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SOME RESULTS INVOLVING FOX´S H-FUNCTION GENERALIZED HYPERGEOMETRIC SERIES, BESSEL FUNCTIONS AND TRIGONOMETRIC SINES

In this paper, we have evaluated an integral involving Fox´s H-function, generalized hypergeometric series, Bessel function and trigonometric sine, and employed it to evaluate one double integral involving Fox´s H-function, generalized hypergeometric series, Bessel functions and trigonometric sines. We have further utilized the integral to establish one Fourier-Bessel series and one double Fourier-Bessel series for the products of generalized hypergeometric functions and trigonometric sines.

1. INTRODUCTION

The object of this paper is to evaluate an integral involving Fox´s H-function, generalized hypergeometric series, Bessel function and trigonometric sine, and utilize it to evaluate a double integral involving Fox´s H-function, generalized hypergeometric series, Bessel functions, and trigonometric sines. We further use the integral to establish a Fourier-Bessel series and a double Fourier-Bessel series for the products of generalized hypergeometric series, the H-function and trigonometric sines.

We also discuss some particular cases of our results and show how our results lead to generalization of many results, some of which are new and others were earlier given by R. L. T a x a k [20].

The subject of expansion formulae and Fourier series of generalized hypergeometric functions occupies a prominent place in the literature of special functions. Certain expansion formulae and

Fourier series of the generalized hypergeometric functions play an important role in the development of the theories of special functions and boundary value problems. It is interesting to note that there is a wide scope of applying the theory of expansion theorems and Fourier series in the fields of boundary value problems and applied mathematics. For example, some results of this paper can be used to obtain certain solutions of the partial differential equation concerning the problem of free oscillations of water in a circular lake ([16], pp. 45-47; [19], pp. 202-203).

The Fourier-Bessel series for generalized hypergeometric functions have been given by S. D. Bajpai [1-5], S. P. Goyal [11] and R. L. Taxak [20], [21]. The references given in this paper [14], [15], [19] together with the sources indicated in these references provide a good converge of the subject. However, it is important to note that so far nobody has attempted to establish single and multiple Fourier-Bessel series for the products of the generalized hypergeometric functions. This paper, therefore appears to be an attempt on the subject of single and multiple Fourier-Bessel series for the products of generalized hypergeometric functions.

The Fox's H-function is a generalization of Meijer's G-function ([7], pp. 206-222) and therefore on specializing the parameters, the H-function may be reduced to almost all special functions appearing in pure and applied mathematics ([15], pp. 144--159). Therefore the results obtained in this paper are of a very general character and hence may encompass several cases of interest. Our results are master or key formulae from which a large number of results can be derived for Meijer's G-function, MacRobert's E-function, Hypergeometric functions, Bessel functions, Legendre functions, Whittaker functions, orthogonal polynomials, trigonometric functions and other related functions.

It is very important to note that operations such as differentiation and integration could almost be performed more readily on the H-function than on the original functions, even though the two are equivalent. Thus the H-function facilitates the analysis by permitting complex expressions to be represented and handled more simply.

The H-function introduced by Fox ([9], p. 408) will be represented and defined as follows:

$$(1.1)\quad H^{u,v}_{p,q}\left[z\left|\begin{matrix}(a_p, e_p)\\(b_q, f_q)\end{matrix}\right.\right] = H^{u,v}_{p,q}\left[z\left|\begin{matrix}(a_1, e_1), ---, (a_p, e_p)\\(b_1, f_1), ---, (b_q, f_q)\end{matrix}\right.\right]$$

$$= \frac{1}{2\pi i}\int_L \chi(s) z^s ds,$$

where L is a suitable Barnes contour and

$$\chi(s) = \frac{\prod_{j=1}^{u}\Gamma(b_j - f_j s)\prod_{j=1}^{v}\Gamma(1 - a_j + e_j s)}{\prod_{j=u+1}^{q}\Gamma(1 - b_j + f_j s)\prod_{j=v+1}^{p}\Gamma(a_j - e_j)}.$$

Asymptotic expansion and analytic continuation of the H-function had been given by B. L. J. B r a a k s m m a [6].

The following formulae are required in the proofs:

$$(1.2)\quad \int_0^\infty x^{\sigma-1}\, \mathcal{J}_m(\omega x) \sin \omega x\, {}_PF_Q\left[\begin{matrix}\alpha_P;\ cx^{2h}\\ \beta_Q\end{matrix}\right]dx$$

$$= \frac{2^{m-1}}{\omega^\sigma}\sum_{\xi=0}^{\infty}\frac{(\alpha_P)_\xi c^\xi \Gamma(\frac{1}{2} - \sigma - 2h\xi)\Gamma(\frac{1}{2} + \frac{1}{2}m + \frac{1}{2}\sigma + h\xi)\omega^{-2h\xi}}{(\beta_Q)_\xi\, \xi!\,\Gamma(1 + m - \sigma - 2h\xi)\Gamma(1 - \frac{1}{2}m - \frac{1}{2}\sigma - h\xi)}$$

where α_P denotes $\alpha_1, \ldots, \alpha_P$; h is a positive integer: $P < Q$ (or $P = Q + 1$ and $|c| < 1$); no one of the β_Q is zero or a negative integer; $-1 < \mathrm{Re}\ m < \mathrm{Re}\ \sigma < \frac{1}{2}$, $\omega > 0$.

The integral (1.2) can easily be established by expressing the generalized hypergeometric series as ([7], p. 181, (1)) and interchanging the order of integration and summation involved in the process, which is justified due to the absolute convergence of the integral and summation involved in the process, and evaluating the integral with the help of [8], p. 328, (10).

$$(1.3)\quad \int_0^\infty x^{\sigma-1}\, \mathcal{J}_m(\omega x) \sin \omega x\ {}_PF_Q\left[\begin{matrix}\alpha_P;\ cx^{2h}\\ \beta_Q\end{matrix}\right] {}_uF_v\left[\begin{matrix}\gamma_u;\ dx^{2k}\\ \delta_v\end{matrix}\right]dx$$

$$= \frac{2^{m-1}}{\omega^\sigma}\sum_{\xi,t=0}^{\infty}\frac{(\alpha_P)_\xi c^\xi (\gamma_u)_t d^t \Gamma(\frac{1}{2} - \sigma - 2h\xi - 2kt)\Gamma(\frac{1}{2} + \frac{1}{2}m + \frac{1}{2}\sigma +}{(\beta_Q)_\xi\, \xi!\,(\delta_v)_t\, t!\,\Gamma(1 + m - \sigma - 2h\xi - 2kt)}$$

$$\frac{+\, h\xi + kt)\omega^{-2(h\xi + kt)}}{\Gamma(1 - \frac{1}{2}m - \frac{1}{2}\sigma - h\xi - kt)},$$

where in addition to the conditions and notations of (1.2), k is a positive integer; $U < V$ (or $U = V + 1$ and $|d| < 1$); no one of the δ_V is zero or a negative integer.

To derive (1.3), we use the series representation for ${}_UF_V$, interchange the order of integration and summation and evaluate the resulting integral with the help of (1.2).

Note 1.1. On applying the above procedure the integral analogous to (1.2) for the products of n generalized hypergeometric series can be derived easily.

The orthogonality property of Bessel functions ([13], p. 291, (6)).

$$(1.4)\qquad \int_0^\infty x^{-1}\, \mathcal{J}_{\nu+2n+1}(x)\, \mathcal{J}_{\nu+2m+1}(x)dx$$

$$= \begin{cases} 0, \text{ if } m \neq n; \\ (4n + 2\nu + 2)^{-1}, \text{ if } m = n,\ \mathrm{Re}\nu + m + n > -1. \end{cases}$$

In what follows for sake of brevity in addition to the notations earlier given in this section, λ and μ are positive numbers and

$$\Phi(\xi) = \frac{(\alpha_P)_\xi\, c^\xi}{(\beta_Q)_\xi\, \xi!};\quad \Psi(t) = \frac{(\gamma_U)_t\, d^t}{(\delta_V)_t\, t!};$$

$$F_1(x) = {}_PF_Q\left[\begin{matrix}\alpha_P;\ cx^{2h}\\ \beta_Q\end{matrix}\right];\quad F_2(x) = {}_UF_V\left[\begin{matrix}\gamma_U;\ dx^{2k}\\ \delta_V\end{matrix}\right];$$

$$H(x) = H^{u,v}_{p,q}\left[z\, x^{2\lambda}\left|\begin{matrix}(a_p, e_p)\\ (b_q, f_q)\end{matrix}\right.\right];$$

$$H_1(m,\xi,t) = H^{u+1,v+1}_{p+3,q+1}\left[\omega^{-2\lambda} z\left|\begin{matrix}(-\frac{1}{2} - \frac{1}{2}m - \frac{1}{2}\sigma - h\xi - kt,\ \lambda),\\ (1 + m - \sigma - 2h\xi - 2kt,\ 2\lambda),\\ (\frac{1}{2} - \sigma - 2h\xi - 2kt,\ 2\lambda),\\ (a_p, e_p)\\ (1 - \frac{1}{2}m - \frac{1}{2}\sigma - h\xi - kt,\ \lambda)\\ (b_q, f_q)\end{matrix}\right.\right];$$

$$H_2(xy) = H^{u,v}_{p,q}\left[zx^{2\lambda}\, y^{2\mu}\middle|\begin{matrix}(a_p,\ e_p)\\(b_q,\ f_q)\end{matrix}\right]$$

$$H_3(m_1,\ m_2,\ \xi_1,\ t_1,\ \xi_2,\ t_2) =$$

$$H^{u+2,v+2}_{p+6,q+2}\left[\omega^{-2(\lambda+\mu)}z\middle|\begin{matrix}(-\tfrac{1}{2}-\tfrac{1}{2}m_1-\tfrac{1}{2}\sigma_1-h\xi_1-kt_1,\ \lambda),\\(a_p,\ e_p),\ (1+m_1-\sigma_1-2h\xi_1-2kt_1, 2\lambda),\\(1+m_2-\sigma_2-2h\xi_2-2kt_2,\ 2\mu),\\(\tfrac{1}{2}-\sigma_1-2h\xi_1-2kt_1,\ 2\lambda),\\(b_q,\ f_q)\end{matrix}\right.$$

$$\left.\begin{matrix}(-\tfrac{1}{2}-\tfrac{1}{2}m_2-\tfrac{1}{2}\sigma_2-h\xi_2-kt_2,\ \mu),\\(1-\tfrac{1}{2}m_1-\tfrac{1}{2}\sigma_1-h\xi_1-kt_1,\ \lambda),\\(1-\tfrac{1}{2}m_2-\tfrac{1}{2}\sigma_2-h\xi_2-kt_2,\ \mu);\\(\tfrac{1}{2}-\sigma_2-2h\xi_2-2kt_2,\ 2\mu),\end{matrix}\right];$$

$$A = \sum_{j=1}^{p} aj - \sum_{j=1}^{q} bj;\quad B = \sum_{j=1}^{v} aj - \sum_{j=v+1}^{p} aj + \sum_{j=1}^{u} bj - \sum_{j=u+1}^{q} bj.$$

2. INTEGRAL

(i) The integral to be evaluated is

$$(2.1)\quad \int_0^\infty x^{\sigma-1}\, \mathcal{T}_m(\omega x)\sin\omega x\, F_1(x)\, F_2(x)\, H(x)dx$$

$$= \frac{2^{m-1}}{\omega^\sigma}\sum_{\xi,t=0}^{\infty} \Phi(\xi)\Psi(t)(\omega)^{-2(h\xi+kt)}\, H_1(m,\xi,t),$$

where $A \leqslant 0$, $B > 0$, $|\arg z| < \frac{1}{2}B\Pi$, $-1 < \mathrm{Re}[m+2\lambda b_j/f_j] < \mathrm{Re}[\sigma + 2\lambda b_j/f_j] < \frac{1}{2}$, $j = 1, \ldots, u$, $\mathrm{Re}\ \omega > 0$ together with the conditions given in (1.2) and (1.3).

P r o o f. To establish (2.1), expressing the H-function in the integrand as a Mellin-Barnes type integral (1.1) and interchanging the order of integrations, which is justified due to the absolute convergence of the integrals involved in the process, we have

$$\frac{1}{2\pi i}\int_L \chi(s)z^s \int_0^\infty x^{\sigma+2\lambda s-1}\, \mathcal{T}_m(\omega x)\, \sin \omega x\, F_1(x)\, F_2(x)dxds.$$

Evaluating the inner-integral with the help of (1.3), we get

$$= \frac{2^{m-1}}{\omega^\sigma} \sum_{\xi,t=0}^{\infty} \Phi(\xi)\Psi(t)(\omega)^{-2(h\xi+kt)}$$

$$\chi \frac{1}{2\pi i}\int_L x(s)z^s \frac{\Gamma(\frac{1}{2}-\sigma-2h\xi-2kt-2\lambda s)\Gamma(\frac{1}{2}+\frac{1}{2}m+\frac{1}{2}\sigma+h\xi+kt+\lambda s)\,\omega^{-2\lambda s}}{\Gamma(1+m-\sigma-2h\xi-2kt-2\lambda s)\Gamma(1-\frac{1}{2}m-\frac{1}{2}\sigma-h\xi-kt-\lambda s)}ds.$$

Now, using (1.1), the value of the integral (2.1) is obtained.

Note 2.2. The integral analogous to (2.1), involving the product of n generalized hypergeometric series, Bessel function and the H-function can be evaluated easily with the help of the result mentioned in Note 1.1.

(ii) Particular cases. In (2.1), putting $d = 0$, we get

$$(2.2)\qquad \int_0^\infty x^{\sigma-1}\, T_m(\omega x)\, \sin \omega x\, F_1(x)\, H(x)dx$$

$$= \frac{2^{m-1}}{\omega^\sigma} \sum_{\xi=0}^{\infty} \Phi(\xi)(\omega)^{-2h\xi} H_1(m,\xi,0),$$

valid under the conditions of (2.1) with $d = 0$.

It is interesting to note that F. Singh and R. C. Varma [18] evaluated an integral involving the product of an associated Legendre function, a generalized hypergeometric series and the H-function ([15], p. 40, (2.9.4)) on making use of finite difference operator E ([17], p. 33 with $w = 1$). It is also interesting to note that K. C. Gupta and G. S. Olkha [12] evaluated an integral involving the product of a generalized hypergeometric series and the H-function using an integral due to G. K. Goyal ([10], p. 202).

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G o y a l ([19], pp. 61-63) presented some integrals based on the technique of Gupta and Olkha.

In view of the above discussion and [14], [15], [19] it appears that our integral is more general and new in addition to the new and simple technique of evaluating such integrals.

In (2.2), setting c = 0, we obtain

$$(2.3)\quad \int_0^\infty x^{\sigma-1}\, \mathcal{J}_m(\omega x) \sin \omega x\, H(x)dx = \frac{2^{m-1}}{\omega^\sigma} H_1(m, 0, 0),$$

valid under the conditions of (2.2) with $c = 0$.

In (2.3), putting $\omega = 1$, $\sigma = \mu$, $m = \nu$, $\lambda = \delta$ and simplifying with the help of (1.1) and ([7], p. 4, (11)), it reduces to a known result due to R. L. T a x a k ([20], p. 94, (2.2)).

3. DOUBLE INTEGRAL

(i) The double integral to be evaluated is

$$(3.1)\quad \int_0^\infty \int_0^\infty x^{\sigma_1-1} y^{\sigma_2-1} \mathcal{J}_{m_1}(\omega x) \sin \omega x\, \mathcal{J}_{m_2}(\omega y) \sin \omega y\, F_1(x)\, F_2(x)$$

$$F_1(y)\, F_2(y) \cdot H_2(xy)dxdy$$

$$= \frac{2^{m_1+m_2-2}}{\omega^{\sigma_1+\sigma_2}} \sum_{\xi_1, t_1=0}^\infty \sum_{\xi_2,\, t_2=0}^\infty \Phi(\xi_1)\Psi(t_1)\, \Phi(\xi_2)\Psi(t_2)$$

$$x(\omega)^{-2(h\xi_1+kt_1+h\xi_2+kt_2)} H_3(m_1, m_2, \xi_1, t_1, \xi_2, t_2),$$

where $A \leqslant 0$, $B > 0$, $|\arg z| < \frac{1}{2}B\pi$, $-1 < \mathrm{Re}[m_1 + 2\lambda b_j/f_j] < \mathrm{Re}[\sigma_1 + 2\lambda b_j/f_j] < \frac{1}{2}$, $-1 < \mathrm{Re}[m_2 + 2\mu b_j/f_j] < \mathrm{Re}[\sigma_2 + 2\mu b_j/f_j] < \frac{1}{2}$, $j = 1, \ldots, u$ together with the conditions given in (1.2) and (1.3).

P r o o f. To establish (3.1), evaluating the x-integral with the help of (2.1) and interchanging the order of integration and summation, we get

$$\frac{2^{m_1-1}}{\omega^{\sigma_1}} \sum_{\xi, t_1=0}^\infty \Phi(\xi_1)\Psi(t_1)(\omega)^{-2(h\xi_1+kt_1)}$$

$$\times \int_0^\infty y^{\sigma_2-1} \mathcal{J}_{m_2}(\omega y) \sin(\omega y) F_1(y) F_2(y)$$

$$\times H^{u+1,v+1}_{p+3,q+1}\left[y^{2\mu}\omega^{-2\lambda} z \,\middle|\, \begin{matrix} (-\tfrac{1}{2}-\tfrac{1}{2}m_1-\tfrac{1}{2}\sigma_1-h\xi_1-kt_1,\lambda), \\ (1+m_1-\sigma_1-2h\xi_1-2kt_1,2\lambda), \\ (\tfrac{1}{2}-\sigma_1-2h\xi_1-2kt_1,2\lambda),(b_q,f_q) \end{matrix} \right.$$

$$\left. \begin{matrix} (a_p,e_p), \\ (1-\tfrac{1}{2}m_1-\tfrac{1}{2}\sigma_1-h\xi_1-kt_1,\lambda); \end{matrix} \right] dy.$$

Now, applying (2.1) to evaluate the y-integral, the value of the integral (3.1) is obtained.

Note 3.1. The multiple integral analogous to (3.1) can be established easily on applying the above technique (n-1) times.

(ii) Particular cases. Putting d = 0 in (3.1), we get

$$(3.2) \quad \int_0^\infty \int_0^\infty x^{\sigma_1-1} y^{\sigma_2-1} \mathcal{J}_{m_1}(\omega x) \sin \omega x \, \mathcal{J}_{m_2}(\omega y) \sin \omega y \, F_1(x) F_1(y)$$

$$H_2(xy)\,dx\,dy$$

$$= \frac{2^{m_1+m_2-2}}{\omega^{\sigma_1+\sigma_2}} \sum_{\xi_1,\xi_2=0}^{\infty} \Phi(\xi_1)\Phi(\xi_2)(\omega)^{-2(h\xi_1+h\xi_2)}$$

$$\times H_3(m_1, m_2, \xi_1, 0, \xi_2, 0),$$

valid under the conditions of (3.1) with d = 0.

In (3.2), setting c = 0, we obtain

$$(3.3) \quad \int_0^\infty \int_0^\infty x^{\sigma_1-1} y^{\sigma_2-1} \mathcal{J}_{m_1}(\omega x) \sin \omega x \, \mathcal{J}_{m_2}(\omega y) \sin \omega y \, H_2(xy)\,dx\,dy$$

$$= \frac{2^{m_1+m_2-2}}{\omega^{\sigma_1+\sigma_2}} H_3(m_1, m_2, 0, 0, 0, 0),$$

valid under the conditions of (3.2) with c = 0.

Note 3.2. The integrals of this section may be employed to establish double and multiple Fourier-Bessel series for the products of generalized hypergeometric series and the H-function.

4. FOURIER-BESSEL SERIES

(i) The Fourier-Bessel series to be established is

$$(4.1)\quad x^{\sigma} \sin \omega x\, F_1(x)F_2(x)H(x)$$

$$= \frac{2^m}{\omega^{\sigma}} \sum_{n=0}^{\infty} N\mathcal{J}_N(\omega x) \sum_{\xi,t=0}^{\infty} \Phi(\xi)\Psi(t)(\omega)^{-2(h\xi+kt)} H_1(N, \xi, t),$$

where $N = \nu + 2n + 1$, $\operatorname{Re} N > 0$ and other conditions of validity are same as in (2.1).

P r o o f. To prove (4.1), let

$$(4.2)\quad f(x) = x^{\sigma} \sin \omega x\, F_1(x)\, F_2(x)\, H(x) = \sum_{n=0}^{\infty} C_n\, \mathcal{J}_{\nu+2n+1}(\omega x).$$

Equation (4.1) is valid, since $f(x)$ is continuous and of bounded variation in the open interval $(0, \infty)$.

Multiplying both sides of (4.2) by $x^{-1}\ J_{\nu+2m+1}(\omega x)$ and integrating with respect to x from 0 to ∞, we get

$$\int_0^{\infty} x^{\sigma-1} \mathcal{J}_{\nu+2m+1}(\omega x) \sin \omega x\, F_1(x)\, F_2(x)\, H(x)dx$$

$$= \sum_{n=0}^{\infty} C_n \int_0^{\infty} x^{-1} \mathcal{J}_{\nu+2m+1}(\omega x)\, \mathcal{J}_{\nu+2n+1}(\omega x) dx.$$

Now using (2.1) and (1.4), we get

$$(4.3)\quad C_m = \frac{2^m}{\omega^{\sigma}} M \sum_{\xi,t=0}^{\infty} \Phi(\xi)\Psi(t)(\omega)^{-2(h\xi+kt)} H_1(M,\xi, t),$$

where $M = \nu + 2m + 1$.
From (4.2) and (4.3), the Fourier-Bessel series (4.1) follows.

(ii) Particular cases. In (4.1), putting $d = 0$, we obtain

$$(4.4)\quad x^{\sigma} \sin \omega x\, F_1(x)\, H(x)$$

$$= \frac{2^m}{\omega^{\sigma}} \sum_{n=0}^{\infty} N\mathcal{J}_N(\omega x) \sum_{\xi=0}^{\infty} \Phi(\xi)(\omega)^{-2h\xi} H_1(N,\xi, 0),$$

valid under the conditions of (4.1) with $d = 0$.

In (4.4), setting $c = 0$, we get

(4.5) $x^{\sigma} \sin \omega x\, H(x)$

$$= \frac{2^m}{\omega^{\sigma}} \sum_{n=0}^{\infty} NH_1(N, 0, 0)\, \mathfrak{J}_N(\omega x),$$

valid under the conditions of (4.4) with $c = 0$.

In (4.5), putting $\omega = 1$, $\sigma = \mu$, $\lambda = \delta$ and simplifying with the help of (1.1) and [7], p. 4, (11). It yields a known result given by R. L. T a x a k ([20], p. 95, (3.2)).

5. DOUBLE FOURIER-BESSEL SERIES

(i) The double Fourier-Bessel series to be established is

(5.1) $x^{\sigma_1} y^{\sigma_2} \sin \omega x \sin \omega y\, F_1(x)\, F_2(x)\, F_1(y)\, F_2(y)\, H_2(xy)$

$$= \frac{2^{m_1+m_2}}{\omega^{\sigma_1+\sigma_2}} \sum_{n_1,n_2=0}^{\infty} N_1 N_2\, \mathfrak{J}_{N_1}(\omega x)\, T_{N_2}(\omega y)$$

$$\times \sum_{\xi_1,t_1=0}^{\infty} \sum_{\xi_2,t_2=0}^{\infty} \Phi(\xi_1)\Psi(t_1)\Phi(\xi_2)\Psi(t_2)(\omega)^{-2(h\xi_1+kt_1+h\xi_2+kt_2)}$$

$$\times H_3(N_1, N_2, \xi_1, t_1, \xi_2, t_2),$$

where $N_1 = \nu_1 + 2n_1 + 1$, $N_2 = \nu_2 + 2n_2 + 1$ and valid under the conditions of (3.1).

P r o o f. To establish (5.1), let

(5.2) $f(x, y) = x^{\sigma_1} y^{\sigma_2} \sin \omega x \sin \omega y\, F_1(x)F_2(x)F_1(y)F_2(y)\, H(xy)$

$$= \sum_{n_1,n_2=0}^{\infty} A_{n_1,n_2}\, \mathfrak{J}_{\nu_1+2n_1+1}(\omega x)\, \mathfrak{J}_{\nu_2+2n_2+1}(\omega y).$$

Equation (5.2) is valid, since $f(x, y)$ is continuous and of bounded variation in the open interval $(0, \infty)$.

The series (5.2) is an example of what is called a double Fourier-Bessel series. Instead of discussing the theory, we show a method to find A_{n_1,n_2} from (5.2). For fixed x, we note that

$$\sum_{n_1=0}^{\infty} A_{n_1,n_2}\, \mathfrak{J}_{\nu_1+2n_1+1}(\omega x)$$

depends only on n_2, furthermore, it must be the coefficient of Fourier-Bessel series in y of f(x, y) over $0 < y < \infty$.

Multiplying both sides of (5.2) by $y^{-1} \mathfrak{J}_{\nu_2+2m_2+1}(\omega y)$ integrating with respect to y from 0 to ∞ and using (2.1) and (1.4), we get

(5.3) $$x^{\sigma_1} \sin \omega x\, F_1(x)\, F_2(x) \frac{2^{m_2}}{\omega^{\sigma_2}} M_2 \sum_{\xi_2, t_2=0} \Phi(\xi_2)\Psi(t_2)(\omega)^{-2(h\xi_2+kt_2)}$$

$$\times H^{u+1,v+1}_{p+3,q+1}\left[\omega^{-2}\mu x^{2\lambda} z \left| \begin{matrix} (-\frac{1}{2} - \frac{1}{2}m_2 - \frac{1}{2}\sigma_2 - h\xi_2 - kt_2, \mu), \\ 1 + m_2 - \sigma_2 - 2h\xi_2 - 2kt_2, 2\mu), \\ (\frac{1}{2} - \sigma_2 - 2h\xi_2 - 2kt_2, 2\mu), (b_q, f_q) \end{matrix} \right.\right.$$

$$\left. \begin{matrix} (a_p, e_p), \\ (1 - \frac{1}{2}m_2 - \frac{1}{2}\sigma_2 - h\xi_2 - kt_2 - \mu); \end{matrix} \right]$$

$$= \sum_{n_1=0}^{\infty} A_{n_1,m_2} \mathfrak{J}_{\nu_1+2n_1+1}(\omega x),$$

where $M_2 = \nu_2 + 2m_2 + 1$.

Multiplying both sides of (5.3) by $x^{-1} \mathfrak{J}_{\nu_1+2m_1+1}(\omega x)$ integrating with respect to x from 0 to ∞ and using (2.1) and (4.1), we obtain.

(5.4) $$A_{m_1,m_2} = \frac{2^{m_1+m_2}}{\omega^{\sigma_1+\sigma_2}} M_1 M_2 \sum_{\xi_1,t_1=0}^{\infty} \sum_{\xi_2,t_2=0}^{\infty} \Phi(\xi_1)\Psi(t_1)\Phi(\xi_2)\Psi(t_2)$$

$$\times (\omega)^{-2(h\xi_1+kt_1+h\xi_2+kt_2)} H_3(M_1, M_2, \xi_1, t_1, \xi_2, t_2),$$

where $M_1 = \nu_1 + 2m_1 + 1$.

From (5.2) and (5.4), the double Fourier-Bessel series (5.1) is obtained.

(ii) Particular cases. In (5.1), putting d = 0, we get

$$(5.5)\quad x^{\sigma_1} y^{\sigma_2} \sin \omega x \sin \omega y\, F_1(x)\, F_1(y)\, H_2(xy)$$

$$= \frac{2^{m_1+m_2}}{\omega^{\sigma_1+\sigma_2}} \sum_{n_1,n_2=0}^{\infty} N_1\, N_2\, \mathcal{J}_{N_1}(\omega x)\, \mathcal{J}_{N_2}(\omega y)$$

$$\times \sum_{\xi_1,\xi_2=0}^{\infty} \Phi(\xi_1)\Phi(\xi_2)(\omega)^{-2h(\xi_1+\xi_2)} \times H_3(N_1, N_2, \xi_1, 0, \xi_2, 0),$$

valid under the conditions of (5.1) with d = 0.

In (5.5), taking c = 0, we obtain

$$(5.6)\quad x^{\sigma_1} y^{\sigma_2} \sin \omega x \sin \omega y\, H_2(xy)$$

$$= \frac{2^{m_1+m_2}}{\omega^{\sigma_1+\sigma_2}} \sum_{n_1,n_2=0}^{\infty} N_1\, N_2\, H_3(N_1, N_2, 0, 0, 0, 0)$$

$$\times \mathcal{J}_{N_1}(\omega x)\, \mathcal{J}_{N_2}(\omega y),$$

valid under the conditions of (5.5) with c = 0.

<u>Note 5.1.</u> Multiple Fourier-Bessel series analogous to (5.1) can be established on applying the above technique repeatedly.

<u>Note 5.2.</u> The results analogous to our main results (2.1), (3.1), (4.1) and (5.1) involving the H-function of several complex variables ([19], pp. 251-255) can be derived easily on following the techniques given in this paper.

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PEWNE REZULTATY DOTYCZĄCE H-FUNKCJI FOXA,
UOGÓLNIONYCH SZEREGÓW HIPERGEOMETRYCZNYCH,
FUNKCJI BESSELA I ROZWINIĘĆ TRYGONOMETRYCZNYCH

Praca poświęcona jest obliczaniu pewnych całek zawierających specjalne funkcje typu H-funkcje Foxa oraz zastosowaniu tych wyników do rozwinięć iloczynu uogólnionych funkcji hipergeometrycznych na pojedynczy i wielokrotny szereg Fouriera-Bessela. Formuły podane są w tzw. postaci zamkniętej i również zawierają H-funkcje.