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TOPOLOGIES RELATED TO (x^a)-POROSITY

In [1] L. Z a j i č e k, using the notion of porosity, defined superporosity, the porosity topology and the *-porosity topology. In our paper we introduce the equivalents of these notions which we get by replacing porosity with (x^{α}) -porosity and (x^{α}, c) -porosity. We examine relationships among the notions which we define in such a way for different α and c.

Let $E \subset \mathbb{R}$ and a < b. By $\lambda(E, a, b)$ and by $\lambda(E, b, a)$ we denote the length of the largest open interval included in (a, b) $\setminus E$. Obviously, $\lambda(E, a, b) = \lambda(\overline{E}, a, b)$.

Let $E \subset R$, $z \in R$, $\alpha \in (0, 1]$ and $c \in (0, \infty]$. Put

$$P_{\alpha}(E, z) = \lim_{h \to 0} \sup \frac{[\lambda(E, z, z + h)]^{\alpha}}{|h|}$$

We say that E is (x^{α}) -porous at z if $p_{\alpha}(E, z) > 0$; (x^{α}, c) --porous at z if $p_{\alpha}(E, z) \ge c$. We say that E is (x^{α}) -superporous at z if EUF is (x^{α}) -porous at z whenever F is (x^{α}) -porous at z. In the same way we define a set which is (x^{α}, c) -superporous at a point. If $\alpha = 1$, then we simply say that E is porous (superporous) at z, instead of saying that E is (x)-porous ((x-superporous)) at z.

Let $P(\alpha)$ and $P(\alpha, c)$ denote the families of all sets (x^{α}) -porous at 0 and (x^{α}, c) -porous at 0, respectively. The properties of the function x^{α} imply

PROPOSITION 1. Let $0 < \alpha < \beta < 1$ and $0 < c \le \infty$. Then P(1, c) \subset P(1) \subset P(β , c) \subset P(β) \subset P(α , c).

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Let SP(α) and SP(α , c) denote the families of all sets (x^{α}) -superporous at 0 and (x^{α}, c) -superporous at 0. Evidently, SP(α) \subset P(α), SP(α , c) \subset P(α , c) and all families SP(α) and SP(α , c) form ideals.

PROPOSITION 2. If 0 < c < 1 and 0 is an accumulation point of E, then E is not (x, c)-superporous at 0.

P r o o f. Put d = c/(2 - c). We may assume that 0 is a right accumulation point of E. Then there is a sequence (x_n) of positive numbers from E, such that $x_{n+1} < (1 - c)x_n$ for every n. Put

$$F = R \setminus \bigcup_{n=1}^{\infty} (x_n - dx_n, x_n + dx_n).$$

Since $\frac{2dx_n}{x_n + dx_n} = \frac{2d}{1 + d} = c$, therefore F is (x, c)-porous at 0.

Let (a, b) be a subinterval of $(0, x_1)$ which does not intersect E U F. Since $x_{n+1} + dx_{n+1} < x_n - dx_n$ for every n, there exists a positive integer n such that (a, b) $\subset (x_n - dx_n, x_n)$ or (a, b) $\subset (x_n, x_n + dx_n)$. In the first case, we have

$$\frac{b-a}{b} = 1 - \frac{a}{b} \le 1 - \frac{x_n - dx_n}{x_n} = d < c$$

and in the second,

$$\frac{b-a}{b} \leq \frac{dx_n}{x_n} = d < c.$$

Hence E U F is not (x, c)-porous at 0 and, consequently, E is not (x, c)-superporous at 0.

PROPOSITION 3. Let $0 < \alpha < 1$ and $0 < c < \infty$. If 0 is an accumulation point of E, then E is not (x^{α}, c) -superporous at 0.

Proof. We may assume that 0 is a right accumulation point of E. Then there exists a sequence (x_n) of positive numbers from E such that $nx_{n+1} < x_n$ for every n. Hence there is a positive integer n_n such that

$$x_{n+1} + \frac{1}{2}(cx_{n+1})^{1/\alpha} < x_n - \frac{1}{2}(cx_n)^{1/\alpha}$$

for $n \ge n_{-}$. Put

$$F = R \setminus \bigcup_{n=n_{\alpha}}^{\infty} (x_n - \frac{1}{2}(cx_n)^{1/\alpha}, x_n + \frac{1}{2}(cx_n)^{1/\alpha}).$$

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$$\frac{[(cx_n)^{1/\alpha}]^{\alpha}}{x_n + \frac{1}{2}(cx_n)^{1/\alpha}} = \frac{c}{1 + \frac{1}{2}c^{1/\alpha}x_n^{(1/\alpha)-1}} \xrightarrow[n \to \infty]{} c,$$

therefore F is (x^{α}, c) -porous at 0.

Let (a, b) be a subinterval of $(0, x_{n_0})$ which does not intersect E U F. Then there is $n \ge n_0$ such that (a, b) $\subset (x_n - \frac{1}{2}(cx_n)^{1/\alpha}, x_n)$ or (a, b) $\subset (x_n, x_n + \frac{1}{2}(cx_n)^{1/\alpha})$. Hence

$$\frac{(b-a)^{\alpha}}{b} \leq \frac{\frac{1}{2^{\alpha}} cx_{n}}{x_{n} - \frac{1}{2} (cx_{n})^{1/\alpha}} = \frac{c}{2^{\alpha} (1 - \frac{1}{2} c^{1/\alpha} x_{n}^{(1/\alpha)-1})}$$

$$\xrightarrow{n \to \infty} \frac{c}{2^{\alpha}} < c.$$

Thus E U F is not (x^{α}, c) -porous at 0 and, consequently, E is not (x^{α}, c) -superporous at 0.

Propositions 2 and 3 guarantee that if we examine (x^{α}, c) -superporosity, we may restrict our considerations to (x, 1)-superporosity and (x^{α}, ∞) -superporosity for $\alpha \in (0, 1)$.

THEOREM 1. No family from the collection $\{SP(\alpha); \alpha \in (0 \ 1]\} \cup \cup \{SP(\alpha, \infty); \alpha \in (0, 1)\} \cup \{SP(1, 1)\}$ is included in any other family from this collection.

To prove Theorem 1, we shall use the following five examples. If $E \subset R$, then $-E = \{-x; x \in E\}$ and |E| denotes the Lebesgue measure of E.

Example 1. Let $\beta \in (0, 1)$ and $p \in (0, 1)$. Put

 $E = \bigcup_{n=1}^{\infty} [p^n - \frac{p^{n/\beta}}{n}, p^n].$

Then the set $A = E \cup (-E)$ is (x^{β}) -superporous at 0. This set is neither (x, 1)-porous at 0 nor (x^{α}) -superporous at 0 or (x^{α}, ∞) -superporous at 0 for $0 < \alpha < \beta$.

Proof. Since

$$\frac{(p^{n} - \frac{p^{n/\beta}}{n}) - p^{n+1}}{p^{n} - \frac{p^{n/\beta}}{n}} = \frac{1 - \frac{p^{(n/\beta) - n}}{n} - p}{1 - \frac{p^{(n/\beta) - n}}{n}} \xrightarrow[n \to \infty]{} 1 - p < 1,$$

thus A is not (x, 1)-porous at 0. On the other hand,

$$\frac{\left(\frac{p^{n/\beta}}{n}\right)^{\alpha}}{p^{n}} = \frac{p^{n(\alpha-\beta)/\beta}}{n^{\alpha}} \xrightarrow[n \to \infty]{} \infty.$$

This means that R\A is (x^{α}, ∞) -porous at 0. Consequently, since $R = A \cup (R \setminus A)$ and R is not (x^{α}) -porous at 0, we conclude that A is neither (x^{α}) -superporous at 0 nor (x^{α}, ∞) -superporous at 0.

Now we show that A is (x^{β}) -superporous at 0. Let B be (x^{β}) -porous at 0. Without loss of generality we may assume that B is (x^{β}) -porous on the right at 0, i.e.

 $\lim_{h \to 0^+} \sup_{h \to 0^+} \frac{\left[\lambda(B, 0, h)\right]^{\beta}}{h} > 0.$ Hence there are a positive number r and a sequence (h_n) tending decreasingly to 0, such that

(1)
$$(h_n - rh_n^{1/\beta}, h_n) \cap B = \emptyset$$
 for every n.

Let (k_n) be a sequence of positive integers such that

(2) $p^{k_n} \le h_n < p^{k_n-1}$ for every n.

Since $\frac{rh_n^{1/\beta}}{h_n} \xrightarrow[n \to \infty]{} 0$, there is a positive integer n_0 with

(3)
$$p^{n+1} \leq h_n - rh_n^{1/\beta}$$
 for $n \geq n_0$.

Let (c, d) be an interval of the maximal length, contained in $(h_n - rh_n^{1/\beta}; h_n) \setminus A \setminus B$. From (1), (2) and (3) we conclude that

the set $(h_n - rh_n^{1/\beta}, h_n) \setminus A \setminus B$ consists of one or two components. So,

$$d - c \ge \frac{1}{2} |(h_n - rh_n^{1/\beta}, h_n) \setminus A \setminus B|$$
$$\ge \frac{1}{2} (rh_n^{1/\beta} - \frac{p^k n^{/\beta}}{k_n} - \frac{p^{(k_n - 1)/\beta}}{k_n - 1})$$

> $\frac{1}{2}(rp^{k_n/\beta} - \frac{2p^{(k_n-1)/\beta}}{k_n-1}) = \frac{1}{2}p^{(k_n-1)/\beta}(rp^{1/\beta} - \frac{2}{k_n-1})$

Hence it follows that

$$\frac{(d-c)^{\beta}}{d} > \frac{p^{k_n^{-1}}(rp^{1/\beta} - \frac{2}{k_n^{-1}})^{\beta}}{2^{\beta}p^{k_n^{-1}}} \quad \xrightarrow{n \to \infty} \frac{r^{\beta}p}{2^{\beta}} > 0.$$

Thus A U B is (x^{β}) -porous at 0 and, therefore, A is (x^{β}) -superporous at 0.

Example 2. Let $\alpha \in (0, 1)$ and $p \in (0, 1)$. We define inductively a sequence (a_n) by putting

$$a_1 = p_1$$

$$a_{n+1} = a_n - a_n^{1/\alpha}$$
 for $n \ge 1$.

Obviously, the sequence (a,) is decreasing and tends to 0. Put

$$E = \bigcup_{n=1}^{\infty} \left[a_n - \frac{a_n^{1/\alpha}}{n}, a_n \right].$$

Then the set $A = E \cup (-E)$ is (x^{α}) -superporous at 0 and is not (x^{α}, ∞) -porous at 0. In consequence, A is neither porous at 0, (x, 1)-porous at 0, (x^{β}) -porous at 0 nor (x^{β}, ∞) -porous at 0 for $\alpha < \beta < 1$.

Proof. Let (c, d) be an interval which does not intersect A. We assume that c > 0 (if c < 0, then the proof is similar). There is a positive integer n such that $(c, d) \subset$

 $\subset (a_{n+1}, a_n - \frac{a_n^{1/\alpha}}{n})$. Hence

$$\frac{(d-c)^{\alpha}}{d} \leq \frac{(a_n - a_{n+1})^{\alpha}}{a_{n+1}} = \frac{a_n}{a_n - a_n^{1/\alpha}} \xrightarrow{n \to \infty}$$

and, consequently, A is not (x^{α}, ∞) -porous at 0.

Now, we show that A is (x^{α}) -superporous at 0. Let B be (x^{α}) -porous at 0. We may assume that B is (x^{α}) -porous on the right at 0. There are a positive number r and a sequence (h_n) tending decreasingly to 0 such that

(1)
$$(h_n - rh_n^{1/\alpha}, h_n) \cap B = \emptyset$$
 for every n.

Let (k_n) be a sequence of positive integers with

(2)
$$a_{k_n} \leq h_n < a_{k_n-1}$$
 for every n

Let (c, d) be an interval of the maximal length, contained in $(h_n - rh_n^{1/\alpha}, h_n) \setminus A \setminus B$. We consider two cases:

(i)
$$h_n - rh_n^{1/\alpha} < a_{k_n+1}$$
.

Then, by (1) and (2), we have

$$d - c \ge (a_{k_n} - \frac{a_{k_n}^{1/\alpha}}{k_n} - a_{k_n+1} = a_{k_n}^{1/\alpha}(1 - \frac{1}{k_n}).$$

Hence

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$$\frac{(d-c)^{\alpha}}{d} \geq \frac{a_{k_{n}}(1-\frac{1}{k_{n}})^{\alpha}}{a_{k_{n}-1}} = (1-a_{k_{n}-1}^{(1/\alpha)-1}(1-\frac{1}{k_{n}})^{\alpha} \xrightarrow{n\to\infty} 1.$$

(ii)
$$a_{k_n+1} \leq h_n - rh_n^{1/\alpha}$$
.

From (1) and (2) it follows that the set $(h_n - rh_n^{1/\alpha}, h_n) \setminus A \setminus B$ consists of one or two components. Thus

$$d - c \ge \frac{1}{2} |(h_n - rh_n^{1/\alpha}, h_n) \setminus A \setminus B|$$
$$\ge \frac{1}{2} (rh_n^{1/\alpha} - \frac{\frac{a_{k_n}^{1/\alpha}}{k_n} - \frac{\frac{a_{k_n}^{1/\alpha}}{k_n}}{k_n^{-1}})$$

>
$$\frac{1}{2}(ra_{k_{n}}^{1/\alpha} - \frac{2a_{k_{n}-1}^{1/\alpha}}{k_{n}-1})$$

$$= \frac{1}{2} a_{k_n-1}^{1/\alpha} [r(1 - a_{k_n-1}^{(1/\alpha)-1})^{1/\alpha} - \frac{2}{k_n-1}]$$

So,

$$\frac{(d-c)^{\alpha}}{d} \ge \frac{(d-c)^{\alpha}}{a_{k_{n}}-1} > \frac{[r(1-a_{k_{n}}^{(1/\alpha)}-1)^{(1/\alpha)} - \frac{2}{k_{n}-1}]^{\alpha}}{2^{\alpha}}$$

$$\xrightarrow{n\to\infty} \frac{r^{\alpha}}{2^{\alpha}}$$
.

From (i) and (ii) we conclude that

$$\limsup_{h \to 0} \frac{\left[\lambda(A \cup B, 0, h)\right]^{\alpha}}{|h|} \ge \min \{1, \frac{r^{\alpha}}{2^{\alpha}}\} > 0.$$

This means that A U B is (x^{α}) -porous at 0 and, therefore, A is (x^{α}) -superporous at 0.

REMARK. Put $E' = \{a_n; n \in N\}$ and $A' = E' \cup (-E')$. It is easy to see that A' has the properties described in Example 2. This set cannot be used in the proof of Theorem 3.

Example 3. Let $\beta \in (0, 1)$ and let (b_n) be a decreasing sequence of positive numbers such that $\lim_{n \to \infty} \frac{b_{n+1}}{b_n} = 0$ and $b_{n+1} < b_n - b_n^{1/\beta}$ for every n. Put $a_n = b_n - b_n^{1/\beta}$ and

$$E = \bigcup_{n=1}^{\infty} [a_n, b_n].$$

Then the set $A = E \cup (-E)$ is (x^{β}, ∞) -superporous at 0. This set is neither (x^{β}) -superporous at 0 nor (x^{α}) -superporous at 0 or (x^{α}, ∞) -superporous at 0 for 0 < α < β .

Proof. Since $\frac{(b_n - a_n)^{\beta}}{b_n} = 1$ for every n, thus R\A is

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(x ^β)-porous at 0. Therefore A is neither (x^{β}) -superporous at 0
nor	(x^{α}) -superporous at 0 or (x^{α}, ∞) -superporous at 0.
que	Let B be (x^{β}, ∞) -porous on the right at 0. There is a sence (h_n) tending decreasingly to 0 such that
(1)	$(h_n - nh_n^{1/\beta}, h_n) \cap B = \emptyset$ for every n.
	Let (k _n) be a sequence of positive inetegers with
(2)	$b_{k_n} \leq h_n < b_{k_n-1}$ for every n.
	We shall consider intervals (c, d) contained in
(h _n	- $nh_n^{1/\beta}$, h_n)\A\B. We examine two cases:
	(i) $h_n - nh_n^{1/\beta} < b_{k_n+1}$.
	Put (c, d) = (b_{k_n+1}, a_{k_n}) . Then, evidently, (c, d) \subset
⊂()	$h_n - nh_n^{1/\beta}, h_n $ (A) and
	$\frac{(d-c)^{\beta}}{(d-c)^{\beta}} = \frac{(a_{k_{n}} - b_{k_{n}+1})^{\beta}}{(d-c)^{\beta}}$
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	cannot be used in the problem of Thereas are
	$[b_{k_{n}}(1 - b_{k_{n}}^{(1/\beta)-1} - \frac{b_{k_{n}+1}}{b_{k_{n}}})]^{\beta}$
	$\rightarrow \xrightarrow{n} \xrightarrow{n} \xrightarrow{n \rightarrow \infty} \infty$

(ii) $b_{k_n+1} \leq h_n - nh_n^{1/\beta}$.

Let (c, d) be an interval of the maximal length, contained in $(h_n - nh_n^{1/\beta}, h_n) \setminus A \setminus B$. Conditions (1) and (2) guarantee that the set $(h_n - nh_n^{1/\beta}, h_n) \setminus A \setminus B$ consists of one or two components. Thus

$$d - c \ge \frac{1}{2} |(h_n - nh_n^{1/\beta}, h_n) \setminus A \setminus B|$$

$$\ge \frac{1}{2} [nh_n^{1/\beta} - (b_{k_n} - a_{k_n}) - (b_{k_n-1} - a_{k_n-1})]$$

$$> \frac{1}{2} (nh_n^{1/\beta} - 2b_{k_n-1}^{1/\beta}) > \frac{1}{2}h_n^{1/\beta} (n - 2).$$

So,

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$$\frac{(d-c)^{\beta}}{d} \ge \frac{(d-c)^{\beta}}{h_{n}} > \frac{(n-2)^{\beta}}{2^{\beta}} \xrightarrow{n \to \infty} \infty$$

From (i) and (ii) it follows that

$$\limsup_{h \to 0} \frac{\left[\lambda(A \cup B, 0, h)\right]^{\beta}}{|h|} = \infty.$$

Hence we conclude that A \cup B is (x^{β}, ∞) -porous at 0 and, consequently, A is (x^{β}, ∞) -superporous at 0.

Example 4. Let $\alpha \in (0, 1)$. Since $\lim_{x \to 0+} \frac{x^{1/\alpha} \log(1/x)}{x} = 0$ there is $b_1 \in (0, 1)$ such that $x^{1/\alpha} \log(1/x) < x$ for each $x \in (0, b_1]$. Put

$$b_{n+1} = b_n - b_n^{1/\alpha} \log(1/b_n)$$
 for $n \ge 1$.

We have thus defined the decreasing sequence (b_n) for which $\lim_{n \to \infty} b_n = 0$ and $\lim_{n \to \infty} (b_{n+1}/b_n) = 1$. Put $a_n = b_n - b_n(b_n - b_{n+1})$ and

$$E = \bigcup_{n=1}^{\infty} [a_n, b_n].$$

Then the set $A = E \cup (-E)$ is (x^{α}, ∞) -superporous at 0 but it is not (x^{β}) -porous at 0 for $\alpha < \beta < 1$. In consequence, A is neither (x^{β}, ∞) -porous at 0 nor porous at 0 or (x, 1)-porous at 0.

Proof. Let (c, d) be an interval which does not intersect A. We may assume that c > 0. There is a positive integer n such that (c, d) \subset (b_{n+1}, a_n). Then

$$\frac{(d - c)^{\beta}}{d} < \frac{(b_n - b_{n+1})^{\beta}}{b_{n+1}}$$
$$= \frac{b_n^{\beta/\alpha} [\log(1/b_n)]^{\beta}}{b_n (1 - b_n^{(1/\alpha) - 1} \log(1/b_n))} \xrightarrow{n \to \infty} 0.$$

This means that A is not (x^{β}) -porous at 0.

Let B be (x^{α}, ∞) -porous on the right at 0. There is a sequence (h_n) tending decreasingly to 0 such that

(1)
$$(h_n - nh_n^{1/\alpha}, h_n) \cap B = \emptyset$$
 for every n.

Let (k_n) be a sequence of positive integers with

(2)
$$b_{k_n} \leq h_n < b_{k_n-1}$$
 for every n.

We consider two cases:

(i)
$$h_n - nh_n^{1/\alpha} < b_{k_n+1}$$
.

Let (c, d) = (b_{k_n+1}, a_{k_n}) . Then, by (1) and (2), (c, d) \subset $\subset (h_n - nh_n^{1/\alpha}, h_n) \setminus A \setminus B$ and

$$\frac{(d - c)^{\alpha}}{d} = \frac{(b_{k_n} - b_{k_n+1})^{\alpha}(1 - b_{k_n})^{\alpha}}{a_{k_n}}$$

$$> \frac{(1 - b_{k_n})^{\alpha} b_{k_n} [\log(1/b_{k_n})]^{\alpha}}{b_{k_n}} \xrightarrow{n \to \infty}$$

(ii) $b_{k_n+1} \le h_n - nh_n^{1/\alpha}$.

Let (c, d) be an interval of the maximal length, contained in $(h_n - nh_n^{1/\alpha}, h_n) \setminus A \setminus B$. From (1) and (2) it follows that the set $(h_n - nh_n^{1/\alpha}, h_n) \setminus A \setminus B$ consists of one or two components. Thus

$$d - c \ge \frac{1}{2} |(h_n - nh_n^{1/\alpha}, h_n) \setminus A \setminus B|$$

$$\ge \frac{1}{2} [nh_n^{1/\alpha} - (b_{k_n} - a_{k_n}) - (b_{k_n-1} - a_{k_n-1})]$$

$$\ge \frac{1}{2} [nb_{k_n}^{1/\alpha} - 2b_{k_n-1}b_{k_n-1}^{1/\alpha} \log(1/b_{k_n-1})]$$

$$= \frac{1}{2} b_{k_n-1}^{1/\alpha} [n(1 - b_{k_n-1}^{(1/\alpha)-1} \log(1/b_{k_n-1}))^{1/\alpha} - 2b_{k_n-1} \log(1/b_{k_n-1})]$$

So,

$$\frac{(d - c)^{\alpha}}{d} \ge \frac{(d - c)^{\alpha}}{b_{k_n - 1}}$$

$$> \frac{\left[n(1 - b_{k_n - 1}^{(1/\alpha) - 1} \log(1/b_{k_n - 1}))^{1/\alpha} - 2b_{k_n - 1} \log(1/b_{k_n - 1})\right]^{\alpha}}{2^{\alpha}}$$

$$\xrightarrow{n \to \infty} \infty.$$

From (i) and (ii) we conclude that $\limsup_{h \to 0} \frac{[\lambda(A \cup B, 0, h)]^{\alpha}}{|h|} = \infty$. Hence A U B is (x^{α}, ∞) -porous at 0 and, consequently, A is (x^{α}, ∞) -superporous at 0.

REMARK. Put $E' = \{b_n; n \in N\}$ and $A' = E' \cup (-E')$. It is easy to see that A' has the properties described in Example 4. This set cannot be used in the proof of Theorem 3.

Example 5. Let te(0, 1) and let (b_n) be a decreasing sequence of positive numbers such that $\lim_{n \to \infty} (b_{n+1}/b_n) = 0$ and $b_{n+1} < tb_n$ for every n. Put

 $E = \bigcup_{n=1}^{\infty} [tb_n, b_n].$

Then the set $A = E \cup (-E)$ is (x, 1)-superporous at 0. This set is

neither superporous at 0 nor (x^{α}) -superporous at 0 or (x^{α}, ∞) --superporous at 0 for 0 < α < 1.

Proof. Since
$$\frac{b_n - tb_n}{b_n} = 1 - t > 0$$
, therefore R\A is

porous at 0. Thus A is neither superporous at 0 nor (x^{α}) -superporous at 0 or (x^{α}, ∞) -superporous at 0.

Let B be (x, 1)-porous on the right at 0. There is a sequence (h_n) tending decreasingly to 0 such that

(1)
$$(\frac{1}{n}h_n, h_n) \cap B = \emptyset$$
 for every n.

Let (k_n) be a sequence of positive integers with

(2)
$$b_{k_n} \leq h_n < b_{k_n-1}$$
 for every n.

We examine two cases:

(i) $\lim_{n \to \infty} \sup \frac{b_{k_n}}{h_n} > 0.$

Put c = max{ h_n/n , b_{k_n+1} } and d = tb_{k_n}. Then, evidently, (c, d) $\subset (h_n/n, h_n) \setminus A \setminus B$ and

$$\frac{d - c}{d} = 1 - \frac{\max\{h_n/n, b_{k_n+1}\}}{tb_{k_n}} \xrightarrow[n \to \infty]{} 1.$$

(ii) $\lim_{n \to \infty} \sup \frac{h_n}{h_n} = 0.$

Put (c, d) = (b_{k_n}, h_n) . Then also (c, d) $\subset (h_n/n, h_n) \setminus A \setminus B$ and moreover,

$$\frac{d - c}{d} = 1 - \frac{b_k}{h_n}$$

Thus, from (i) and (ii) it follows that $\lim_{h \to 0} \sup \frac{\lambda(A \cup B, 0, h)}{|h|} = 1$ Hence A U B is (x, 1)-porous at 0 and, therefore, A is (x, 1)--superporous at 0.

Topologies related to (x^{α}) -porosity

In the sequel, $A \not\subset B$ will mean that A is not contained in B.

Proof of Theorem 1. Let $0 < \alpha < \beta < \gamma < 1$.

The conditions $SP(\beta) \not\subset SP(\alpha)$ and $SP(\beta) \not\subset SP(\alpha, \infty)$ follow from Example 1.

Example 2 implies the conditions $SP(\beta) \notin SP(\beta, \infty)$, $SP(\beta) \notin SP(\gamma)$, $SP(\beta) \notin SP(\gamma, \infty)$, $SP(\beta) \notin SP(1)$ and $SP(\beta) \notin SP(1, 1)$.

The conditions $SP(\beta, \infty) \not\subset SP(\beta)$, $SP(\beta,\infty) \not\subset SP(\alpha)$ and $SP(\beta,\infty) \not\subset \varphi$ $\not\subset SP(\alpha,\infty)$ result from Example 3.

Example 4 implies the conditions $SP(\beta, \infty) \notin SP(\gamma)$, $SP(\beta, \infty) \notin \varphi$ $\notin SP(\gamma, \infty)$, $SP(\beta, \infty) \notin SP(1)$ and $SP(\beta, \infty) \notin SP(1, 1)$.

The conditions SP(1) \notin SP(1, 1), SP(1) \notin SP(α) and SP(1) \notin \notin SP(α , ∞) follow from Example 1.

The conditions SP(1, 1) \notin SP(1), SP(1, 1) \notin SP(α) and SP(1, 1) \notin SP(α , ∞) are implied by Example 5.

Let $\alpha \in (0, 1]$ and $c \in (0, \infty]$. We say that a set $G \subset R$ is (x^{α}) -porosity open if R\G is (x^{α}) -superporous at each point of G; (x^{α}, c) -porosity open if R\G is (x^{α}, c) -superporous at each point of G. Since the family of all (x^{α}) -superporous sets at a fixed point z forms an ideal, the family of all (x^{α}) -porosity topology and denote by T_{α} . Similarly, the family $T_{\alpha,c}$ of all (x^{α}, c) -porosity topology. We call it the (x^{α}) -porosity topology rooting topology. Evidently, all topologies T_{α} and $T_{\alpha,c}$ are finer than the Euclidean topology. Properties 2 and 3 imply that all topologies $T_{\alpha,c}$ for $\alpha \in (0, 1)$ and $c \in (0, \infty)$ and all topology. On the other hand, Examples 1-5 imply:

THEOREM 2. No topology from the collection $\{T_{\alpha}; \alpha \in (0, 1]\} \cup \cup \{T_{\alpha, \infty}; \alpha \in (0, 1)\} \cup \{T_{1,1}\}$ is included in any other topology from this collection.

Proof. We only show that $T_{1,1} \notin T_1$. Let A be the set defined in Example 5. Then the set $G = R \setminus A$ is (x, 1)-porosity open and it is not porosity open. The remaining conditions are proved analogously (compare the proof of Theorem 1).

Let τ_1 and τ_2 be topologies on a set X. We say that τ_1 and

 τ_2 are S-related if, for any set $A \subset X$, we have $\operatorname{int}_{\tau_1} A \neq \emptyset$ if and only if $\operatorname{int}_{\tau_2} A \neq \emptyset$.

Let a set $A \subset R$ be (x^{α}) -superporous $((x^{\alpha}, c)$ -superporous) at a point z. It is easy to see that also \overline{A} is (x^{α}) -superporous $((x^{\alpha}, c)$ -superporous) at z. Hence, as in L. Z a j i č e k s paper [1] we can prove that all topologies T_{α} and $T_{\alpha,c}$ are S-related to the Euclidean topology (see [1], Proposition 3).

Indeed, let U = int $T_{\alpha} G \neq \emptyset$ and $z \in U$. Then R\A is (x^{α}) --superporous at z. Consequently, R\int G = R\int U = $\overline{R \setminus U}$ is (x^{α}) --superporous at z, whence int $G \neq \emptyset$.

Put $T^*_{\alpha} = \{G \mid P; G \text{ is } (x^{\alpha}) \text{-porosity open and } P \text{ is a first category set}\},$

 $T_{\alpha,C}^* = \{G \mid P; G \text{ is } (x^{\alpha}, c) \text{-porosity open and } P \text{ is a first category set} \}.$

The families T_{α}^{*} and $T_{\alpha,c}^{*}$ form topologies. Theorem 1 from [1] implies that they are category density topologies (i.e. they have the form $\{\phi(A) \setminus P\}$; A has the Baire property and P is of the first category} where ϕ is an operator of lower density). We call them the $(x^{\alpha})^{*}$ -porosity topology and the $(x^{\alpha}, c)^{*}$ -porosity topology, respectively. From Examples 1-5 we conclude:

THEOREM 3. No topology from the collection $\{T_{\alpha}^{*}; \alpha \in (0, 1]\} \cup \cup \{T_{\alpha,\infty}^{*}; \alpha \in (0, 1)\} \cup \{T_{1,1}^{*}\}$ is included in any other topology from this collection.

Proof. It is sufficient to show that $T_{1,1}^* \notin T_1^*$ (the remaining conditions are proved similarly). Let A be the set defined in Example 5. Then the set $G = R \setminus A$ is (x, 1)-porosity open, hence it belongs to $T_{1,1}^*$.

Suppose to the contrary that G belongs to T_1^{\star} . Then there are a porosity open set U and a first category set P such that G = = U\P. Thus R\U = A\P is superporous at 0 and, consequently, $\overline{A} = \overline{A \setminus P}$ is superporous at 0, This contradicts Example 5.

Topologies related to (x^a)-porosity

REMARK. The topology T_1^* is equal to the I-density topology defined by W. Wilczyński (see [2]).

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TOPOLOGIE ZWIĄZANE Z (x^C)-POROWATOŚCIĄ

W pracy rozważane są uogólnienia superporowatości, topologii porowatości i topologii *-porowatości otrzymane przez zamianę w odpowiednich definicjach porowatości na (x^{α}) -porowatość lub (x^{α}, c) -porowatość. Badane są związki między tymi pojęciami dla różnych α i c.

1. Each price of a balance of