ACTA UNIVERSITATIS LODZIENSIS
FOLIA MATHEMATICA 5, 1992

M. I. Krastanov, V. M. Veliov

LOCAL CONTROLLABILITY OF STATE CONSTRAINED LINEAR SYSTEMS

This paper deals with a comprehensive solution of the ILC problem for a linear system with a state constraint. There is considered the system $\dot{x} \in Ax + V$, $x(0) = 0$, $\langle x, l\rangle \geq 0$, where $x \in R^n$, $V \subset R^n$, $l \neq 0$, A is a matrix of corresponding dimension.

1. INTRODUCTION

The property of a linear control system to be small time locally controllable (abbreviated further as ILC - Instantaneously Locally Controllable) plays an important role and was comprehensively studied in a great number of papers, among which we shall mention Brammer [2], Bianchini [1], Sussmann [4] and Veliov [5], where the ILC problem was completely solved for arbitrary control constraints. However, to the authors' knowledge, there are no results concerning the ILC property of a state constrained linear system, despite that this is the case in many economical and physical models.

In the present paper we give a comprehensive solution of the ILC problem for a linear system with a state constraint. The exact formulation is the following. We consider the system

(1.1) $\dot{x} \in Ax + V, \quad x(0) = 0, \langle x, l\rangle \geq 0,$

where $x \in R^n$, $V \subset R^n$, $l \neq 0$, A is a matrix of corresponding dimension. Denote by R(t) the reachable set of (1.1) on [0, t], that is, the set of all points x(t) where $s \to x(s)$ is an absolutely continuous function on [0, t] satisfying the relations in (1.1) for almost every $s \in [0, t]$. Since R(t) is a subset of the half-space

$$\Pi_+ = \{x:\ \langle x, l\rangle \geq 0\},$$

the ILC property can be defined as

$$0 \in \mathrm{int}_{\Pi_+} R(t) \quad \text{for every} \quad t > 0,$$

where int_{Π_+} is the interior in Π_+.

We show in the next section that the above property is very strong and holds only under quite restrictive conditions. For this reason, in sections 3 and 4 we characterize the "expansion cone" of $R(\cdot)$ at $t = 0$, which gives a comprehensive information about the evolution of the reachable set at $t = 0$. In doing this we use ideas of F r a n c o w s k a [3], W a l c z a k [8], V e l i o v and K r a s t a n o v [6], [7].

2. NECESSARY CONDITIONS FOR ILC

The results of this section will point out the importance of the existence of a vector $v \in V$ such that $\langle v, l\rangle < 0$. If such a vector does not exist, then system (1.1) is ILC only under rather specific conditions which are easily checkable, as seen below.

PROPOSITION 2.1. Let system (1.1) be ILC and let $\langle v, l\rangle \geq 0$ for every $v \in V$. Then the following conditions are fulfilled:

(i) l is an eigenvector of the matrix A^* (* means transposition);

(ii) there is $\bar{v} \in V$ such that $\langle \bar{v}, l\rangle > 0$;

(iii) the system

$$\dot{x} \in Ax + V \cap \Pi_o, \quad x(0) = 0,$$

is ILC when is considered on

$$\Pi_o = \{x:\ \langle x, l\rangle = 0\}$$

(in view of (i), Π_o is invariant with respect to A).

P r o o f.

(i) For an arbitrary trajectory $x(\cdot)$ of (1.1) on $[0, t]$,

$$\langle \dot{x}(s), l\rangle = \langle Ax(s), l\rangle + \langle v, l\rangle \geq \langle Ax(s), l\rangle.$$

If $\langle x(t), l\rangle = 0$, since $\langle x(s), l\rangle \geq 0$, we have

$$0 = \langle x(t), l\rangle = \langle x(s), l\rangle + \int_s^t \langle \dot{x}(\tau), l\rangle d\tau \geq \int_s^t \langle Ax(\tau), l\rangle\, d\tau$$

for every $s \le t$, which implies $\langle Ax(t), l\rangle \le 0$. Since (1.1) is supposed to be ILC, we conclude that the equality $\langle x, l\rangle = 0$ must imply $\langle Ax, l\rangle \le 0$ or, equivalently, $\langle Ax, l\rangle = 0$, which holds if and only if $A^*l = \alpha l$ for some α.

(ii) If $\langle v, l\rangle = 0$ for every $v \in V$, then

$$\langle \dot{x}(s), l\rangle = \langle Ax(s), l\rangle = \alpha\langle x(s), l\rangle,$$

which shows that $\langle x(s), l\rangle = 0$ for every trajectory $x(\cdot)$ of (1.1).

(iii) If $x(\cdot)$ is a trajectory of (1.1) on $[0, t]$, corresponding to some $v(\cdot)$, $v(s) \in V$, then $y(s) = \langle x(s), l\rangle$ and $w(s) = \langle v(s), l\rangle$ satisfy

$$\dot{y} = \alpha y + w, \quad y(0) = 0, \quad w(s) \ge 0,$$

which apparently gives that if $y(t) = 0$, then $y(s) = w(s) = 0$ for every $s \in [0, t]$, which means that $v(s) \in V \cap \Pi_o$. Hence, the ILC of (1.1) implies (iii).

REMARK 2.1. Necessary and sufficient conditions for (iii) are given in the above-mentioned papers [1], [2], [4], [5].

In the "unconstrained" case, $V = BU$, $U \subseteq R^r$, $0 \in \operatorname{int} U$, B - $(n \times r)$ - matrix, the inequality $\langle v, l\rangle \ge 0$ holds for every $v \in V$ if and only if $B^*l = 0$. In this case, property (ii) in Proposition 2.1 is never satisfied, which implies the following

COROLLARY 2.1. If $B^*l = 0$, then the system

$$(2.1) \qquad \dot{x} \in Ax + BU, \quad \langle x, l\rangle \ge 0$$

is not ILC.

If $B^*l \ne 0$, then a necessary and sufficient condition for the ILC of (1.1) can be extracted from the more general results in [6] and [7], but it will also be obtained as a consequence of the results in the next sections.

3. DIRECTIONS OF EXPANSION OF THE REACHABLE SET

Consider in R^n system (2.1), where $U \subseteq R^r$, $0 \in \operatorname{int} U$ and A and B are of corresponding dimensions. As seen in the previous sections, the condition $B^*l \ne 0$ is necessary for the ILC of (2.1). This condition, however, is not fulfilled by many systems of interest. For this reason, we shall use the notion of direction of expansion of the reachable set to characterize its local behaviour at $t = 0$.

DEFINITION 3.1. The vector $p \in R^n$ will be called direction of expansion of $R(\cdot)$ if there is strictly positive function $\varphi(\cdot)$ defined for $t > 0$, such that

$$\lim_{t \to 0} \operatorname{dist}(\varphi(t)p, \; R(t))/\varphi(t) = 0.$$

The function $\varphi(\cdot)$ can be considered as a lower estimate of the speed of expansion of $R(\cdot)$ in the direction p.

It is also reasonable to apply the same definition to the mapping $t \to R(t) \cap \Pi_o$, insted of $R(t)$, coming in this way to the notion of direction of expansion of $R(t)$ in Π_o. By D (or D_o) we shall denote the set of all directions of expansion of $R(\cdot)$ (resp. in Π_o).

The above notion can be illustrated by the simplest example

$$\dot{x} = y, \quad x \geq 0,$$
$$\dot{y} = u.$$

Here $D = \{(p, q); \; p \geq 0\}$, while $D_o = \{(0, -1)\}$.

We shall also mention that $p \in D$ implies that for any $\alpha \geq 0$, $\varepsilon > 0$ and $t > 0$, the inclusion $\alpha p \in R(t) + O(\varepsilon)$ holds, provided that the set U is sufficiently large (depending on p, α, t and ε), ($O(\varepsilon)$ denotes the ball with radius ε, centered at the origin of the respective space).

LEMMA 3.1.

(i) D and D_o are convex cones;

(ii) D and D_o are closed;

(iii) if $p \in \operatorname{int} D$ or $p \in \operatorname{int}_{\Pi_o} D$, then

$\Psi(t)p \in R(t)$, (resp. $\Psi(t)p \in R(t) \cap \Pi_o$) for some strictly positive function $\Psi(\cdot)$, (i.e. in this case p is a direction of expansion of $R(\cdot)$ in a stronger sense, as all vectors from D in the examp, except for (0, 1).

P r o o f.

(i) Follows apparently from the definition and from the convexity of R(t).

(ii) Let $p_k \in D$ and $\lim_{k \to \infty} p_k = p \neq 0$. Let $\varphi_k(\cdot)$ be the function from Definition 3.1. Then, given $t > 0$, we have $\varphi_k(t)p_k = x_k(t) + o_k(t)$ for some $x_k(t) \in R(t)$ and $o_k(\cdot)$, such that $\alpha_k(t) =$

$= o_k(t)/\varphi_k(t)$ tends to zero with t. We shall define a sequence $s_1, s_2, \ldots$ as follows: s_1 is such that $\alpha_1(s) \leq 1$ for $s \in [0, s_1]$ and inductively, $s_{k+1} < s_k/2$ is such that $\alpha_{k+1}(s) \leq 1/(k+1)$ for $s \in [0, s_{k+1}]$. Define

$$\alpha(s) = \begin{cases} \alpha_k(s) & \text{if } s \in (s_{k+1}, s_k), \\ 1 & \text{if } s > s_1. \end{cases}$$

Since $\{s_k\}$ tends to zero, $\alpha(s)$ is correctly defined for $s > 0$. Moreover, for $s < s_k$, we have $\alpha(s) = \alpha_{k'}(s) \leqslant 1/k' \leq 1/k$ for some $k' \geq k$, which means that $\alpha(s)$ goes to zero with s. Now, define, for $t < s_1$, the integer $k(t)$ by the requirement $t \in (s_{k(t)+1}, s_{k(t)}]$ and the function $\varphi(t) = \varphi_{k(t)}(t)$. We have

$$\varphi(t)p = \varphi_{k(t)}(t)p = \varphi_{k(t)}(t)p_{k(t)} + \varphi_{k(t)}(t)(p - p_{k(t)})$$
$$= x_{k(t)}(t) + o_{k(t)}(t) + \varphi(t)(p - p_{k(t)}).$$

Since $x_{k(t)}(t) \in R(t)$, it remains to prove that

$$\{o_{k(t)}(t) + \varphi(t)(p - p_{k(t)})\}/\varphi(t) \xrightarrow[t \to o]{} 0.$$

For the second term, this follows from the obvious property $k(t) \to +\infty$ as $t \to 0$. Moreover,

$$o_{k(t)}/\varphi(t) = o_{k(t)}/\varphi_{k(t)}(t) = \alpha_{k(t)}(t) = \alpha(t) \to 0,$$

which completes the proof of (ii).

(iii) The proof uses in a standard way the convexity and the monotonicity of $R(t)$.

As a consequence of Lemma 3.1 (iii) we obtain that if $D = \Pi_+$ and $D_o = \Pi_o$, then (2.1) is ILC. If only $D = \Pi_+$, then (2.1) is only "approximately" ILC, in particular, $\mathrm{cl}\, R(t) = \Pi_+$ for every $t > 0$ if $U = R^r$ (as in the example considered above).

In the next section we shall describe constructively the cones D and D_o.

4. CHARACTERIZATION OF THE EXPANSION CONES

Let k be the smalest integer such that

$$\langle A^k B\bar{u}, l\rangle > 0 \text{ for some } \bar{u} \in R^r.$$

If such a k does not exist, then $R(t) \subseteq \Pi_o$ for every t and the set $D = D_o$ can be characterized in the standard way (there is no more state constraint) related to the Kalman rank condition. Thus we can suppose that the integer k exists.

Denote $b = B\bar{u}$ and define the matrix C by

(4.1) $\quad Cu = Bu - \rho\langle A^k Bu, l\rangle b$

where $\rho = 1/\langle A^k Bu, l\rangle$. Consider the system

(4.2) $\quad \dot{x} \in Ax + CU + b[-1, 1]$.

Obviously, the sets D and D_o do not depend on the "size" of the set U since $0 \in \operatorname{int} U$, and that is the only difference between (2.1) and (4.2). We shall consider the system (4.2) which is in a more convenient form, because of the relations

$$\langle Cu, l\rangle = \ldots = \langle A^k Cu, l\rangle = 0, \quad u \in R^r.$$

Now, introduce the matrix P by

$$Px = Ax - \rho \langle A^{k+1}x, l\rangle b.$$

By $[C]_j$ we shall denote the j-th column of C.

THEOREM 4.1. The expansion cones D_o and D of the reachable set of system (2.1) are

$$D_o = \operatorname{cone} \{\pm P^i[C]_j, \quad i = 0, \ldots, n-2, \quad j = 1, 2, \ldots, r,$$
$$\pm P^m b, \quad m = 0, \ldots, k-2, \quad -P^{k-1}b, \quad P^{k+1}b\},$$
$$D = \operatorname{cone} \{D_o, P^{k-1}b, P^k b\}.$$

(here cone denotes the convex conic hull).

P r o o f.

1. First, we shall prove that D_o and D contain all the vectors listed above, having Lemma 3.1. in mind.

Define a subspace S of Π_o by

$$S = \{x \in R^n; \quad \langle x, l\rangle = \ldots = \langle A^k x, l\rangle = 0\}.$$

From (4.1) we get $Cu \in S$ for every $u \in R^r$. Moreover, the space S is invariant with respect to P, which follows from the equalities

$$\langle A^i Px, l\rangle = \langle A^{i+1}x, l\rangle - \rho\langle A^{k+1}x, l\rangle\langle A^i b, l\rangle = 0$$

for $i = 0, \ldots, k$ and $x \in S$. Hence the system

$$\dot{x} \in Px + CU, \; x(0) = 0,$$

can be considered as a system on S. Moreover, if $x(\cdot)$ is its trajectory, then

$$\dot{x}(s) \in Ax(s) - \rho\langle A^{k+1}x(s), l\rangle b + CU \subseteq Ax(s) + CU + b[-1, 1]$$

if only $x(s)$ is sufficiently small. From this, by a standard argument, we conclude that

$$\pm P^i[C]_j \in D_o, \quad i = 0, \ldots, n-2, \quad j = 1, \ldots, r.$$

Now, take a measurable $v(\cdot)$ on $[0, 1]$, $|v(s)| \le 1$, and denote $v_t(s) = v(s/t)$ for $t > 0$. Let us apply $v_t(\cdot)$ on $[0, t]$ as a selection of $[-1, 1]$ in (4.2). The solution $x_t(\cdot)$, corresponding to the zero-selection of U, is then a trajectory of (4.2) and is given by

$$x_t(s) = \int_0^s e^{P(s-\tau)} b \cdot v_t(\tau)d\tau = t \int_0^{s/t} e^{P(s-\theta t)} b, \; v(\theta)d\theta.$$

Denoting $\omega = s/t$, we have

$$x_t(s) = t\int_0^\omega e^{Pt(\omega-\theta)} b \cdot v(\theta)d\theta = \sum_{i=0}^\infty t^{i+1} P^i b \int_0^\omega \frac{(\omega-\theta)^i}{i!} v(\theta)d\theta.$$

Denote $\beta_i = \int_0^1 \frac{(1-\theta)^i}{i!} v(\theta)d\theta$. Then $x(t) = \sum_{i=0}^\infty t^{i+1}\beta_i P^i b$. The trajectory $x_t(\cdot)$ satisfies the state constraint in (2.1) if and only if

$$\langle x(s), l\rangle = \sum_{i=0}^\infty t^{i+1}\langle P^i b, l\rangle \int_0^\omega \frac{(\omega-\theta)^i}{i!} v(\theta)d\theta$$

$$= t^{k+1} \int_0^\omega \frac{(\omega-\theta)^k}{k!} v(\theta)d\theta \, \langle P^k b, l\rangle \ge 0.$$

for every $\omega \in [0, 1]$. We used here the relations $\langle P^i b, l\rangle = 0$ for $i \ne k$, which easily follows from the definition of P and the property of k.

Thus, we obtained that the trajectory $x_t(\cdot)$ on $[0, t]$, corresponding to $v(\cdot)$, satisfies the state constraint if and only if

$$\int_0^\omega \frac{(\omega-\theta)^k}{k!} v(\theta)d\theta \geqslant 0, \quad \omega \in [0, 1].$$

Moreover, $x_t(t) \in \Pi_o$ if and only if $\beta_k = 0$.

Now, take $m \in \{0, \ldots, k-2\}$ and an arbitrary $k+1$ - times differentiable function $\varphi(\cdot)$ on $[0, 1]$, satisfying the relations

(i) $\varphi(0) = \varphi'(0) = \dots = \varphi^{(k)}(0) = 0;$

(ii) $\varphi(1) = 0, \ \varphi^{(k-m)}(1) = \delta, \ \varphi^{(k-m+1)}(1) = \dots = \varphi^{(k)}(1) = 0$

where δ is equal to 1 or to -1.

(iii) $\varphi(\omega) \geq 0, \quad \omega \in [0, 1].$

Such a function obviously exists since $m \leq k - 2$. Observe that for $m = k - 1$, $\varphi(\cdot)$ with the above properties also exists but if $\delta = -1$. Defining $v(s) = \varphi^{(k+1)}(s)$, we obviously have

$$\varphi(\omega) = \int_0^\omega \frac{(\omega - \theta)^k}{k!} v(\theta) d\theta$$

and, moreover, $\beta_i = \varphi^{(k-i)}(1), \quad i = 0, \dots, k$. From (ii) and (iii) we conclude that

$$x_t(t) = \delta t^{m+1} P^m b + o(t^{m+1}) \in R(t) \cap \Pi_o,$$

which means that $\pm P^m b \in D_o$, $m = 0, \dots, k - 2$ and $-P^{k-1} b \in D_o$.

If we remove the requirement $\varphi(1) = 0$ in (ii), then obviously $\varphi(\cdot)$ exists also for $m = k - 1$ and $\delta = 1$. Hence $P^{k-1} b \in D$. Taking $m = k$ and replacing (ii) by

(ii') $\varphi(1) = 1, \ \varphi(1) = \dots = \varphi^{(k)}(1) = 0,$

we get $P^k b \in D$.

The inclusion $P^{k+1} b \in D_o$ can be proved in the same way by using a $(k + 2)$-times differentiable function $\Psi(\cdot)$ on $[0, 1]$, satisfying the relations

(i) $\Psi(0) = \Psi'(0) = \dots = \Psi^{(k+1)}(0);$

(ii) $\Psi(1) = 1, \quad \Psi'(1) = \dots = \Psi^{(k+1)}(1) = 0$

(iii) $\Psi'(\omega) \geq 0, \quad$ for $\omega \in [0, 1].$

Thus, the first part of the proof is completed.

2. We shall prove that D_o and D do not contain more vectors than is claimed by Theorem 4.1.

Denote

$$S_o = \text{Lin}\,\{Cu, PCu, \dots, P^{n-2}Cu; \ u \in R^r\},$$

$$M^m = \text{Lin}\,\{b, \dots, P^{m-1}b\}.$$

Let q be an arbitrary element of D (or of D_o). By definition, there are functions $\varphi(\cdot)$ and $o(\cdot)$, $\varphi(t) > 0$, $o(\cdot)/\varphi(t)$, tending to zero with t, such that

$$\varphi(t)q + o(t) \in R(t) \quad (\text{resp. } R(t) \cap \Pi_o).$$

Let $u_t(\cdot): [0, t] \to U$ and $v_t(\cdot): [0, t] \to [-1, 1]$ be such that the corresponding trajectory $x_t(\cdot)$ of (4.2) satisfies the state constraint $\langle x_t(s), l\rangle \geq 0$, $s \in [0, t]$ (resp $\langle x_t(t), l\rangle = 0$, in addition) and

$$\varphi(t)q + o(t) = x_t(t).$$

Changing the variable in the Cauchy formula for $x_t(\cdot)$ and denoting $\omega = s/t$, we get

$$(4.3) \quad x_t(s) = t \int_0^{\omega} e^{tP(\omega-\Theta)} b, v_t(t\Theta)d\Theta + \xi(t, \omega)$$

$$= \sum_{i=0}^{\infty} t^{i+1} P^i b \, \Psi_i(t, \omega) + \xi(t, \omega)$$

where $\xi(t, \omega) \in S_o$ and $\Psi_i(t, \omega)$ are defined in an obvious way. The inequality $\langle x_t(s), l\rangle \geq 0$ is satisfied if and only if

$$(4.4) \quad \Psi_k(t, \omega) = \int_0^{\omega} \frac{(\omega - \Theta)^k}{k!} v_t(t\Theta)d\Theta \geq 0, \quad \omega \in [0, 1].$$

Then $\frac{\partial}{\partial\omega}\Psi_{k+1}(t, \omega) = \Psi_k(t, \omega) \geq 0$ and $\Psi_{k+1}(t, 1) = \max \{\Psi_{k+1}(t, \omega);\ \omega \in [0, 1]\}$. Hence

$$\Psi_{k+2}(t, \omega) = \int_0^{\omega} \Psi_{k+1}(t, \tau)d\tau \leq \int_0^{\omega} \Psi_{k+1}(t, 1)d\tau \leq \Psi_{k+1}(t, 1)$$

and, successively,

$$\Psi_i(t, \omega) \leq \Psi_{k+1}(t, 1), \quad i \geq k + 1.$$

Taking (4.3) into account, we obtain

$$\varphi(t)q + o(t) = \sum_{i=0}^{k+1} t^{i+1} P^i b \, \Psi_i(t, 1) + t^{k+3} \Psi_{k+1}(t, 1)\zeta(t) + $$

$$+ \zeta(t, 1)$$

where $\zeta(t)$ is bounded. From this we easily get $q \in M^{k+2} + S_o$, and since $\Psi_{k+1}(t, 1) \geq 0$, in fact, $q \in \text{cone}\,\{M^{k+1} + S_o, P^{k+1}b\}$, which completes the proof of the representation of D.

If $q \in D_o$, then $\Psi_k(t, 1) = 0$ and, because of (4.4) and

$\Psi_{k-1}(t, 1) = \frac{\partial}{\partial\omega} \Psi_k(t, 1)$, we obtain $\Psi_{k-1}(t, 1) \leq 0$, which easily implies that $q \in \text{cone}\,\{M^{k-1} + S_o, -P^{k-1}b, P_k b\}$. The proof is completed.

REFERENCES

[1] B i a n c h i n i R. M., *Instant controllability of linear autonomous systems,* J. Optim. Theory and Appl., 39(2), (1983), 237-250.

[2] B r a m m e r R. F., *Differential controllability and the solution of linear inequalities,* Part 1, IEEE Trans. Automat. Control AC 20 (1975), 128-131.

[3] F r a n c o w s k a H., *Théorème d'application ouverte pour des correspondences,* CRAS (Paris) 302 (1986), 559-562.

[4] S u s s m a n n H. J., *Small time local controllability and continuity of the optimal time function for linear systems,* J. Opt. Theory and Appl., 53(2) (1987), 281-296.

[5] V e l i o v V. M., *On the controllability of control constrained systems* (to appear in: Mathematica Balcanica).

[6] V e l i o v V. M., K r a s t a n o v M. I., *Controllability of piece linear systems.* Systems & Control Letters. 7 (1986), 335-341.

[7] V e l i o v V. M., K r a s t a n o v M. I., *Controllability of discontinuous control systems.* IIASA Working Paper, 1988.

[8] W a l c z a k S., *A note on the controllability of nonlinear systems,* Math. Systems Theory, 17 (1984), 351-356.

Department of Mathematics
University of Sofia

M. I. Krastanov, V. M. Veliov

LOKALNA STEROWALNOŚĆ UKŁADÓW LINIOWYCH
Z OGRANICZENIEM NA STAN

W pracy rozważa się problem momentalnej lokalnej sterowalności liniowego układu z ograniczeniem na współrzędne stanu.