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Marek Balcerzak

ORTHOGONAL σ-IDEALS AND ALMOST DISJOINT FAMILIES

Two σ ideals $\mathfrak{I}$ and $\mathfrak{J}$ of subsets of an uncountable set X are called orthogonal if there are $A \in \mathfrak{I}$ and $B \in \mathfrak{J}$ such that $A \cup B = X$. For a family $\mathfrak{M}$ of σ-ideals on X, we formulate three problems concerning orthogonality. We solve them in the case when $\mathfrak{M}$ consists of all σ-ideals generated by almost disjoint families on ω_1.

1. ORTHOGONALITY OF σ-IDEALS

Recall that measures μ and ν defined on a σ-algebra Σ of subsets of a given set X are orthogonal if there is a set $A \in \Sigma$ such that $\mu(A) = 0$ and $\nu(X \setminus A) = 0$. This definition can obviously be reformulated in the language of σ-ideals (consisting of sets on which the measures vanish). That leads to a general definition of orthogonal σ-ideals (cf. [6], [15]).

We shall use the standard set-theoretical notation (see [7]). Throughout the paper, we assume that X is an uncountable set, and that each σ-ideal of subsets of X contains all singletons and does not contain X. We then use the phrase "a σ-ideal on X". By $[X]^{\leq\omega}$ we denote the family of all countable subsets of X. Thus each σ-ideal on X contains $[X]^{\leq\omega}$. We say that a σ-ideal $\mathfrak{I}$ on X is *generated by* $\mathcal{F} \subseteq \mathcal{P}(X)$ if each set in $\mathfrak{I}$ is contained in the union of a countable subfamily of $\mathcal{F}$. Two σ-ideals $\mathfrak{I}$ and $\mathfrak{J}$ on X are called *orthogonal* (abbr. $\mathfrak{I} \perp \mathfrak{J}$) if there is $E \in \mathfrak{I}$ such that $X \setminus E \in \mathfrak{J}$. We then say that $\mathfrak{J}$ is an *orthogonal complement* of $\mathfrak{I}$. It is obvious that if $\mathfrak{I}, \mathfrak{I}', \mathfrak{J}, \mathfrak{J}'$ are σ-ideals on X fulfilling $\mathfrak{I} \subseteq \mathfrak{I}'$ and $\mathfrak{J} \subseteq \mathfrak{J}'$, then $\mathfrak{I} \perp \mathfrak{J}$ implies $\mathfrak{I}' \perp \mathfrak{J}'$.

The well-known fact that the real line $\mathbb{R}$ is a union of a Lebesgue null set and a set of the first category (briefly: a meager set) yields a classical example of orthogonal σ-ideals (cf. [12]; for the generalizations, see [8]). New nontrivial examples of orthogonal σ-ideals were given in [11], [9], [13], [14] and [1], [2]. The orthogonality of σ-ideals appears in the Sierpiński-Erdös duality theorem (see [12]) which, besides the classical case concerning Lebesgue null sets and meager sets, can be applied to other pairs of σ-ideals (see [2] - [4]). Note that in this theorem (originally formulated for $X = \mathbb{R}$) the Continuum Hypothesis (CH) is assumed. Some special properties of orthogonal ideals are observed in [6] and [15].

In the paper we propose the studying of a few problems connected with the orthogonality of σ-ideals. At first, consider the following natural question: has a σ-ideal $\mathcal{I}$ on X an orthogonal complement $\mathcal{J}$? The answer is no if $\mathcal{I} = [X]^{\leqslant\omega}$. If $\mathcal{I} \neq [X]^{\leqslant\omega}$, the answer is yes since it suffices to fix an uncountable $A \in \mathcal{I}$ and define $\mathcal{J}$ as the family of all $E \subseteq X$ such that $E \cap A$ is countable. The above question becomes interesting if one searches for orthogonal complements in more restrictive families of σ-ideals.

Let $\mathfrak{M}$ be a fixed nonempty family of σ-ideals on X. We say that $\mathcal{I} \in \mathfrak{M}$ is o r t h o g o n a l i z a b l e in $\mathfrak{M}$ if it has an orthogonal complement in $\mathfrak{M}$. The set of all σ-ideals orthogonalizable in $\mathfrak{M}$ will be denoted by $\mathrm{ORT}(\mathfrak{M})$.

PROBLEM A. Establish $\mathrm{ORT}(\mathfrak{M})$.

We say that a set $E \subseteq X$ r e a l i z e s o r t h o g o n a l i t y in $\mathfrak{M}$ if there are $\mathcal{I}$ and $\mathcal{J}$ in $\mathfrak{M}$ such that $E \in \mathcal{I}$ and $X \setminus E \in \mathcal{J}$. The family of all sets realizing orthogonality in $\mathfrak{M}$ will be denoted by $\mathrm{REA}(\mathfrak{M})$. Obviously, neither countable nor co-countable sets belong to $\mathrm{REA}(\mathfrak{M})$.

PROBLEM B. Establish $\mathrm{REA}(\mathfrak{M})$.

From the definitions we easily deduce that

$$\mathrm{ORT}(\mathfrak{M}) = \bigcup \{\{\mathcal{I} \in \mathfrak{M}: E \in \mathcal{I}\}: E \in \mathrm{REA}(\mathfrak{M})\}.$$

The following lemma describes simple relations when two families $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are considered.

LEMMA 1.1. If $\mathfrak{M}_1 \subseteq \mathfrak{M}_2$, then $\mathrm{ORT}(\mathfrak{M}_1) \subseteq \mathrm{ORT}(\mathfrak{M}_2)$ and $\mathrm{REA}(\mathfrak{M}_1) \subseteq \mathrm{REA}(\mathfrak{M}_2)$. □

We say that $\mathcal{I} \in \mathrm{ORT}(\mathfrak{M})$ is **sharper** than $\mathcal{J} \in \mathrm{ORT}(\mathfrak{M})$ (abbr. $\mathcal{I} \prec \mathcal{J}$) if there is $\mathcal{J}' \in \mathfrak{M}$ such that $\mathcal{J} \perp \mathcal{J}'$ and $\mathcal{I} \subseteq \mathcal{J} \cap \mathcal{J}'$ (cf. [2]). Obviously, the relation $\prec$ is antireflexive and transitive. Observe that $\mathcal{I} \prec \mathcal{J}$ implies $\mathcal{I} \subsetneq \mathcal{J}$.

PROBLEM C. Find all pairs $\langle \mathcal{I}, \mathcal{J} \rangle$ from $\mathrm{ORT}(\mathfrak{M}) \times \mathrm{ORT}(\mathfrak{M})$ such that $\mathcal{I} \subsetneq \mathcal{J}$ implies $\mathcal{I} \prec \mathcal{J}$.

The studying of Problems A, B and C for various fixed families $\mathfrak{M}$ is a project of a research. In the present paper, we start that research with the case of σ-ideals generated by almost disjoint families on ω_1.

2. THE σ-IDEALS GENERATED BY ALMOST DISJOINT FAMILIES ON ω_1

An uncountable set $\mathcal{F} \subseteq \mathcal{P}(\omega_1)$ is called an **almost disjoint family** (abbr. adf) on ω_1 if $|A| = \omega_1$ for each $A \in \mathcal{F}$ and if $|A \cap B| < \omega_1$ for any distinct $A, B \in \mathcal{F}$. It is well known that each adf on ω_1 is not maximal with respect to inclusion and (by Zorn's lemma) it can be extended to a maximal adf of size $> \omega_1$ (see [7]). The size of a maximal adf on ω_1 depends on special axioms of set theory (see [5, 7]).

Let T be the set of all cardinalities of adfs on ω_1 and, for $\varkappa \in T$, let $\mathcal{A}(\varkappa)$ denote the set of all σ-ideals which can be generated by adfs on ω_1 of size $\varkappa$ (note that $\varkappa > \omega$ for all $\varkappa \in T$). Then define $\mathcal{A} = \bigcup \{\mathcal{A}(\varkappa) : \varkappa \in T\}$.

Here we study Problems A, B and C when $\mathfrak{M}$ equals $\mathcal{A}(\varkappa)$ or $\mathcal{A}$. Problems A and B seem rather self-evident. We solve them adding an observation about isomorphisms between the respective orthogonal σ-ideals.

A bijection f from X onto X is called an **involution** if $f = f^{-1}$. We say that σ-ideals $\mathcal{I}$ and $\mathcal{J}$ on X are **bi-isomorphic** (abbr. $\mathcal{I} \approx \mathcal{J}$) if there is an involution f from X to X such that $f_*[\mathcal{I}] = \mathcal{J}$ where $f_*: \mathcal{P}(X) \to \mathcal{P}(X)$ is given by $f_*(E) = f[E]$ for $E \in \mathcal{P}(X)$ (cf. [4]).

LEMMA 2.1. For each $E \subseteq \omega_1$ such that $|E| = |\omega_1 \setminus E| = \omega_1$ and for each adf $\mathcal{F}$ on ω_1 containing E, there is an involution f from ω_1 to ω_1 such that $\omega_1 \setminus E$ belongs to the adf $f_*[\mathcal{F}]$.

P r o o f. Consider any bijection g from E onto $\omega_1\setminus E$. Then $f\colon \omega_1 \to \omega_1$ equal to g on E and to g^{-1} on $\omega_1\setminus E$ is the desired involution. □

PROPOSITION 2.2. Let $\varkappa \in T$. For each set $E \subseteq \omega_1$ such that $|E| = |\omega_1\setminus E| = \omega_1$ and for each σ-ideal $\mathfrak{I} \in \mathcal{A}(\varkappa)$ such that $E \in \mathfrak{I}$, there is $\mathfrak{J} \in \mathcal{A}(\varkappa)$ fulfilling $\omega_1\setminus E \in \mathfrak{J}$ and $\mathfrak{I} \approx \mathfrak{J}$.

P r o o f. Let $\mathfrak{F}$ be any adf on ω_1 of size $\varkappa$, generating $\mathfrak{I}$. We can always modify $\mathfrak{F}$ so that $E \in \mathfrak{F}$. Thus assume that $E \in \mathfrak{F}$. Let $\mathfrak{J}$ be the σ-ideal generated by $f_*[\mathfrak{F}]$ where f is the involution from Lemma 2.1. □

COROLLARY 2.3. Let $\varkappa \in T$.

(a) $\mathrm{ORT}(\mathcal{A}(\varkappa)) = \mathcal{A}(\varkappa)$, $\mathrm{ORT}(\mathcal{A}) = \mathcal{A}$;

(b) $\mathrm{REA}(\mathcal{A}(\varkappa)) = \mathrm{REA}(\mathcal{A}) = \{E \subseteq \omega_1\colon |E| = |\omega_1\setminus E| = \omega_1\}$. □

Now, let us turn to Problem C.

LEMMA 2.4. If $\mathfrak{I}$ and $\mathfrak{J}$ belong to $\mathcal{A}$, and $\mathfrak{I} \subseteq \mathfrak{J}$, then, for each adf $\mathfrak{F}$ generating $\mathfrak{I}$, there is an adf $\mathcal{H}$ generating $\mathfrak{J}$ such that, for each $A \in \mathfrak{F}$, there is $B \in \mathcal{H}$ containing A.

P r o o f. Consider any adf $\mathcal{G}$ generating $\mathfrak{J}$. For each $A \in \mathfrak{F}$, choose a countable family $\mathcal{G}_A \subseteq \mathcal{G}$ such that $A \subseteq \bigcup \mathcal{G}_A$. The family

$$\mathcal{H} = \{\bigcup \mathcal{G}_A\colon A \in \mathfrak{F}\} \cup (\mathcal{G}\setminus\bigcup\{\mathcal{G}_A\colon A \in \mathfrak{F}\})$$

is as desired. □

LEMMA 2.5. For $\mathfrak{I}$ and $\mathfrak{J}$ from $\mathcal{A}$, fulfilling $\mathfrak{I} \subseteq \mathfrak{J}$, let $\mathfrak{F}$ and $\mathcal{H}$ have the meanings as in 2.4. If $A \in \mathcal{H}\setminus\mathfrak{I}$, then at least one of the conditions holds:

(1) there is $B \subseteq A$, $B \notin \mathfrak{J}$, such that $\mathfrak{F} \cup \{B\}$ is an adf on ω_1

(2) there are $B \subseteq A$, $B \notin \mathfrak{I}$, and an uncountable adf $\mathfrak{F}_A \subseteq \mathfrak{F}$ such that $\bigcup\mathfrak{F}_A = B$.

P r o o f. Define $\mathfrak{F}_A = \{E \in \mathfrak{F}\colon E \subseteq A\}$. Consider two cases. They will give (1) and (2), respectively.

<u>Case</u> 1. $\bigcup\mathfrak{F}_A \in \mathfrak{I}$. Put $B = A\setminus\bigcup\mathfrak{F}_A$. It suffices to show that $|B \cap E| \leqslant \omega$ for all $E \in \mathfrak{F}$. It $E \in \mathfrak{F}_A$, then $B \cap E = \emptyset$. If $E \in \mathfrak{F}\setminus\mathfrak{F}_A$, then $E \subseteq C$ for some $C \in \mathcal{H}\setminus\{A\}$, by the properties of $\mathcal{H}$ established in 2.4. Hence $|B \cap E| \leqslant |B \cap C| \leqslant |A \cap C| \leqslant \omega$.

<u>Case</u> 2. $\bigcup\mathfrak{F}_A \notin \mathfrak{I}$. Put $B = \bigcup\mathfrak{F}_A$. We get (2) immediately. □

LEMMA 2.6. For any $\mathcal{I}$ and $\mathcal{J}$ from $\mathcal{A}$, such that $\mathcal{I} \subsetneq \mathcal{J}$, there exists an adf generating $\mathcal{J}$ such that, for each $A \in \mathcal{H} \setminus \mathcal{I}$, there is $\mathcal{J}' \in \mathcal{A}$ for which $\mathcal{I} \subseteq \mathcal{J}'$ and $\omega_1 \setminus A \in \mathcal{J}'$ (thus $\mathcal{J} \perp \mathcal{J}'$).

P r o o f. Fix an adf generating $\mathcal{I}$ and choose an adf $\mathcal{H}$ generating $\mathcal{J}$ according to 2.4. Let $A \in \mathcal{H} \setminus \mathcal{I}$. Now, use 2.5. If (1) holds, consider any uncountable adf $\mathcal{F}^*$ such that $\bigcup \mathcal{F}^* = B$ and define $\mathcal{J}'$ as the σ-ideal generated by $\{\omega_1 \setminus B\} \cup \mathcal{F}^*$. Thus $\mathcal{J}' \in \mathcal{A}$ and $\omega_1 \setminus A \in \mathcal{J}'$. To show that $\mathcal{I} \subseteq \mathcal{J}'$, consider any $E \in \mathcal{I}$. Then $E \subseteq \bigcup_{n<\omega} E_n$ for some $E_n \in \mathcal{F}$, $n < \omega$. Since $|\bigcup_{n<\omega} E_n \cap B| \leqslant \omega$, it is obvious that $E \in \mathcal{J}'$. If (2) holds, let $\mathcal{J}'$ be the σ-ideal generated by the adf $\{\omega_1 \setminus B\} \cup \mathcal{F}_A$. Thus the assertion is clear. □

REMARK 2.7. Observe that Lemmas 2.4-2.6 and their proofs work when $\mathcal{A}$ is replaced by $\mathcal{A}(\omega_1)$.

From Lemma 2.6 and Remark 2.7 we derive:

PROPOSITION 2.8. The relation $\prec$ considered on $\mathcal{A}$ (resp. $\mathcal{A}(\omega_1)$) is identical with $\subsetneq$. □

That solves Problem C for $\mathcal{A}$ and $\mathcal{A}(\omega_1)$. For $\mathcal{A}(\varkappa)$ where $\varkappa \in T \setminus \{\omega_1\}$, it remains open.

Note that our results from this section can easily be extended to the case when ω_1 is replaced by any uncountable cardinal λ. Then the definition of an almost disjoint family must be modified in an obvious manner and the family $\mathcal{A}$ would consist of all λ-additive ideals on λ generated by adfs on λ (the λ-additivity of $\mathcal{I}$ means that $\bigcup_{\alpha<\varkappa} A_\alpha \in \mathcal{I}$ whenever $\{A_\alpha \colon \alpha < \varkappa\} \subseteq \mathcal{I}$ and $\varkappa < \lambda$).

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Institute of Mathematics
University of Łódź

Marek Balcerzak

σ-IDEAŁY ORTOGONALNE I RODZINY PRAWIE ROZŁĄCZNE

Dwa σ-ideały $\mathcal{I}$ i $\mathcal{J}$ podzbiorów nieprzeliczalnego zbioru X nazywają się ortogonalne, gdy istnieją $A \in \mathcal{I}$ i $B \in \mathcal{J}$ takie, że $A \cup B = X$. Dla rodziny $\mathfrak{M}$ σ-ideałów na X, sformułowano trzy problemy dotyczące ortogonalności. Podano rozwiązania w przypadku, gdy $\mathfrak{M}$ składa się z σ-ideałów generowanych przez prawie rozłączne rodziny na ω_1.