]. Let $G \subset R$ be an open set. Then 0 is an $\mathcal{J}$-dispersion point of G if and only if, for every natural number n, there exist a natural number k and a real number $\delta > 0$ such that, for each $h \in (0, \delta)$ and for each $i \in \{1, ..., n\}$ there exist two natural numbers $j_r, j_l \in \{1, ..., k\}$ such that

$$G \cap ((\frac{i-1}{n} + \frac{j_r - 1}{n \cdot k}) \cdot h, (\frac{i-1}{n} + \frac{j_r}{n \cdot k}) \cdot h) = \emptyset$$

and

$$G \cap (-(\frac{i-1}{n} + \frac{j_l}{n \cdot k}) \cdot h, -(\frac{i-1}{n} + \frac{j_l - 1}{n \cdot k}) \cdot h) = \emptyset$$

We shall use the above lemmas for $x \in R$ by translating the set if necessary.

Definition 3. Let f be a function having the Baire property defined on the closed interval [a, b]. We shall call upper symmetric $\mathcal{J}$-approximate derivative of f at a point $c \in (a, b)$, $\overline{f}^{s}_{\mathcal{J}\text{-ap}}(c)$, the greatest lower bound of all the numbers $\alpha(+\infty$ included) for which the set

$$\{t : \frac{f(c + t) - f(c - t)}{2t} \leqslant \alpha\}$$

has 0 as a point of $\mathcal{J}$-density.

Similarly we can define lower symmetric $\mathcal{J}$-approximate derivative $\underline{f}^{s}_{\mathcal{J}\text{-ap}}(c)$. When these derivatives are equal, their common value is termed symmetric $\mathcal{J}$-approximate derivative of f at c and is written $f^{s}_{\mathcal{J}\text{-ap}}(c)$.

At the end points a and b, we mean $\overline{f}^{s}_{\mathcal{J}\text{-ap}}(a) = \overline{f}_{\mathcal{J}\text{-ap}}(a)$ and $\underline{f}^{s}_{\mathcal{J}\text{-ap}}(b) = \underline{f}_{\mathcal{J}\text{-ap}}(b)$ where $\overline{f}_{\mathcal{J}\text{-ap}}$, $\underline{f}_{\mathcal{J}\text{-ap}}$ is ordinary upper (lower) $\mathcal{J}$-approximate derivatives (see [3]).

Theorem 2. If $f : R \to R$ is an $\mathcal{J}$-approximately continuous function then $\overline{f}^{s}_{\mathcal{J}\text{-ap}}(x)$ and $\underline{f}^{s}_{\mathcal{J}\text{-ap}}(x)$ have the property of Baire.

P r o o f. First we observe that $\underline{f}^{s}_{\mathcal{J}\text{-ap}}(x) = -(\overline{-f})^{s}_{\mathcal{J}\text{-ap}}(x)$. Therefore it is sufficent to show that for each $a \in R$, a set $A = \{x : \overline{f}^{s}_{\mathcal{J}\text{-ap}}(x) < a\}$ has the property of Baire.

Let $H(x, h) = \dfrac{f(x + h) - f(x - h)}{2h}$ for all $x \in R$ and $h > 0$. Let $a \in R$ and $\{a_m\}_{m \in N}$ be an arbitrary sequence such that, for each $m \in N$, $a_m < a_{m+1} < a$ and $\lim_{m \to \infty} a_m = a$. For each $m \in N$, we shall denote $B_m(x) = \{h > 0 : H(x, h) > a_m\}$ and $A_m = \{x : B_m(x)$ has an $\mathcal{J}$-dispersion point at 0}. It is obvious that $A = \bigcup_{m=1}^{\infty} A_m$. Since for each $x \in R$ the function $H(x, h) : R^{+} \to R$ is an $\mathcal{J}$-continuous function therefore for each $x \in R$ and $m \in N$ we have $B_m(x) \in \tau$, $\operatorname{int} B_m(x) \neq \emptyset$ and $A_m = \{x : \operatorname{int} B_m(x)$ has an $\mathcal{J}$-dispersion point at 0}.

Let $m \in N$ and $x \in R$. By *lemma 2* we have that for each $n \in N$, there exist $k \in N$ and $p \in N$ such that, for each $\delta \in (0, \frac{1}{p})$ and for each $i \in \{1, ..., n\}$, there exists $j \in \{1, ..., k\}$ such that,

$$\left(\frac{(i - 1) \cdot k + j - 1}{n \cdot k} \cdot \delta, \ \frac{(i - 1) \cdot k + j}{n \cdot k} \cdot \delta\right) \cap \operatorname{int} B_m(x) = \emptyset$$

Thus

$$A = \bigcup_{m=1}^{\infty} \bigcap_{n=1}^{\infty} \bigcup_{k=1}^{\infty} \bigcup_{p=1}^{\infty} \bigcap_{\delta \in (0,\frac{1}{p})} \bigcap_{i=1}^{\infty} \bigcup_{j=1}^{\infty} D(m, n, k, p, \delta, i, j),$$

where $D(m, n, k, p, \delta, i, j) = \{x : (\frac{(i-1)\cdot k + j - 1}{n \cdot k} \cdot \delta, \frac{(i-1)\cdot k + j}{n \cdot k} \cdot \delta) \cap \operatorname{int} B_m(x) = \emptyset\}$.

Let $m, n, k, p \in N$, $\delta \in (0, \frac{1}{p})$, $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, k\}$. We shall show that $D = D(m, n, k, p, \delta, i, j)$ is a closed set with respect to τ-topology.

Let $x_o \notin D$. Then there exists an open interval (α, β) such that $(\alpha, \beta) \subset \operatorname{int} B_m(x_o) \cap (\frac{(i-1)\cdot k + j - 1}{n \cdot k} \cdot \delta, \frac{(i-1)\cdot k + j}{n \cdot k} \cdot \delta)$. By $\mathcal{G}$-continuity of the function $H(x_o, h)$ on (α, β) (see [4]), we know that there exists $h_o \in (\alpha, \beta)$ such that, $H(x_o, h)$ is continuous at h_o. Since $H(x_o, h_o) > a_m$ then there exist $\varepsilon_o > 0$ and $\eta > 0$ such that,

(*) if $|h - h_o| < \eta$ then $\frac{f(x_o + h) - f(x_o - h)}{2h} > a_m + \varepsilon_o > a_m$.

Let $\varepsilon < 2\varepsilon_o h_o$. We shall consider the point $x_o - h_o$. The function f is $\mathcal{G}$-continuous at $x_o - h_o$ and therefore $x_o - h_o$ belongs to $\{t : |f(t) - f(x_o - h)| < \frac{\varepsilon}{2}\}$ which is open with respect to topology τ. If $C = \operatorname{int}\{t : |f(t) - f(x_o - h_o)| < \frac{\varepsilon}{2}\} \cup \{x_o - h_o\}$ then $C \in \tau$. Let $E = C \cap (-C + 2x_o - 2h_o)$. We observe that $E \in \tau$, $x_o - h_o \in E$, for each $t \in E$, $-t + 2x_o - 2h_o \in E$ and

(**) for each $t_1, t_2 \in E$, $|f(t_1) - f(t_2)| < \varepsilon$.

In the similar way we can find the set $F \in \tau$ such that $x_o + h_o \in F$, for each $t \in F$, $-t + 2x_o - 2h_o \in F$ and

(***) for each $t_1, t_2 \in F$, $|f(t_1) - f(t_2)| < \varepsilon$.

Let $E_1 = (E \cap (x_o - h_o - \eta, x_o - h_o]) + h_o$ and $x \in E_1$. Then $x - x_o \in (-\eta, 0]$ and $x - h_o \in E$. Therefore, if $h = x - x_o$ then $x_o - h_o + h \in E$ and $x_o - h_o - h \in E$. Then, by (*) and (**), we have that

$$\frac{f(x + h_o) - f(x - h_o)}{2h_o} = \frac{f(x_o + h_o + h) - f(x_o - h_o + h)}{2h_o} >$$

$$\frac{f(x_o + h_o + h) - f(x_o - h_o - h) - \varepsilon}{2h_o} > \frac{f(x_o + h_o + h) - f(x_o - h_o - h)}{2(h_o + h)}$$

$$- \frac{\varepsilon}{2h_o} > a_m + \varepsilon_o - \frac{\varepsilon}{2h_o} > a_m.$$

Therefore $x \notin D$.

Now, let $F_1 = (F \cap [x_o + h_o, x_o + h_o + \eta)) - h_o$ and $x \in F_1$. In the similar way, by (*) and (***) we can show that $x \notin D$.

Let $M = F_1 \cup E_1$. Then $M \in \tau$, $x_o \in M$ and $M \subset R \setminus D$. Thus D is a closed set with respect to τ-topology and by it, D has the property of Baire.

Now, it is obvious that A has the property of Baire and proof of theorem is complete.

In the similar way we can prove the following proposition:

Proposition 1. If $f : R \to R$ is a continuous function then $\overline{f}^s_{\mathcal{J}\text{-ap}}(x)$ and $\underline{f}^s_{\mathcal{J}\text{-ap}}(x)$ belong to third class of Baire.

Theorem 3. Let f be a monotone function defined on open interval I. Then for each $x \in I$ $\underline{f}^s_{\mathcal{J}\text{-ap}}(x) = \underline{f}^s(x)$ and $\overline{f}^s_{\mathcal{J}\text{-ap}}(x) = \overline{f}^s(x)$.

P r o o f. We shall assume that f is a nondecreasing function.

First we observe that for each $x \in I$, $\underline{f}^s(x) \leq \underline{f}^s_{\mathcal{J}\text{-ap}}(x)$. Now we suppose that there exists $x_o \in I$ such that,

$$k_1 = \underline{f}^s(x_o) < \underline{f}^s_{\mathcal{J}\text{-ap}}(x_o) = k_2.$$

Let $0 < \varepsilon < \frac{k_2 - k_1}{2}$ and $B = \{h > 0 : \frac{f(x_o + h) - f(x_o - h)}{2h} \geq k_2 - \frac{\varepsilon}{2}\}$.

Since $\underline{f}^s_{\mathcal{J}\text{-ap}}(x_o) = k_2$, therefore 0 is a right-hand $\mathcal{J}$-density point of the set B. Thus, by *lemma 1*,

(*) for each $n \in N$, there exists $\delta_n > 0$ such that, for each $0 < h < \delta_n$ and for each $0 \leq l \leq n-1$ $[\frac{l}{n} \cdot h, \frac{l + 1}{n} h] \cap B \neq \emptyset$.

By assumption that $\underline{f}^s(x_o) = k_1 < k_2 - 2\varepsilon$, we know that there exists a sequence $\{h_n\}_{n \in N}$ such that, $\lim_{n \to \infty} h_n = 0$ and for each $n \in N$, $h_n > 0$, $x_o + h_n \in I$ and

$$(**) \quad \frac{f(x_o + h_n) - f(x_o - h_n)}{2h_n} < k_2 - 2\varepsilon.$$

We shall consider closed intervals $J_n = [\frac{f(x_o + h_n) - f(x_o - h_n)}{2(k_2 - \varepsilon)}, h_n]$ for each $n \in N$ and we shall show that for each $n \in N$ $B \cap J_n = \emptyset$. Indeed, if $h \in J_n$ then $\frac{f(x_o + h_n) - f(x_o - h_n)}{2(k_2 - \varepsilon)} \leqslant h \leqslant h_n$. Thus, by monotonicity of the function f we have that

$$f(x_o + h) \leqslant f(x_o + h_n) \leqslant f(x_o - h_n) + 2h(k_2 - \varepsilon) \leqslant f(x_o - h_n) + 2h(k_2 - \varepsilon)$$

and

$$\frac{f(x_o + h) - f(x_o - h)}{2h} \leqslant k_2 - \varepsilon.$$

Therefore $h \notin B$.

Now, let $n_o > \frac{k_2 - \varepsilon}{\varepsilon}$, $\delta_{n_o} \in R$ such that δ_{n_o} and n_o satisfy the condition (*). We choose $h_{n_1} \in \{h_n\}_{n \in N}$ such that $h_{n_1} < \delta_{n_o}$. Then, by (**), we know that the length of the interval J_{n_1} equals $h_{n_1} - \frac{f(x_o + h_{n_1}) - f(x_o - h_{n_1})}{2(k_2 - \varepsilon)}$ and by simple computations we know that it is not less then $\frac{h_{n_1}}{n_o}$. Thus $[\frac{n_o - 1}{n_o} \cdot h_{n_1}, h_{n_1}] \subset J_{n_1}$ and by above $[\frac{n_o - 1}{n_o} \cdot h_{n_1}, h_{n_1}] \cap B = \emptyset$ which gives a contradiction.

To prove of $\bar{f}^s_{\mathcal{J}\text{-ap}} \equiv \bar{f}^s$ we observe that for each $x \in I$ $\bar{f}^s_{\mathcal{J}\text{-ap}}(x) \leqslant \bar{f}^s(x)$. Now, we suppose that there exists $x_o \in I$ such that

$$k_1 = \bar{f}^s_{\mathcal{J}\text{-ap}}(x_o) < \bar{f}^s(x_o) = k_2.$$

Let $0 < \varepsilon < \frac{k_2 - k_1}{2}$ and $C = \{h > 0 : \frac{f(x_o + h) - f(x_o - h)}{2h} \leqslant k_1 + \frac{\varepsilon}{2}\}$. Since $\bar{f}^s_{\mathcal{J}\text{-ap}}(x_o) = k_1$, therefore 0 is a right-hand $\mathcal{J}$-density point of the set C.

Thus, by *lemma 1*

(***) for each $n \in N$ there exists $\delta_n > 0$ such that, for each $0 < h < \delta_n$ and for each $0 \leqslant l \leqslant n-1$ $[\frac{l}{n} \cdot h, \frac{l + 1}{n} \cdot h] \cap C \neq \emptyset$.

By assumption that $\overline{f}^s(x_o) = k_2 > k_1 + 2\varepsilon$, we know that there exists a sequence $\{h_n\}_{n \in N}$ such that, $\lim_{n\to\infty} h_n = 0$ and for each $n \in N$, $h_n > 0$, $x_o + h_n \in I$ and

$$(****)\quad \frac{f(x_o + h_n) - f(x_o - h_n)}{2h_n} > k_1 + 2\varepsilon.$$

We shall consider the closed intervals $J'_n = [h_n, \frac{f(x_o + h_n) - f(x_o - h_n)}{2(k_1 + \varepsilon)}]$ and in the similar way as above we can prove that for all $n \in N$ $C \cap J'_n = \emptyset$.

Now, let $n'_o > \frac{k_1 + \varepsilon}{\varepsilon}$ and $\delta'_{n_o} \in R$ such that δ'_{n_o} with n_o satisfy the condition (***). We choose $h'_{n_1} \in \{h_n\}_{n \in N}$ such that $h'_{n_1} < \delta'_{n_o}$. Then, by (****), we know that length of the interval J'_{n_1} is not less then $\frac{h'_{n_1}}{n_o}$. Thus $[\frac{n'_o - 1}{n_o} \cdot h'_{n_1}, h'_{n_1}] \cap C = \emptyset$ which gives a contradiction and proof of the theorem is complete.

Corollary. If f is a monotone and symmetrically $\mathcal{T}$-approximately differentiable function on an open interval I then f is a symmetrically differentiable function.

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𝒯-APROKSYMATYWNA SYMETRYCZNA POCHODNA

W pracy tej podana jest definicja 𝒯-aproksymatywnej symetrycznej pochodnej funkcji. $f : R \to R$ i udowodnione są pewne własności tej pochodnej, które zachodzą również dla aproksymatywnej symetrycznej pochodnej. A mianowicie pokazano, że przy założeniu 𝒯-ciągłości funkcji f 𝒯-aproksymatywne symetryczne pochodne górna i dolna posiadają własności Baire'a oraz że pochodne te są równe pochodnej symetrycznej odpowiednio górnej i dolnej w przypadku gdy funkcja f jest funkcją monotoniczną określoną na przedziale otwartym.