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ON DENSITY OF PAIRS OF NATURAL NUMBERS

R. Creighton Buck in [1], using the notion of a density of the set $A = \{a_n : n \in N\}$ where $\{a_n\}_{n \in N}$ is an increasing sequence of positive integers, has introduced a measure defined on some non-trivial class of subsets of the set of positive integers.

In this paper we develop the notion of density for sets of pairs of positive integers. Using the notion of two-dimensional density, we define a measure and we study its properties.

R. Creighton Buck in [1] using the notion of a density of the set $A = \{a_n : n \in N\}$, where $\{a_n\}_{n \in N}$ is an increasing sequence of positive integers, has introduced a measure defined on some non-trivial class of subsets of the set of positive integers. Recall that the density of a set A is equal to $\lim_{n \to \infty} \frac{n}{a_n}$, if this limit exists.

In this paper we shall develop the notion of density for sets of pairs of positive integers. Roughly speaking, the density of a subset B of N x N will mean the limit of the quotient having in the numerator the number of elements of B which lie in the square with vertices (0, 0), (0, n), (n, 0), (n, n) and in the denominator the number n^2 (obviously if it is the number of all integer points in the above square). If $A \subset N$, then $A^{\urcorner} = A_1^{\urcorner} \cup A_2^{\urcorner}$, where by definition $A_1^{\urcorner} = \{\langle a_j, k\rangle : j \in N, 1 \leq k \leq a_j\}$, $A_2 = \{\langle k, a_j\rangle : j \in N, 1 \leq k < a_j\}$. Observe that the density of $A_1^{\urcorner}$ exists if and only if the density of $A_2^{\urcorner}$ exists, and in this case both densities are equal and equal to half of density of $A^{\urcorner}$. In theorem 1 we shall show that if $A \subset N$ possesses a density, then an associated set $A^{\urcorner}$ has the same density. It is a starting point for the development of a measure defined on a family of subsets of N x N and to construct a measure theory similar to that in [1]. In the part of this paper devoted to these problems we shall omit all proofs which are similar to those in Buck's paper.

Let $\{a_k\}_{k \in N}$ be an increasing sequence of positive integers

Definition 1. Let $n \in N$ thus

$i(n) = \text{card}\,\{a_k : a_k \leq n,\ k \in N\}$.

Then the following lemma is true.

Lemma 1. Let $A = \{a_k : k \in N\}$. Then

$$\lim_{n \to \infty} \frac{n}{a_n} = D(A) \text{ if an only if } \lim_{n \to \infty} \frac{i(n)}{n} = D(A).$$

P r o o f. The sufficient condition. For each positive integer $k \in [a_n, a_{n+1}]$ inequalities

$$\frac{i(k)}{a_n} \geq \frac{i(k)}{k} > \frac{i(k)}{a_{n+1}}$$

hold.

Thus

$$\frac{n}{a_n} \geq \frac{i(k)}{k} > \frac{n}{a_{n+1}}$$

By the assumption $\frac{n}{a_n}$ and $\frac{n}{a_{n+1}}$ with $n \to \infty$ tend to D(A) therefore by the theorem about three sequences we have

$$\lim_{k \to \infty} \frac{i(k)}{k} = D(A).$$

The necessary condition.

We assume that there exists $\lim_{n \to \infty} \frac{i(n)}{n} = D(A)$. It is easy to see that $\lim_{n \to \infty} \frac{n}{a_n}$ as a limit of the subsequence of a sequence $\{\frac{i(n)}{n}\}_{n \in N}$ also exists and equals D(A).

We give the denotations which will be valid on in this paper.

Let $\{a_n\}_{n \in N}$ be an increasing sequence of positive integers for which $\lim_{n \to \infty} \frac{n}{a_n} = D(A)$.

Definition 2. Let $n \in N$ then

$I(n) = I_1(n) + I_2(n)$,

where

$I_1(n) = \text{card}\,\{\langle a_j, k\rangle : j \in N,\ a_j \leq n,\ 1 \leq k \leq a_j\}$,

$I_2(n) = \text{card}\,\{\langle k, a_j\rangle : j \in N,\ a_j \leq n,\ 1 \leq k < a_j\}$.

For any positive integer the inequalities

$$I_1(n) - n \leqslant I_2(n) \leqslant I_1(n)$$

hold. From it we have the following lemma.

Lemma 2. If there exists $\lim_{n\to\infty} \frac{2I_1(n)}{n(n+1)}$, then there exists $\lim_{n\to\infty} \frac{2I_2(n)}{n(n+1)}$ and equals $\lim_{n\to\infty} \frac{2I_1(n)}{n(n+1)}$.

We give the following definition.

Definition 3. Let $b_1, b_2 \in R^+$, then

$$i(b_1, b_2) = \text{card}\,\{m : b_1 \leqslant a_m \leqslant b_2$$

Lemma 3. If $\eta > 0$, then

$$\lim_{n\to\infty} \frac{i(n, n(1+\eta)}{n\eta + 1} = D(A).$$

P r o o f. Let $\varepsilon > 0$. Put $\varepsilon_1 < \min\left(\frac{\varepsilon}{3}, \frac{\varepsilon}{6}\eta\right)$. There exists a positive integer N such that for $n > N$.

$$\left|\frac{i(n)}{n} - D(A)\right| < \varepsilon_1.$$

Increasing N, if it is necessary, we can assume that for $n > N$ an additional condition $\frac{1}{n\eta + 1} < \frac{\varepsilon}{3}$ is fulfilled. We shall show that

$$\left|\frac{i(n, n(1+\eta))}{n\eta + 1} - D(A)\right| < \varepsilon$$

for $n > N$. Observe that if $n > N$ then $n(1+\eta) > N$. Thus the following inequalities

$$(D(A) - \varepsilon_1)(n-1) < i(n-1) < D(A) + \varepsilon_1)(n-1)$$

$$(D(A) - \varepsilon_1)[n(1+\eta)] < i([n(1+\eta)]) < (D(A) + \varepsilon_1)[n(1+\eta)]$$

hold.
Hence we have

$$-(D(A) + \varepsilon_1)(n-1) < -i(n-1) < -(D(A) - \varepsilon_1)(n-1) \quad (1)$$

$$(D(A) - \varepsilon_1)(n(1+\eta) - 1) < i([n(1+\eta)]) < (D(A) + {} + \varepsilon_1\, n(1+\eta) \quad (2)$$

Adding sides (1) and (2) we have

$$(D(A) - \varepsilon_1)(n(1+\eta) - 1) - (D(A) + \varepsilon_1)(n-1) < {} < i([n(1+\eta)]) - i(n-1) < D(A) + \varepsilon_1)(n(1+\eta)) + {} - (D(A) - \varepsilon_1)(n-1).$$

Hence

$$D(A)\, n + D(A)\, n\, \eta - D(A) - \varepsilon_1 n - \varepsilon_1 n\eta + \varepsilon_1 - D(A)\, n +$$

$$+ D(A) - \varepsilon_1 n + \varepsilon_1 < i(n, n\,(1 + \eta))$$

$$D(A)\, n\, \eta + D(A)\, n + \varepsilon_1 n\, \eta + \varepsilon_1 n + \varepsilon_1 n - \varepsilon_1 - D(A) n + D(A)$$

and after reducing we have

$$D(A)\, n\, \eta + 2\varepsilon_1 - \varepsilon_1 n\, \eta - 2\varepsilon_1 n < i(n, n(1 + \eta)) <$$

$$< D(A)\, (n\, \eta + 1) + \varepsilon_1 (n\, \eta - 1) + \varepsilon_1 2n. \qquad (3)$$

We divide the sides (3) by $n\,\eta + 1$

$$D(A) - \frac{D(A)}{n\,\eta + 1} - \varepsilon_1 - 2\,\frac{\varepsilon_1}{\eta} < \frac{i(n, n(1 + \eta))}{n\,\eta + 1} < D(A) + \varepsilon_1 + \frac{2\,\varepsilon_1}{\eta}.$$

Hence

$$D(A) - \frac{\varepsilon}{3} - \frac{\varepsilon}{3} - \frac{\varepsilon}{3} < \frac{i(n, n(1 + \eta))}{n\,\eta + 1} < D(A) + \frac{\varepsilon}{3} + \frac{\varepsilon}{3}.$$

Finally we have

$$\left|\frac{i(n, n(1 + \eta))}{n\,\eta + 1} - D(A)\right| < \varepsilon \qquad \text{q.e.d.}$$

We give the following definition.

Definition 4. $I_1(n, n(1 + \eta)) = I_1(n, (1 + \eta)) - I_1(n - 1)$.

Lemma 4. For any $\varepsilon > 0$ there exists $\eta > 0$ and a positive integer N_1 such that for $n > N_1$

$$D(A) - \varepsilon < \frac{2I_1(n, n(1 + \eta))}{n(n\,\eta + 1)\,(2 + \eta)} < D(A) + \varepsilon$$

P r o o f. For any positive integer n the inequalities

$$i(n, n(1 + \eta))\, n \leqslant I_1(n, n(1 + \eta)) \leqslant i(n, n(1 + \eta)) \cdot$$

$$\cdot\, n(1 + \eta) \qquad (4)$$

hold. For any $0 < \varepsilon < 1$ put $\mathfrak{z} = \frac{\varepsilon}{4}$. We choose $\eta < 2\mathfrak{z}$. Then the inequality

$$\frac{\eta}{2 + \eta} < \mathfrak{z} \qquad (5)$$

is satisfied.

From *lemma 3* for η chosen earlier and for the number $\mathfrak{z} > 0$ there exists N_1 such that if $n > N_1$ then

$$D(A) - \mathfrak{z} < \frac{i(n, n(1 + \eta))}{n\,\eta + 1} < D(A) + \mathfrak{z} \qquad (6)$$

By inequality (4) applied for $n > N_1$ and (6) we have

$(D(A) - \gamma)\ (n\eta + 1)\ n < I_1(n, n(1 + \eta)) < (D(A) + \gamma)\ (n\eta + 1)\ n(1 + \eta).$ (7)

We divide the sides (7) by $n(n\eta + 1)\frac{2 + \eta}{2}$. We obtain

$$(D(A) - \gamma)\frac{2}{2 + \eta} < \frac{2I_1(n, n(1 + \eta))}{n(n\eta + 1)(2 + \eta)} < (D(A) + \gamma)\frac{2 + 2\eta}{2 + \eta}.$$

Hence by (5) we have

$$(D(A) - \gamma)(1 - \gamma) < \frac{2I_1(n, n(1 + \eta))}{n(n\eta + 1)(2 + \eta)} < (D(A) + \gamma)(1 + \gamma).$$

Next

$$D(A) - \gamma - D(A)\gamma + \gamma^2 < \frac{2I_1(n, n(1 + \eta))}{n(n\eta + 1)(2 + \eta)} < D(A) + \gamma D(A) + \gamma + \gamma^2,$$

therefore

$$D(A) - 2\gamma < \frac{2I_1(n, n(1 + \eta))}{n(n\eta + 1)(2 + \eta)} D(A) + 3\gamma.$$

Finally

$$D(A) - \varepsilon < \frac{2I_1(n, n(1 + \eta))}{n(n\eta + 1)(2 + \eta)} < D(A) + \varepsilon$$ q.e.d.

Theorem 1. By the preceding assumptions

$$\lim_{n \to \infty} \frac{2I_1(n)}{n(n + 1)} = D(A).$$

P r o o f. Suppose $\varepsilon > 0$. We want to find N_2 such that for $n > N_2$

$$D(A) - \varepsilon < \frac{2I_1(n)}{n(n + 1)} < D(A) + \varepsilon. \quad (8)$$

From *lemma 4* for $\varepsilon/4$ we take $0 < \eta < 1$ and $N_1 \in N$. For $n > N_1$ we have

$$I_1(N_1, n) \leqslant I_1(n) \leqslant I_1(N_1, n) + \frac{N_1(N_1 - 1)}{2}$$

Having divided the sides by $\frac{n(n + 1)}{2}$, we have

$$\frac{2I_1(N_1, n)}{n(n + 1)} \leqslant \frac{2I_1(n)}{n(n + 1)} \leqslant \frac{2I_1(N_1, n)}{n(n + 1)} + \frac{N_1(N_1 - 1)}{n(n + 1)}$$

We take such N_2 so that for $n > N_2$ the inequality

$$\frac{2N_1(2N_1 - 1)}{n(n + 1)} \quad (9)$$

holds.

If $N_1 \leq k \leq 2N_1$ and $n > 2N_1$, then

$$I_1(k, n) \leq I_1(n) \leq I_1(k, n) + \frac{k(k-1)}{2}.$$

Dividing the sides by $\frac{n(n+1)}{2}$, we have

$$\frac{2I_1(k, n)}{n(n+1)} \leq \frac{2I_1(n)}{n(n+1)} \leq \frac{2I_1(k, n)}{n(n+1)} + \frac{k(k-1)}{n(n+1)}.$$

Since we have taken such N_2 so that (9) may hold for $n > N_2$ thus irrespective of what $k \in (N_1, 2N_1)$ is like, we have

$$\frac{k(k-1)}{n(n+1)} < \frac{\varepsilon}{4} \quad \text{for} \quad n > N_2.$$

Now we shall show that for $n > N_2$ (8) holds. Let us make a decreasing sequence

$n_o = n > N_2$,

n_1 such a positive integer so that $[n_1(1 + \eta)] = n_o$,

$n_2 = n_1 - 1$,

n_3 such a positive integer so that $[n_3(1 + \eta)] = n_2$ and so on until we obtain for the first time a number $n_{2i-1} < 2N_1 + 1$. It is clear that $n_{2i-1} \geq N_1$. If it were $n_{2i-1} < N_1$ and $n_{2i-2} > 2N_1$ then $\frac{n_{2i-2}}{n_{2i-1}} > 2$ which is impossible because $1 + \eta < 2$. We denote the number n_{2i-1} as n_{2i_o-1}. We have then

$$I_1(n_{2i_o-1}, n_o) = \sum_{i=1}^{i_o} I_1(n_{2i-1}, n_{2i-2}).$$

Considering the construction of the numbers n_j we have

$$(D(A) - \tfrac{\varepsilon}{4}) \frac{n_{2i-2}^2 - n_{2i-1}^2 + n_{2i-2} + n_{2i-1}}{2} < I_1(n_{2i-1}, n_{2i-2}) <$$

$$< (D(A) + \tfrac{\varepsilon}{4}) \frac{n_{2i-2}^2 - n_{2i-1}^2 + n_{2i-2} + n_{2i-1}}{2}.$$

Summing up $i = 1, 2, \ldots, i_o$, we obtain

$$(D(A) - \tfrac{\varepsilon}{4} \frac{n^2 + n - n_{2i_o-1}^2}{2} < I_1(n_{2i_o-1}, n) <$$

$$< (D(A) + \frac{\varepsilon}{4}) \frac{n^2 + n - n^2_{2i_o-1} + n_{2i_o-1}}{2}.$$

We divide by $\frac{n(n+1)}{n}$ and we have

$$(D(A) - \frac{\varepsilon}{4}) (1 - \frac{n^2_{2i_o-1} - n_{2i_o-1}}{n(n+1)}) < \frac{2I_1(n_{2i_o-1}, n)}{n(n+1)} <$$

$$(D(A) + \frac{\varepsilon}{4}) (1 - \frac{n^2_{2i_o-1} - n_{2i_o-1}}{n(n+1)}).$$

Hence

$$(D(A) - \frac{\varepsilon}{4}) (1 - \frac{\varepsilon}{4}) < \frac{2I_1(n_{2i_o-1}, n)}{n(n+1)} < D(A) + \frac{\varepsilon}{4}.$$

Thus

$$D(A) - \frac{\varepsilon}{2} < \frac{2I_1(n)}{n(n+1)} < D(A) + \frac{\varepsilon}{2}$$

which proves (8).

Let an increasing sequence of positive integers $\{a_n\}_{n\in N}$ denote arithmetic progression of the form $\{a \cdot n + b\}_{n \in N}$ where a, $b \in N$ are any constants for that sequence. Let us consider $A \subset \subset N \times N$ where

$$A = \{\langle a_j, k\rangle : j \in N, 1 \leqslant k \leqslant a_j\} \cup \{\langle k, a_j\rangle : j \in N,$$
$$1 \leqslant k < a_j\}. \tag{10}$$

Let us make the class D_o.

Definition 5. Let the class D_o be a family of sets of the form (10), or of finite unions of such sets, or of the sets which differ from these by finite sets.

Let us have denotations which will be used on. A dot placed above the symbol for a relation will be used to indicate that the relation holds modulo the class of finite sets. Thus $A \dot{\subset} B$ means that if finite sets are deleted from both sets we will have $A \subset B$, while $A \doteq \emptyset$ means that A itself is finite.

Obviously if $A, B \subset N$ and $A \dot{\subset} B$ then $A^{\urcorner} \dot{\subset} B^{\urcorner}$. The class D_o has the following properties.

A 1. If $A \in D_o$, then $A' \in D_o$ where A' is the complement of A.

A 2. If $A \in D_o$, $B \in D_o$, then $A \cup B$ and $A \cap B$ belong to D_o.

A 3. If $A \in D_o$ and $A \doteq B$, then $B \in D_o$.

Definition 6. If A is a set of the form

$$\{\langle a \cdot j + b, k\rangle : j \in N, \ 1 \leqslant k \leqslant a \cdot j + b\} \cup$$
$$\cup \{\langle k, a \cdot j + b\rangle : j \in N, \ 1 \leqslant k < a \cdot j + b\}$$

then $\Delta(A) = \frac{1}{a}$.

If A is the union of the disjoint sets $A_1, A_2, \ldots, A_r$, $r \in N$ of the form as above, then

$$\Delta(A) = \Delta(A_1) + \Delta(A_2) + \ldots + \Delta(A_r).$$

If $\Delta(A)$ is defined and $A \doteq B$, then $\Delta(B) = \Delta(A)$.

The function Δ has the following properties.

B 1. If A and B belong to D_o and $A \dot{\subset} B$, then $\Delta(A) \leqslant \Delta(B)$.

B 2. If A and B belong to D_o and $A \cap B \doteq \emptyset$, then $\Delta(A \cup B) = \Delta(A) + \Delta(B)$.

B 3. If A and B belong to the class D_o, then $\Delta(A \cup B) + \Delta(A \cap B) = \Delta(A) + \Delta(B)$.

We now define an outer measure on N x N.

Definition 7. If $S \subset N \times N$, then $\mu(S) = \inf \Delta(A)$ for $A \dot{\supset} S$ and $A \in D_o$.

The function μ has the following properties.

C 1. If $S_1 \subset N \times N$, $S_2 \subset N \times N$, $S_1 \dot{\subset} S_2$, then $\mu(S_1) \leqslant \mu(S_2)$.

C 2. If $S_1, S_2 \subset N \times N$, then $\mu(S_1 \cup S_2) \leqslant \mu(S_1) + \mu(S_2)$.

C 3. If $A \in D_o$, then $\mu(A) = \Delta(A)$.

Definition 8. $D\mu$ is the class of all sets $S \subset N \times N$ for which $\mu(S) + \mu(S') = 1$.

This class is the Carathéodory extension of D_o since the definition above is equivalent to either of the following

(i) S belongs to $D\mu$ if for any set X

$$\mu(X) = \mu(X \cap S) + \mu(X \cap S')$$

(ii) S belongs to $D\mu$ if, given $\varepsilon > 0$, there exist sets A and B in D_o with $A \dot{\subset} S \dot{\subset} B$ and $\Delta(B) - \Delta(A) = \Delta(B - A) < \varepsilon$.

Let us denote

$$D\mu = \{S : \mu(S) + \mu(S') = 1\}$$

$$\bar{D}\mu = \{S : \forall_{X \subset N \times N} \ \mu(X) = \mu(X \cap S) + \mu(X \cap S')\}$$

$$\hat{D}\mu = \{S \subset N \times N : \forall_{\varepsilon > 0} \ \exists_{A, B \in D_o} \ (A \dot{\subset} S \dot{\subset} B \text{ and } \Delta(B) - \Delta(A) = \Delta(B - A) < \varepsilon)\}$$

One can prove

$$D\mu = \overline{D}\mu = \hat{D}\mu$$

For the class $\hat{D}\mu$ the following properties hold.

D 1. If $S \in \hat{D}\mu$, then $S' \in \hat{D}\mu$.

D 2. If S_1 and S_2 belong to $\hat{D}\mu$, then so do $S_1 \cap S_2$ and $S_1 \cup S_2$.

D 3. If S_1 and S_2 are any two sets of $\hat{D}\mu$, then $\mu(S_1 \cup S_2) + \mu(S_1 \cap S_2) = \mu(S_1) + \mu(S_2)$.

Now we shall try to show that the class $D\mu$ property contains the class D_o. An immediate consequence of the definition of D_o is that if $A \in D_o$ and $\Delta(A) = 0$, then $A \doteq \emptyset$. We shall prove that the class $D\mu$ contains infinite sets of measure zero.

Theorem 2. Let P_o be a set of primes such that $\sum_{p \in P_o} \frac{1}{p} = \infty$. Let S be a set of positive integers having the property that if $p \in P_o$, no more then a finite number of integers of S are divisible by p. Let us form from the elements belonging to the set S an increasing sequence $\{a_n\}_{n \in N}$ and next the set S^* of the form (10). Then $\mu(S^*) = 0$.

P r o o f. Let λ be a product of primes of P_o and let A_k be the arithmetic progression $\{\lambda n + k\}_{n \in N}$ for $k = 1, 2, \ldots, \lambda$. Every positive integer of A_k is divisible by (λ, k), and this in turn is either 1, or a product of primes of P_o. Let us form the sets A_k^* of the form (10) for particular sequences A_k. Consider the set $S^* \cap A_k^*$ in case $(\lambda, k) \neq 1$. First we consider the set $S \cap A_k$. Each element of this set is divisible by at least one prime of P_o which is also a divisor of λ.

By the hypothesis only a finite number of terms of S is divisible by any one prime of P_o, and hence by any of the finite collection of primes dividing λ. We conclude that the set $S \cap A_k$ is finite, thus also the set $S^* \cap A_k^*$ is finite.

It is clear

$$N \times N = \bigcup_{k \leq \lambda} A_k^*$$

and thus

$$S^* = \bigcup_{k \leq \lambda} (S^* \cap A_k^*) = \bigcup_{k \leq \lambda}{}' (S^* \cap A_k^*)$$

where the dash indicates that the union is to be taken only for k with $(\lambda, k) = 1$. Hence we have

$$S^* \dot{\subset} \bigcup_{k \leqslant \lambda} A_k^*$$

and by *lemma 1* and *theorem 1* we have

$$\mu(S^*) \leqslant \sum_{k \leqslant \lambda} \Delta(A_k^*) = \frac{\phi(\lambda)}{\lambda}$$

where as usual $\phi(m)$ is Euler's phi function and is equal to the number of primes to m and less than m. Let us now choose λ as $\prod_{p_o < m} p_o$ where the subscript indicates that we are considering only primes belonging to P_o. Since $\phi(\lambda) = \prod_{p_o \leqslant m} (1 - \frac{1}{p_o}) \leqslant \lambda \exp\{-\sum_{p_o \leqslant m} \frac{1}{p}\}$, we have

$$\mu(S^*) \leqslant \exp\{-\sum_{p_o \leqslant m} \frac{1}{p_o}\}.$$

Using the hypothesis of the set P_o, we obtain $\mu(S^*) = 0$.

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O GĘSTOŚCI ZBIORU PAR LICZB NATURALNYCH

R. Creighton Buck w [1] korzystając z pojęcia gęstości zbioru $A = \{a_n : n \in N\}$ gdzie $\{a_n\}_{n \in N}$ jest rosnącym ciągiem liczb naturalnych, wprowadził pojęcie miary definiowanej na pewnej klasie podzbiorów zbioru liczb naturalnych.

Praca niniejsza przenosi pojęcie gęstości na odpowiednie zbiory par liczb naturalnych. Korzystając z dwuwymiarowej gęstości definiuje się miarę i bada jej własności.